微積分解法2016 レポート5回目

問

Arcsin x のマクローリン展開を
4次のオーダーまでもとめなさい。

$$\mathrm{Arcsin}\, x = x + \frac{1}{\boxed{(1)}} x^3 + \frac{3}{\boxed{(2)}} x^5 + \cdots$$

$\boxed{(1)} = 6$

$\boxed{(2)} = 40$

(解1)

$$f(x) = \mathrm{Arcsin}\, x$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f''(x) = x(1-x^2)^{-\frac{3}{2}}$$

$$f'''(x) = (1-x^2)^{-\frac{3}{2}} + 3x^2(1-x^2)^{-\frac{5}{2}}$$

$$f^{(4)}(x) = 9x(1-x^2)^{-\frac{5}{2}} + 15x^2(1-x^2)^{-\frac{7}{2}}$$

$$f^{(5)}(x) = 9(1-x^2)^{-\frac{5}{2}}$$

$$+ (45x^2+30x)(1-x^2)^{-\frac{7}{2}}$$

$$+ 105x^3(1-x^2)^{-\frac{9}{2}}$$

よって

$f(0) = 0,\ f'(0) = 1,\ f''(0) = 0$

$f'''(0) = 1,\ f^{(4)}(0) = 0,\ f^{(5)}(0) = 9$

よって

$$\mathrm{Arcsin}\,x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots$$

$$= x + \frac{1}{3!}x^3 + \frac{9}{5!}x^5 + \cdots$$

$$= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots$$

(解2)

$$(\mathrm{Arcsin}\,x)' = \frac{1}{\sqrt{1-x^2}}$$

$g(z) = (1+z)^{\alpha}$ のマクローリン展開は、

$$g(0) = 1$$

$$g'(0) = \alpha(1+z)^{\alpha-1}$$

$$g''(0) = \alpha(\alpha-1)(1+z)^{\alpha-2}$$

$$\vdots$$

$$g^{(n)}(0) = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+z)^{\alpha-n}$$

$$g(z)=1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}z^{n}$$

よって

$$\frac{1}{\sqrt{1-x^2}}=(1-x^2)^{-\frac{1}{2}}$$

$$=1+\sum_{n=1}^{\infty}\frac{\overbrace{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-n+1\right)}^{n\text{コ}}}{n!}(-x^2)^{n}$$

$$=1+\sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdots(2n-1)}{2^{n}\cdot n!}x^{2n}$$

$$=1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}x^{2n}$$

$$\mathrm{Arcsin}\,x=\int\frac{d}{dx}\mathrm{Arcsin}\,x\,dx$$

$$=\int\frac{1}{\sqrt{1-x^2}}dx$$

$$=\int\left(1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}x^{2n}\right)dx$$

$$=x+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{x^{2n+1}}{2n+1}+C$$

$\mathrm{Arcsin}\, 0 = 0$ なので

$0 = \mathrm{Arcsin}\, 0 = C$

$\therefore\ C = 0$

よって

$$\mathrm{Arcsin}\, x = x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{x^{2n+1}}{2n+1}$$

$$= x + \frac{1!!}{2!!}\frac{1}{3}x^3 + \frac{3!!}{4!!}\frac{1}{5}x^5 + \cdots$$

$$= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots$$