積分版 2017

問12、変数分離型微分方程式をとけ.

(1) $x^2 \dfrac{dy}{dx} + y^2 = 0$

(2) $y \dfrac{dy}{dx} = x(1+y^2)$

(3) $(1+x)y + (1+y)x \dfrac{dy}{dx} = 0$

(4) $xy(1+x^2)\dfrac{dy}{dx} = 1-y^2$

問13、次の微分方程式を解け、

$$\frac{dy}{dx} = \frac{2x^3y - y^4}{x^4 - 2xy^3}$$

ヒント. $\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$ のとき、

$y = u \cdot x$ とおくと.

$$\frac{du}{dx} = \frac{f(u)-u}{x}.$$

変数分離型微分方程式となる.

テスト範囲外

問12

(1) $x^2 \dfrac{dy}{dx} + y^2 = 0$

$$-\frac{1}{x^2} = \frac{1}{y^2}\frac{dy}{dx}$$

$$-\int \frac{1}{x^2}dx = \int \frac{1}{y^2}\frac{dy}{dx}dx = \int \frac{1}{y^2}dy$$

$$-\frac{1}{y} = \frac{1}{x} + C$$

$$\therefore \underline{Cxy + x + y = 0}$$

(2) $y\dfrac{dy}{dx} = x(1+y^2)$

$$\int x\,dx = \int \frac{y}{1+y^2}\frac{dy}{dx}dx = \int \frac{y}{1+y^2}dy$$

$$\frac{1}{2}x^2 = \frac{1}{2}\log(1+y^2) + C'$$

$$e^{x^2} = (1+y^2)e^{2C'}$$

$(C = e^{-2C'})$

$$\therefore \underline{1+y^2 = Ce^{x^2}}$$

(3) $(1+x)y + (1+y)x\dfrac{dy}{dx} = 0$

$$-\frac{1+x}{x} = \frac{1+y}{y}\frac{dy}{dx}$$

$$-\int\left(\frac{1}{x}+1\right)dx = \int\left(\frac{1}{y}+1\right)\frac{dy}{dx}\,dx$$

$$= \int\left(\frac{1}{y}+1\right)dy$$

$$-\log|x| - x + C = \log|y| + y$$

$\therefore\ x + y + \log|xy| + C$

(4) $$xy(1+x^2)\frac{dy}{dx} = 1-y^2$$

$$\frac{1}{x(x^2+1)} = \frac{y}{1-y^2}\frac{dy}{dx}$$

$$\int\frac{dx}{x(x^2+1)} = \int\frac{y}{1-y^2}dy \quad \cdots (☆)$$

さて、被積分関数の部分分数展開は、

・ $$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$ を解く

$$1 = A(x^2+1)+(Bx+C)x$$
$$= (A+B)x^2 + C\cdot x + A$$

$A=1,\ C=0,\ A+B=0$ より、

$A=1, B=-1, C=0$

よって

$$\frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{-x}{x^2+1}$$

・ $$\frac{y}{1-y^2} = \frac{1}{2}\left(\frac{1}{1-y} - \frac{1}{1+y}\right)$$

よって、(☆) は、

$$左辺 = \int\frac{dx}{x(x^2+1)}$$

$$= \int\left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx$$

$$= \log|x| - \frac{1}{2}\log(x^2+1) + C_1$$

右辺 $= \int \frac{y}{1-y^2} dy$

$= \frac{1}{2} \int \left(\frac{1}{1-y} - \frac{1}{1+y} \right) dy$

$= \frac{1}{2} \left(-\log|1-y| - \log|1+y| \right) + C_2$

$= -\frac{1}{2} \log|1-y^2| + C_2$

まとめて(*)は

$$-\frac{1}{2} \log|1-y^2| = \log|x| - \frac{1}{2} \log(1+x^2) + C_1 - C_2$$

$$|1-y^2| = \frac{1}{x^2}(1+x^2) e^{2(C_1-C_2)}$$

$$\underline{x^2|1-y^2| = C(1+x^2)} \qquad (C = e^{2(C_1-C_2)})$$