微積分解法 2019 (レポート7回目)

問.

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{\boxed{(1)}}$$

$$\int_0^1 \frac{dx}{x^3+1} = \frac{1}{\boxed{(2)}}\log 2 + \frac{\pi}{\boxed{(2)}\sqrt{\boxed{(2)}}}$$

$\boxed{(1)} = 4$

$\boxed{(2)} = 3$

$$I = \int_0^1 \sqrt{1-x^2}\, dx$$

$x = \sin\theta$ とする. $\frac{dx}{d\theta} = \cos\theta$

$$\begin{aligned} x &: 0 \longmapsto 1 \\ \theta &: 0 \longmapsto \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta)\, d\theta \\ &= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} \\ &= \underline{\frac{\pi}{4}} \end{aligned}$$

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

と部分分数展開される。

$$\frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} = \frac{A(x^2-x+1)+(x+1)(Bx+C)}{(x+1)(x^2-x+1)}$$

$$= \frac{(A+B)x^2+(-A+B+C)x+(A+C)}{x^3+1}$$

であるから，

$$A+B=0,\ -A+B+C=0,\ A+C=1$$

$$\Longleftrightarrow$$

$$A=\frac{1}{3},\ B=-\frac{1}{3},\ C=\frac{2}{3}$$

$$\frac{Bx+C}{x^2-x+1} = -\frac{1}{3}\frac{x-2}{x^2-x+1}$$

$$= -\frac{1}{6}\frac{2x-1}{x^2-x+1} + \frac{1}{2}\frac{1}{x^2-x+1}$$

$$= -\frac{1}{6}\frac{(x^2-x+1)'}{x^2-x+1} + \frac{2}{3}\frac{1}{\left\{\frac{2}{\sqrt{3}}(x-\frac{1}{2})\right\}^2+1}$$

よって

$$\int \frac{Bx+C}{x^2-x+1}\,dx = -\frac{1}{6}\log(x^2-x+1) + \frac{1}{\sqrt{3}}\int\frac{dy}{y^2+1}$$

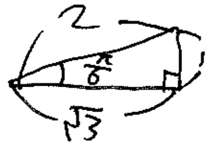
$$= -\frac{1}{6}\log(x^2-x+1) + \frac{1}{\sqrt{3}}\mathrm{Arctan}\left(\frac{2}{\sqrt{3}}(x-\frac{1}{2})\right)$$

$y = \frac{2}{\sqrt{3}}(x-\frac{1}{2})$

$dy = \frac{2}{\sqrt{3}}dx$

$$I=\int_0^1 \frac{dx}{x^3+1} = \left[\frac{1}{3}\log(x+1)\right]_0^1 - \frac{1}{6}\left[\log(x^2-x+1)\right]_0^1$$
$$+\frac{1}{\sqrt{3}}\left[\operatorname{Arctan}\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right)\right]_0^1$$

$$\operatorname{Arctan}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$
$$\operatorname{Arctan}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$



であるから、

$$I = \underline{\frac{1}{3}\log 2 + \frac{\pi}{3\sqrt{3}}}$$