特論 2016　解答例　①

問7.

$\zeta^N = 1 \quad (\zeta \neq 1) \quad (N = 2, 3, 4, \cdots)$ は

$\zeta^{N-1} + \zeta^{N-2} + \cdots + \zeta + 1 = 0$

をみたすことを示せ.

$$\zeta^N - 1 = (\zeta - 1)(\zeta^{N-1} + \zeta^{N-2} + \cdots + \zeta + 1) = 0.$$

$\zeta \neq 1$ であるから,

$$\zeta^{N-1} + \zeta^{N-2} + \cdots + \zeta + 1 = 0.$$

問8.

$e^{ad(Y)}(X) = e^Y \cdot X e^{-Y}$ を示せ.

ただし、両辺とも収束するとする.

まず、

$$ad(Y)^n(X) = \sum_{k=0}^{n} {}_nC_k (-1)^{n-k} Y^k X Y^{n-k} \quad \text{---}(☆)$$

を示す. n についての帰納法.

$n=1$ 　$LHS = ad(Y)(X) = [Y, X]$

$RHS = {}_1C_0 (-1) XY + {}_1C_1 (-1)^0 YX = [Y, X]$

となり一致.

あるnまで正しいならば、

$$
\begin{aligned}
&\mathrm{ad}^{n+1}(Y)(X) \\
&= \mathrm{ad}(Y)\ \mathrm{ad}^{n}(Y)(X) \\
&= \left[Y, \sum_{k=0}^{n} {}_nC_k (-1)^{n-k} Y^k X Y^{n-k}\right] \\
&= \sum_{k=0}^{n} {}_nC_k (-1)^{n-k} Y^{k+1} X Y^{n-k} \\
&\qquad - \sum_{k=0}^{n} {}_nC_k (-1)^{n-k} Y^{k} X Y^{n+1-k} \\
&= \sum_{k=1}^{n+1} {}_nC_{k-1} (-1)^{n+1-k} Y^{k} X Y^{n+1-k} \\
&\quad + \sum_{k=0}^{n} {}_nC_k (-1)^{n+1-k} Y^{k} X Y^{n+1-k} \\
&= \sum_{k=0}^{n+1} {}_{n+1}C_k (-1)^{n+1-k} Y^{k} X Y^{n+1-k}
\end{aligned}
$$

よって (☆) は $n+1$ でも成立.

ただし、

$$
{}_nC_0 = 1 = {}_{n+1}C_0
$$

$$
{}_nC_n = 1 = {}_{n+1}C_{n+1}
$$

$$
{}_nC_{k-1} + {}_nC_k = {}_{n+1}C_k \quad (k=1,2,3,\cdots,n)
$$

を用いた.

(☆)を用いると、

$$e^{\mathrm{ad}(Y)}(X)$$

$$= \sum_{=0}^{\infty} \frac{1}{n!} \mathrm{ad}(Y)^n (X)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} {}_nC_k (-1)^{n-k} Y^k X Y^{n-k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!} Y^k \cdot X Y^{n-k}$$

2重和に関する式

$$\sum_{m=0}^{\infty} a_m \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k}$$

に留意すれば

$$\text{与式} = \sum_{m=0}^{\infty} \frac{1}{m!} Y^m \cdot X \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} Y^n$$

$$= e^{Y} \cdot X \cdot e^{-Y}$$

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問9

$K, K^* > 0$ は、

$\sinh(2K)\cdot\sinh(2K^*) = 1$ をみたす.

このとき、

$\cosh(2K^*) = \cosh(2K)\sinh(2K^*)$

$\cosh(2K) = \cosh(2K^*)\sinh(2K)$

を示せ.

K, K^* は立場が対称なので、

$\cosh(2K) = \cosh(2K^*)\sinh(2K)$

のみ示せば十分.

$\sinh(2K)\cdot\sinh(2K^*) = 1$ を

$e^{2K^*} >$ について解く.

$$(e^{2K}-e^{-2K})(e^{2K^*})^2 - 4e^{2K^*} - (e^{2K}-e^{-2K}) = 0.$$

$$e^{2K^*} = \frac{2 \pm \sqrt{D}}{e^{2K}-e^{-2K}}, \quad D = (e^{2K}+e^{-2K})^2.$$

$e^{2K^*} >$ であるから

$$e^{2K^*} = \frac{2+\sqrt{D}}{e^{2K}-e^{-2K}} = \frac{(e^{K}+e^{-K})^2}{(e^{K}+e^{-K})(e^{K}-e^{-K})} = \frac{e^{K}+e^{-K}}{e^{K}-e^{-K}}$$

$$\cosh(2K^*) = \frac{1}{2}\left(e^{2K^*} + e^{-2K^*}\right)$$

$$= \frac{1}{2}\left(\frac{e^{K}+e^{-K}}{e^{K}-e^{-K}} + \frac{e^{K}-e^{-K}}{e^{K}+e^{-K}}\right)$$

$$= \frac{\cosh(2K)}{\sinh(2K)}$$

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問10

次の式を示せ.

$$x = \int_0^{2\pi} \log\left\{2\left(\cosh x - \cos\theta\right)\right\}\frac{d\theta}{2\pi}$$

$$(x > 0)$$

Let us set

$$f(x) = \int_0^{2\pi} \log\{2(\cosh x - \cos\theta)\}\frac{d\theta}{2\pi} \qquad (x>0).$$

We have

$$\frac{df}{dx}(x) = \int_0^{2\pi} \frac{d}{dx}\log\{2(\cosh x - \cos\theta)\}\frac{d\theta}{2\pi}$$

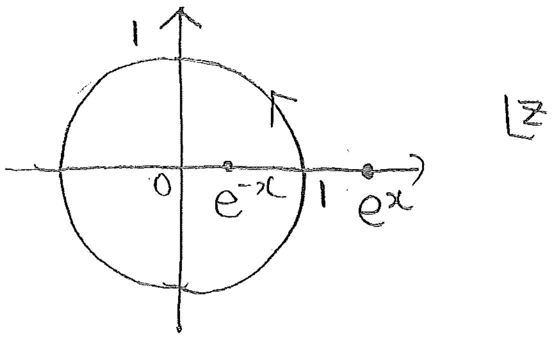
$$= \int_0^{2\pi} \frac{\sinh x}{\cosh x - \cos\theta}\frac{d\theta}{2\pi}$$

$$= \int_0^{2\pi} \frac{(e^{-x}-e^{x})}{(e^{i\theta}-e^{x})(e^{i\theta}-e^{-x})}\cdot\frac{e^{i\theta}}{2\pi}\,d\theta.$$

$$= (e^{-x}-e^{x})\int_C \frac{1}{(z-e^{x})(z-e^{-x})}\frac{dz}{2\pi i}$$

where $z = e^{i\theta}$ and

the integration contour C is given by



Because

$$\int_C \frac{dz}{(z-e^{x})(z-e^{-x})} = 2\pi i \operatorname*{Res}_{z=e^{-x}} \frac{1}{(z-e^{x})(z-e^{-x})}$$

$$= 2\pi i/(e^{-x}-e^{x}),$$

We have

$$\frac{df}{dx}(x) = 1 .$$

Hence we have

$$f(x) = x + C .$$

Below we determine the constant C.

$$f(x) = \left(\int_{\pi}^{2\pi} + \int_{0}^{\pi} \right) \frac{d\theta}{2\pi} \log\{2(\cosh x - \cos\theta)\}$$

$$= \int_0^{\pi} \frac{d\theta}{2\pi} \log 2(\cosh x - \cos\theta)$$

$$+ \int_0^{\pi} \frac{d\theta}{2\pi} \log 2(\cosh x + \cos\theta)$$

$$= \int_0^{\pi} \frac{d\theta}{2\pi} \log 2^2(\cosh^2 x - \cos^2\theta)$$

$$= \int_0^{\pi} \frac{d\theta}{2\pi} \log 2(\cosh 2x - \cos 2\theta)$$

$$= \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{2\pi} \log 2(\cosh 2x - \cos\theta)$$

$$= \frac{1}{2} f(2x)$$

$$\therefore \quad f(2x) = 2f(x)$$

$$f(2x) = 2x + C$$

$$2f(x) = 2x + 2C$$

$$\therefore \quad C = 0 \qquad \therefore \quad \boxed{f(x) = x}$$