数学I 2019 レポート 8回目

問.

$$f(x,y,z) = x^2+y^2+z^2$$

$$D = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq a^2\}$$

$a \geq 0$ は定数.

$$I = \iiint_D f(x,y,z)\, dxdydz$$ を求めよう。

極座標

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{pmatrix}$$ に変換する.

ヤコビアンは

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = r^{\boxed{(1)}} \sin\theta$$

変数変換公式により、

$$I = \iiint_{\tilde{D}} r^{\boxed{(2)}} \sin\theta \cdot drd\theta d\varphi$$

$$= \frac{\boxed{(3)}}{5}\pi a^{\boxed{(4)}}$$

$\boxed{(1)} = 2$

$\boxed{(2)} = 4$

$\boxed{(3)} = 4$

$\boxed{(4)} = 5$

$$I = \iiint_{\tilde{D}} f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) \times r^2 \sin\theta \, dr d\theta d\varphi$$

$$= \iiint_{\tilde{D}} r^4 \sin\theta \, dr d\theta d\varphi$$

ただし、

$$\tilde{D} = \{(r, \theta, \varphi) \mid 0 \leq r \leq a, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi\}$$

被積分関数が変数分離しているので、

$$I = \int_0^a r^4 dr \times \int_0^\pi \sin\theta \, d\theta \times \int_0^{2\pi} 1 \, d\varphi$$

$$= \left[\frac{r^5}{5}\right]_{r=0}^{r=a} \left[-\cos\theta\right]_{\theta=0}^{\theta=\pi} \left[\varphi\right]_{\varphi=0}^{\varphi=2\pi}$$

$$= \frac{a^5}{5} \times 2 \times 2\pi = \underline{\frac{4}{5}\pi a^5}$$

$\frac{\pi}{10}a^5$