A remark on the integrals of motion associated with level k realization of the elliptic algebra $U_{q,p}(\widehat{sl_2})$

September 26, 2008

T.KOJIMA [α] and J.SHIRAISHI [β]

[α] *Department of Mathematics, College of Science and Technology, Nihon University, Surugadai, Chiyoda-ku, Tokyo 101-0062, JAPAN*

[β] *Graduate School of Mathematical Science, University of Tokyo, Komaba, Megro-ku, Tokyo, 153-8914, JAPAN*

Abstract

We give one parameter deformation of level k free field realization of the screening current of the elliptic algebra $U_{q,p}(\widehat{sl_2})$. By means of these free field realizations, we construct infinitly many commutative operators, which are called the nonlocal integrals of motion associated with the elliptic algebra $U_{q,p}(\widehat{sl_2})$ for level k. They are given as integrals involving a product of the screening current and elliptic theta functions. This paper give level k generalization of the nonlocal integrals of motion given in [1].

1 Introduction

One of the results in V.Bazhanov, S.Lukyanov, Al.Zamolodchikov [4] is construction of field theoretical analogue of the commuting transfer matrix $\mathbf{T}(z)$, acting on the highest weight representation of the Virasoro algebra. Their commuting transfer matrix $\mathbf{T}(z)$ is the trace of the image of the universal R-matrix associated with the quantum affine symmetry $U_q(\widehat{sl_2})$. This construction is very simple and the commutativity $[\mathbf{T}(z), \mathbf{T}(w)] = 0$ is direct consequence of the Yang-Baxter equation. They call the coefficients of the Taylor expansion of $\mathbf{T}(z)$ the nonlocal integrals of motion. The higher-rank generalization of [4] is considered in [5, 6]. The elliptic deformation of the nonlocal integrals of motion is considered in [1]. V.Bazhanov, S.Lukyanov, Al.Zamolodchikov [4] constructed the continuous transfer matrix $\mathbf{T}(z)$ by taking the trace of the image of the universal R-matrix associated with $U_q(\widehat{sl_2})$. However it is not so easy to calculate the image of the elliptic version of the universal R-matrix, which is obtaind by using the twister [10]. Hence the construction method of the elliptic version [1] should be completely different from those in [4]. Instead of considering the transfer matrix $\mathbf{T}(z)$, the authors [1] give the integral representation of the integrals of motion directly. The commutativity of the integrals of motion is not consequence of the Yang-Baxter equation. It is consequence of the commutative subalgebra of the Feigin-Odesskii algebra [11]. The higher-rank generalization of [1] is considered in [2, 3]. This paper is a continuation of [1, 2, 3]. This paper give level k generalization of the nonlocal integrals of motion given in [1].

The organization of this paper is as following. In section 2 we give one parameter "s" deformation of the level k free field realization of the screening current of the elliptic algebra $U_{q,p}(\widehat{sl_2})$. In section 3 we construct infinitly many commutative operators, which are called the nonlocal integrals of motion associated with the elliptic algebra $U_{q,p}(\widehat{sl_2})$ for level k. In section 3 we state main theorem and give conjecture. In appendix we summarize the normal ordering of basic operators.

2 Elliptic current

In this section we give one parameter "s" deformation of the level k free field realization of the elliptic algebra $U_{q,p}(\widehat{sl_2})$. We fix complex numbers x, r, r^*, s, $(|x| < 1, \mathrm{Re}(r), \mathrm{Re}(r^*) > 0, s \neq 2)$, and $k = r - r^* \neq 0, -2$. We use symbols

$$[n] = \frac{x^n - x^{-n}}{x - x^{-1}}, \quad [n]_+ = x^n + x^{-n}.$$

We set the parameter τ, τ^*

$$x = e^{-\pi\sqrt{-1}/r\tau} = e^{-\pi\sqrt{-1}/r^*\tau^*}. \tag{2.1}$$

Let us use parametrization $z = x^{2u}$. The symbol $[u]_r$ stands for the Jacobi elliptic theta function

$$[u]_r = x^{\frac{u^2}{r} - u}\Theta_{x^{2r}}(z), \quad [u]_{r^*} = x^{\frac{u^2}{r^*} - u}\Theta_{x^{2r^*}}(z), \tag{2.2}$$

where we have used

$$\Theta_p(z)=(z;p)_\infty(p/z;p)_\infty(p;p)_\infty,\quad (z;p)_\infty=\prod_{n=0}^\infty(1-p^nz). \tag{2.3}$$

The theta function $[u]_r$ enjoys the quasi-periodicity property

$$[u+r]_r=-[u]_r,\quad [u+r\tau]_r=-e^{-\pi\sqrt{-1}\tau-\frac{2\pi\sqrt{-1}}{r}u}[u]_r. \tag{2.4}$$

2.1 Bosons

We set the bosons $\alpha_m^j,\widetilde{\alpha}_m^j,(j=1,2;m\in\mathbb{Z}_{\neq 0})$,

$$\begin{aligned}
[\alpha_m^j,\alpha_n^j] &= -\frac{1}{m}\frac{[2m][rm]}{[km][(r-k)m]}\delta_{m+n,0},\quad (j=1,2), && (2.5)\\
[\alpha_m^1,\alpha_n^2] &= \frac{1}{m}\left(\frac{x^{(-r+k)m}([sm]-[(s-2)m])}{[(r-k)m]}+\frac{x^{km}([sm]+[(s-2)m])}{[km]}\right)\delta_{m+n,0}, && (2.6)\\
[\widetilde{\alpha}_m^j,\widetilde{\alpha}_n^j] &= -\frac{1}{m}\frac{[2m][(r-k)m]}{[km][rm]}\delta_{m+n,0},\quad (j=1,2), && (2.7)\\
[\widetilde{\alpha}_m^1,\widetilde{\alpha}_n^2] &= \frac{1}{m}\left(\frac{x^{rm}(-[sm]+[(s-2)m])}{[rm]}+\frac{x^{km}([sm]+[(s-2)m])}{[km]}\right)\delta_{m+n,0}, && (2.8)\\
[\alpha_m^j,\widetilde{\alpha}_n^j] &= -\frac{1}{m}\frac{[2m]}{[km]}\delta_{m+n,0},\quad (j=1,2), && (2.9)\\
[\alpha_m^1,\widetilde{\alpha}_n^2] &= \frac{1}{m}\frac{[sm]+[(s-2)m]}{[km]}\delta_{m+n,0}, && (2.10)\\
[\widetilde{\alpha}_m^1,\alpha_n^2] &= \frac{1}{m}\frac{[sm]+[(s-2)m]}{[km]}\delta_{m+n,0}. && (2.11)
\end{aligned}$$

We set the bosons β_m^j,γ_m^j, $(j=1,2;m\in\mathbb{Z}_{\neq 0})$,

$$\begin{aligned}
[\beta_m^j,\beta_n^j] &= \frac{[2m][(k+2)m]}{m}\delta_{m+n,0},\ (j=1,2), && (2.12)\\
[\beta_m^1,\beta_n^2] &= -\frac{[(k+2)m]([sm]+[(s-2)m])}{m}\delta_{m+n,0}, && (2.13)\\
[\gamma_m^j,\gamma_n^j] &= \frac{1}{m}\frac{[2m]}{[km]}\delta_{m+n,0},\quad (j=1,2), && (2.14)\\
[\gamma_m^1,\gamma_n^2] &= -\frac{1}{m}\frac{[sm]+[(s-2)m]}{[km]}\delta_{m+n,0}. && (2.15)
\end{aligned}$$

We set the zero-mode operators $P_0,Q_0,\ h,\alpha$ and $h_0,h_1,h_2,\alpha_0,\alpha_1,\alpha_2$,

$$[P_0,iQ_0]=1,\ [h,\alpha]=2, \tag{2.16}$$

$$[h_0,\alpha_0]=[h_1,\alpha_2]=[h_2,\alpha_1]=(2-s),\quad [h_1,\alpha_1]=[h_2,\alpha_2]=0. \tag{2.17}$$

We set the Fock space $\mathcal{F}_{K,L}$, $(K,L\in\mathbb{Z})$.

$$\mathcal{F}_{K,L}=\bigoplus_{n,n_0,n_1,n_2\in\mathbb{Z}}\mathbb{C}[\alpha_{-m}^j,\widetilde{\alpha}_{-m}^j,\beta_{-m}^j,\gamma_{-m}^j,(j=1,2;m\in\mathbb{N}_{\neq 0})]\otimes|K,L\rangle_{n,n_0,n_1,n_2},$$

$$(2.18)$$

$$|K,L\rangle_{n,n_0,n_1,n_2} = e^{\left(L\sqrt{\frac{2r}{r-k}}-K\sqrt{\frac{2(r-k)}{r}}\right)iQ} \otimes e^{n\alpha} \otimes e^{n_0\alpha_0} \otimes e^{n_1\alpha_1} \otimes e^{n_2\alpha_2}. \tag{2.19}$$

Upon specialization $s \to 2$, simplification occures.

$$\alpha_m^2 = -\alpha_m^1, \quad \tilde{\alpha}_m^1 = \frac{[(r-k)m]}{[rm]}\alpha_m^1, \quad \tilde{\alpha}_m^2 = -\frac{[(r-k)m]}{[rm]}\alpha_m^1, \tag{2.20}$$

$$\beta_m^2 = -\beta_m^1, \quad \gamma_m^2 = -\gamma_m^1, \quad h_0 = h_1 = h_2 = \alpha_0 = \alpha_1 = \alpha_2 = 0. \tag{2.21}$$

The bosons $\alpha_m^1, \beta_m^1, \gamma_m^1$ are the same bosons which were introduced to construct the elliptic current associated with the elliptic algebra $U_{q,p}(\widehat{sl_2})$ and the deformed Virasoro algebra $Vir_{q,t}$ [7, 8, 9]. In order to construct infinitly many commutative operators, we introduce one prameter s deformation of the bosons in [7, 8, 9]. This additional parameter s plays an important role in proof of the main theorem.

2.2 Elliptic current

We introduce the operators $C_j(z), C_j^\dagger(z)$, $(j=1,2)$ acting on the Fock space $\mathcal{F}_{J,K}$.

$$C_1(z) = e^{-\sqrt{\frac{2r}{k(r-k)}}iQ_0} e^{-\sqrt{\frac{2r}{k(r-k)}}P_0 \log z} : \exp\left(-\sum_{m\neq 0}\alpha_m^1 z^{-m}\right):, \tag{2.22}$$

$$C_2(z) = e^{\sqrt{\frac{2r}{k(r-k)}}iQ_0} e^{\sqrt{\frac{2r}{k(r-k)}}P_0 \log z} : \exp\left(-\sum_{m\neq 0}\alpha_m^2 z^{-m}\right):, \tag{2.23}$$

$$C_1^\dagger(z) = e^{\sqrt{\frac{2(r-k)}{kr}}iQ_0} e^{\sqrt{\frac{2(r-k)}{kr}}P_0 \log z} : \exp\left(\sum_{m\neq 0}\tilde{\alpha}_m^1 z^{-m}\right):, \tag{2.24}$$

$$C_2^\dagger(z) = e^{-\sqrt{\frac{2(r-k)}{kr}}iQ_0} e^{-\sqrt{\frac{2(r-k)}{kr}}P_0 \log z} : \exp\left(\sum_{m\neq 0}\tilde{\alpha}_m^2 z^{-m}\right):. \tag{2.25}$$

Here $: * :$ represents normal ordering. We set the operators $\widetilde{\Psi}_{j,I}(z), \widetilde{\Psi}_{j,II}(z), \widetilde{\Psi}_{j,I}^\dagger(z), \widetilde{\Psi}_{j,II}^\dagger(z)$, $(j=1,2)$ acting on the Fock space $\mathcal{F}_{J,K}$.

$$\begin{aligned}\widetilde{\Psi}_{j,I}(z) &= \exp\left(-(x-x^{-1})\sum_{m>0}\frac{x^{\frac{km}{2}}}{[m]_+}\beta_m^j z^{-m}\right) \quad (2.26)\\ &\times \exp\left(-\sum_{m>0}x^{-\frac{km}{2}}\gamma_{-m}^j z^m\right)\exp\left(-\sum_{m>0}x^{\frac{km}{2}}\frac{[(k+1)m]_+}{[m]_+}\gamma_m^j z^{-m}\right), \ (j=1,2),\end{aligned}$$

$$\widetilde{\Psi}_{j,II}(z) = \exp\left((x-x^{-1})\sum_{m>0}\frac{x^{\frac{km}{2}}}{[m]_+}\beta_{-m}^j z^m\right) \tag{2.27}$$

$$\times \quad \exp\left(-\sum_{m>0} x^{\frac{km}{2}} \frac{[(k+1)m]_+}{[m]_+} \gamma_{-m}^j z^m\right) \exp\left(-\sum_{m>0} x^{-\frac{km}{2}} \gamma_m^j z^{-m}\right), \ (j=1,2),$$

$$\widetilde{\Psi}_{j,I}^{\dagger}(z) = \exp\left((x-x^{-1}) \sum_{m>0} \frac{x^{-\frac{km}{2}}}{[m]_+} \beta_m^j z^{-m}\right) \tag{2.28}$$

$$\times \quad \exp\left(\sum_{m>0} x^{\frac{km}{2}} \gamma_{-m}^j z^m\right) \exp\left(\sum_{m>0} x^{-\frac{km}{2}} \frac{[(k+1)m]_+}{[m]_+} \gamma_m^j z^{-m}\right), \ (j=1,2),$$

$$\widetilde{\Psi}_{j,II}^{\dagger}(z) = \exp\left(-(x-x^{-1}) \sum_{m>0} \frac{x^{-\frac{km}{2}}}{[m]_+} \beta_{-m}^j z^m\right) \tag{2.29}$$

$$\times \quad \exp\left(\sum_{m>0} x^{-\frac{km}{2}} \frac{[(k+1)m]_+}{[m]_+} \gamma_{-m}^j z^m\right) \exp\left(\sum_{m>0} x^{\frac{km}{2}} \gamma_m^j z^{-m}\right), \ (j=1,2).$$

We set the operators $\Psi_{j,I}(z), \Psi_{j,II}(z), \Psi_{j,I}^{\dagger}(z), \Psi_{j,II}^{\dagger}(z)$, $(j=1,2)$ acting on the Fock space $\mathcal{F}_{J,K}$.

$$\Psi_{1,I}(z) = \widetilde{\Psi}_{1,I}(z) e^{\alpha+\alpha_0+\alpha_1} x^{\frac{h}{2}+h_0+h_1} z^{-\frac{h}{k}}, \tag{2.30}$$

$$\Psi_{1,II}(z) = \widetilde{\Psi}_{1,II}(z) e^{\alpha+\alpha_0+\alpha_1} x^{-\frac{h}{2}+h_0-h_1} z^{-\frac{h}{k}}, \tag{2.31}$$

$$\Psi_{2,I}(z) = \widetilde{\Psi}_{2,I}(z) e^{-\alpha-\alpha_0+\alpha_2} x^{-\frac{h}{2}+h_0+h_2} z^{\frac{h}{k}}, \tag{2.32}$$

$$\Psi_{2,II}(z) = \widetilde{\Psi}_{2,II}(z) e^{-\alpha-\alpha_0+\alpha_2} x^{\frac{h}{2}+h_0-h_2} z^{\frac{h}{k}}, \tag{2.33}$$

$$\Psi_{1,I}^{\dagger}(z) = \widetilde{\Psi}_{1,I}^{\dagger}(z) e^{-\alpha-\alpha_0+\alpha_1} x^{\frac{h}{2}-h_0-h_1} z^{\frac{h}{k}}, \tag{2.34}$$

$$\Psi_{1,II}^{\dagger}(z) = \widetilde{\Psi}_{1,II}^{\dagger}(z) e^{-\alpha-\alpha_0+\alpha_1} x^{-\frac{h}{2}-h_0+h_1} z^{\frac{h}{k}}, \tag{2.35}$$

$$\Psi_{2,I}^{\dagger}(z) = \widetilde{\Psi}_{2,I}^{\dagger}(z) e^{\alpha+\alpha_0+\alpha_2} x^{-\frac{h}{2}-h_0-h_2} z^{-\frac{h}{k}}, \tag{2.36}$$

$$\Psi_{2,II}^{\dagger}(z) = \widetilde{\Psi}_{2,II}^{\dagger}(z) e^{\alpha+\alpha_0+\alpha_2} x^{\frac{h}{2}-h_0+h_2} z^{-\frac{h}{k}}. \tag{2.37}$$

Definition 2.1 *We set the operators $E_j(z), F_j(z)$, $(j=1,2)$, which can be regarded as one parameter deformation of the level k elliptic currents associated with the elliptic algebra $U_{q,p}(\widehat{sl_2})$ [7, 9].*

$$E_j(z) = C_j(z)\Psi_j(z), \quad F_j(z) = C_j^{\dagger}(z)\Psi_j^{\dagger}(z), \quad (j=1,2), \tag{2.38}$$

where we have set

$$\Psi_j(z) = \frac{1}{x-x^{-1}}(\Psi_{j,I}(z) - \Psi_{j,II}(z)), \quad \Psi_j^{\dagger}(z) = \frac{-1}{x-x^{-1}}(\Psi_{j,I}^{\dagger}(z) - \Psi_{j,II}^{\dagger}(z)), \quad (j=1,2). \tag{2.39}$$

We have following proposition as direct consequence of the normal orderings of the basic operators summarized in appendix.

Proposition 2.2 *The elliptic currents $E_j(z)$, $(j=1,2)$ satisfy the following commutation relations.*

$$[u_1-u_2]_{r-k}[u_1-u_2-1]_{r-k} E_j(z_1)E_j(z_2)$$

$$= [u_2-u_1]_{r-k}[u_2-u_1-1]_{r-k}E_j(z_2)E_j(z_1), \ (j=1,2), \tag{2.40}$$

$$\left[u_1-u_2+\frac{s}{2}\right]_{r-k}\left[u_1-u_2-\frac{s}{2}+1\right]_{r-k}E_1(z_1)E_2(z_2)$$
$$= \left[u_2-u_1+\frac{s}{2}\right]_{r-k}\left[u_2-u_1-\frac{s}{2}+1\right]_{r-k}E_2(z_2)E_1(z_1). \tag{2.41}$$

The elliptic currents $F_j(z)$, $(j=1,2)$ satisfy the following commutation relations.

$$[u_1-u_2]_r[u_1-u_2+1]_rF_j(z_1)F_j(z_2)$$
$$= [u_2-u_1]_r[u_2-u_1+1]_rF_j(z_2)F_j(z_1), \ (j=1,2), \tag{2.42}$$

$$\left[u_1-u_2-\frac{s}{2}\right]_r\left[u_1-u_2+\frac{s}{2}-1\right]_rF_1(z_1)F_2(z_2)$$
$$= \left[u_2-u_1-\frac{s}{2}\right]_r\left[u_2-u_1+\frac{s}{2}-1\right]_rF_2(z_2)F_1(z_1). \tag{2.43}$$

The currents $E_j(z)$ and $F_j(z)$ satisfy

$$[E_j(z_1),F_j(z_2)] = \frac{x^{(-1)^j(s-2)}}{x-x^{-1}}\left(:C_j(z_1)C_j^\dagger(z_2)\Psi_{j,I}(z_1)\Psi_{j,I}^\dagger(z_2):\delta\left(\frac{x^kz_2}{z_1}\right)\right. \tag{2.44}$$
$$\left. - :C_j(z_1)C_j^\dagger(z_2)\Psi_{j,II}(z_1)\Psi_{j,II}^\dagger(z_2):\delta\left(\frac{x^{-k}z_2}{z_1}\right)\right), \quad (j=1,2).$$

Here we have used the delta-function $\delta(z)=\sum_{n\in\mathbb{Z}}z^n$.

Upon specialization $s=2$ the currents $E_1(z),F_1(z)$ degenerate to elliptic currents in [9]. We set $E_j^{DV}(z)=E_j(z)|_{s=2}$, $F_j^{DV}(z)=F_j(z)|_{s=2}$, $(j=1,2)$.

3 Integrals of motion

In this section we construct infinitly many commutative operators $\mathcal{G}_m^*, \mathcal{G}_m$, $(m\in\mathbb{N})$, which we call the nonlocal integrals of motion for level k.

3.1 Nonlocal integrals of motion

Let us set the theta function $\vartheta_\alpha^*(u)$, $\vartheta_\alpha(u)$, $(\alpha\in\mathbb{C})$ by

$$\vartheta^*(u+1)=\vartheta^*(u), \quad \vartheta^*(u+r^*\tau^*)=e^{-2\pi\sqrt{-1}\tau^*-\frac{2\pi\sqrt{-1}}{r^*}(2u-\sqrt{\frac{2rr^*}{k}}P_0-\frac{r^*}{k}h)}\vartheta^*(u), \tag{3.1}$$
$$\vartheta(u+1)=\vartheta(u), \quad \vartheta(u+r\tau)=e^{-2\pi\sqrt{-1}\tau-\frac{2\pi\sqrt{-1}}{r}(2u-\sqrt{\frac{2rr^*}{k}}P_0-\frac{r}{k}h)}\vartheta(u). \tag{3.2}$$

Let us use the parametrization $z_j^{(t)}=x^{2u_j^{(t)}}$, $(t=1,2;j=1,2,\cdots,m)$.

Definition 3.1 *We define the operator $\mathcal{G}_m^*$ for the regime $\mathrm{Re}(r)>k$ and $0<\mathrm{Re}(s)<2$ by*

$$\mathcal{G}_m^* = \int\cdots\int_{C^*}\prod_{j=1}^m\frac{dz_j^{(1)}}{z_j^{(1)}}\prod_{j=1}^m\frac{dz_j^{(2)}}{z_j^{(2)}}E_1(z_1^{(1)})E_1(z_2^{(1)})\cdots E_1(z_m^{(1)})E_2(z_1^{(2)})E_2(z_2^{(2)})\cdots E_2(z_m^{(2)})$$

$$\times \frac{\displaystyle\prod_{t=1,2}\prod_{1\leqq i<j\leqq m}\left[u_i^{(t)}-u_j^{(t)}\right]_{r-k}\left[u_j^{(t)}-u_i^{(t)}+1\right]_{r-k}}{\displaystyle\prod_{1\leqq i,j\leqq m}\left[u_i^{(1)}-u_j^{(2)}-\frac{s}{2}\right]_{r-k}\left[u_j^{(2)}-u_i^{(1)}-\frac{s}{2}+1\right]_{r-k}}\vartheta^*\left(\sum_{j=1}^m(u_j^{(2)}-u_j^{(1)})\right), \qquad (3.3)$$

were the integral contour C^* *encircles* $z_j^{(t)}=0$, $(t=1,2;j=1,2,\cdots,m)$ *in such a way that*

$$|z_j^{(t)}|=1, \quad (t=1,2;j=1,2,\cdots,m).$$

We define the operator $\mathcal{G}_m$ *for the regime* $\mathrm{Re}(r)>0$ *and* $0<\mathrm{Re}(s)<2$ *by*

$$\mathcal{G}_m = \int\cdots\int_C\prod_{j=1}^m\frac{dz_j^{(1)}}{z_j^{(1)}}\prod_{j=1}^m\frac{dz_j^{(2)}}{z_j^{(2)}}F_1(z_1^{(1)})F_1(z_2^{(1)})\cdots F_1(z_m^{(1)})F_2(z_1^{(2)})F_2(z_2^{(2)})\cdots F_2(z_m^{(2)})$$
$$\times \frac{\displaystyle\prod_{t=1,2}\prod_{1\leqq i<j\leqq m}\left[u_i^{(t)}-u_j^{(t)}\right]_{r}\left[u_j^{(t)}-u_i^{(t)}-1\right]_{r}}{\displaystyle\prod_{1\leqq i,j\leqq m}\left[u_i^{(1)}-u_j^{(2)}+\frac{s}{2}\right]_{r}\left[u_j^{(2)}-u_i^{(1)}+\frac{s}{2}-1\right]_{r}}\vartheta\left(\sum_{j=1}^m(u_j^{(1)}-u_j^{(2)})\right), \qquad (3.4)$$

were the integral contour C^* *encircles* $z_j^{(t)}=0$, $(t=1,2;j=1,2,\cdots,m)$ *in such a way that*

$$|z_j^{(t)}|=1, \quad (t=1,2;j=1,2,\cdots,m).$$

We call the operators $\mathcal{G}_m^*$ *and* $\mathcal{G}_m$ *the nonlocal integrals of motion for level* k.

The definition of the opeartors $\mathcal{G}_m^*$, $\mathcal{G}_m$ for generic $s\in\mathbb{C},(s\neq 2)$ should be understood as analytic continuation. In the limit $s\to 2$, the contour C^*, C pinch at $z_j^{(t)}=z_i^{(t')}$. Hence the definition of $\mathcal{G}_m^*,\mathcal{G}_m$ do not hold for $s=2$. We give modified definition of $\mathcal{G}_m^*,\mathcal{G}_m$ for $s=2$, below. We note that parameter $s\neq 2$ plays an important role in proof of main theorem 3.3.

Definition 3.2 *We define the operator* $\mathcal{G}_m^{DV*}$ *for the regime* $\mathrm{Re}(r)>k$ *and* $s=2$ *by*

$$\mathcal{G}_m^{DV*} = \int\cdots\int_{C^*_{Arg}}\prod_{j=1}^m\frac{dz_j^{(1)}}{z_j^{(1)}}\prod_{j=1}^m\frac{dz_j^{(2)}}{z_j^{(2)}}E_1^{DV}(z_1^{(1)})\cdots E_1^{DV}(z_m^{(1)})E_2^{DV}(z_1^{(2)})\cdots E_2^{DV}(z_m^{(2)})$$
$$\times \frac{\displaystyle\prod_{t=1,2}\prod_{1\leqq i<j\leqq m}\left[u_i^{(t)}-u_j^{(t)}\right]_{r-k}\left[u_j^{(t)}-u_i^{(t)}+1\right]_{r-k}}{\displaystyle\prod_{1\leqq i,j\leqq m}\left[u_i^{(1)}-u_j^{(2)}-1\right]_{r-k}\left[u_j^{(2)}-u_i^{(1)}\right]_{r-k}}\vartheta^*\left(\sum_{j=1}^m(u_j^{(2)}-u_j^{(1)})\right), \qquad (3.5)$$

were the integral contour C^*_{Arg} *encircles* $z_j^{(t)}=0$, $(t=1,2;j=1,2,\cdots,m)$ *in such a way that*

$$|x^2z_m^{(2)}|,|x^{2r^*}z_m^{(2)}|<|z_1^{(1)}|<|z_1^{(2)}|<|z_2^{(1)}|<|z_2^{(2)}|<\cdots<|z_m^{(1)}|<|z_m^{(2)}|.$$

We define the operator $\mathcal{G}_m^{DV}$ *for the regime* $\mathrm{Re}(r)>0$ *and* $s=2$ *by*

$$\mathcal{G}_m^{DV} = \int\cdots\int_{C_{Arg}}\prod_{j=1}^m\frac{dz_j^{(1)}}{z_j^{(1)}}\prod_{j=1}^m\frac{dz_j^{(2)}}{z_j^{(2)}}F_1^{DV}(z_1^{(1)})\cdots F_1^{DV}(z_m^{(1)})F_2^{DV}(z_1^{(2)})\cdots F_2^{DV}(z_m^{(2)})$$

$$\times\frac{\prod_{t=1,2}\prod_{1\leqq i<j\leqq m}\left[u_i^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_i^{(t)}-1\right]_r}{\prod_{1\leqq i,j\leqq m}\left[u_i^{(1)}-u_j^{(2)}+1\right]_r\left[u_j^{(2)}-u_i^{(1)}\right]_r}\vartheta\left(\sum_{j=1}^m(u_j^{(1)}-u_j^{(2)})\right), \tag{3.6}$$

were the integral contour C_{Arg} *encircles* $z_j^{(t)}=0$, $(t=1,2;j=1,2,\cdots,m)$ *in such a way that*

$$|x^2z_m^{(2)}|,|x^{2r}z_m^{(2)}|<|z_1^{(1)}|<|z_1^{(2)}|<|z_2^{(1)}|<|z_2^{(2)}|<\cdots<|z_m^{(1)}|<|z_m^{(2)}|.$$

3.2 Main result

The following is main theorem of this paper.

Theorem 3.3 *For the regime* $s\neq 2$ *and* $\mathrm{Re}(r)>k$, *we have*

$$[\mathcal{G}_m^*,\mathcal{G}_n^*]=0,\quad (m,n\in\mathbb{N}). \tag{3.7}$$

For the regime $s\neq 2$ *and* $\mathrm{Re}(r)>0$, *we have*

$$[\mathcal{G}_m,\mathcal{G}_n]=0,\quad (m,n\in\mathbb{N}). \tag{3.8}$$

We skech proof of theorem 3.3. Proof is given as the same manner as level $k=1$ case [1, 3]. By symmetrization of the screenings $E_j(z)$, the commutation relation $[\mathcal{G}_m^*,\mathcal{G}_n^*]=0$ is reduced to the following sufficient condition of the theta functions, which is shown by induction as the same manner as [1, 3]. We note that this symmetrization procedure holds only for $s\neq 2$.

$$\sum_{\substack{K\cup K^c=\{1,2,\cdots,n+m\}\\|K|=n,|K^c|=m}}\sum_{\substack{L\cup L^c=\{1,2,\cdots,n+m\}\\|L|=n,|L^c|=m}}\vartheta^*(\sum_{j\in K^c}u_j^{(2)}-\sum_{j\in L^c}u_j^{(1)})\vartheta^*(\sum_{j\in K}u_j^{(2)}-\sum_{j\in L}u_j^{(1)})$$
$$\times\prod_{\substack{i\in K^c\\p\in K^c}}\prod_{\substack{j\in K^c\\q\in K^c}}\frac{[u_j^{(2)}-u_p^{(1)}-\frac{s}{2}]_{r-k}[u_i^{(1)}-u_q^{(2)}-\frac{s}{2}]_{r-k}[u_p^{(1)}-u_j^{(2)}-\frac{s}{2}+1]_{r-k}[u_q^{(2)}-u_i^{(1)}-\frac{s}{2}+1]_{r-k}}{[u_i^{(1)}-u_p^{(1)}]_{r-k}[u_j^{(2)}-u_q^{(2)}]_{r-k}[u_p^{(1)}-u_i^{(1)}+1]_{r-k}[u_q^{(2)}-u_j^{(2)}+1]_{r-k}}$$
$$=\sum_{\substack{K\cup K^c=\{1,2,\cdots,n+m\}\\|K|=n,|K^c|=m}}\sum_{\substack{L\cup L^c=\{1,2,\cdots,n+m\}\\|L|=n,|L^c|=m}}\vartheta^*(\sum_{j\in K^c}u_j^{(2)}-\sum_{j\in L^c}u_j^{(1)})\vartheta^*(\sum_{j\in K}u_j^{(2)}-\sum_{j\in L}u_j^{(1)})$$
$$\times\prod_{\substack{i\in K^c\\p\in K^c}}\prod_{\substack{j\in K^c\\q\in K^c}}\frac{[u_q^{(2)}-u_i^{(1)}-\frac{s}{2}]_{r-k}[u_p^{(2)}-u_j^{(1)}-\frac{s}{2}]_{r-k}[u_i^{(1)}-u_q^{(2)}-\frac{s}{2}+1]_{r-k}[u_j^{(2)}-u_p^{(1)}-\frac{s}{2}+1]_{r-k}}{[u_p^{(1)}-u_i^{(1)}]_{r-k}[u_q^{(2)}-u_j^{(2)}]_{r-k}[u_i^{(1)}-u_p^{(1)}+1]_{r-k}[u_q^{(2)}-u_j^{(2)}+1]_{r-k}}. \tag{3.9}$$

Naively, when we take the limit $s \to 2$, it seems that we have $[\mathcal{G}_m^{DV*}, \mathcal{G}_n^{DV*}] = 0$. However, very precicely, in order to take the limit $s \to 2$, we have to consider special treatment which we call "renormalized" limit in [1]. Here we state only conjecture on the operator $\mathcal{G}_m^{DV*}$. Theorem 3.3 give a supporting argument of the following conjecture.

Conjecture 3.4 *For the regime* $s = 2$ *and* $\mathrm{Re}(r) > k$ *we have*

$$[\mathcal{G}_m^{DV*}, \mathcal{G}_n^{DV*}] = 0 \quad (m, n \in \mathbb{N}). \tag{3.10}$$

For the regime $s = 2$ *and* $\mathrm{Re}(r) > 0$ *we have*

$$[\mathcal{G}_m^{DV}, \mathcal{G}_n^{DV}] = 0, \quad (m, n \in \mathbb{N}). \tag{3.11}$$

In this paper we gave one parameter "s" deformation of level k free field realization of the screening current of the elliptic algebra $U_{q,p}(\widehat{sl_2})$. By means of these free field realizations, we constructed infinitly many commutative operators, which we call the nonlocal integrals of motion associated with the elliptic algebra $U_{q,p}(\widehat{sl_2})$ for arbitrary level $k \neq 0, -2$. They are given as integrals involving a product of the screening current and Jacobi elliptic theta functions. The construction of the local integrals of motion $\mathcal{I}_m$ for arbitrary level k is open problem. The construction of the local integrals of motion $\mathcal{I}_m$ for level 1 only is summarized in [1, 2, 3].

Acknowledgements

We would like to thank the organizing committee of the X-th International Conference on Geometry, Integrability and Quantization in Sts.Constantine and Elena, Bulgaria. We would like to thank Professors V Bazhanov, P Bouwknegt, A Chervov, V Gerdjikov, F Goehmann, K Hasegawa, M Jimbo, A Kluemper, P Kulish, W-X Ma, V Mangazeev and I Mladenov for their interst in this work. This work is partly supported by Grant-in Aid for Young Scientist **B** (18740092) from JSPS.

A Normal Ordering

In appendix we summarize the normal orderings of the basic operatrors.

$$C_j(z_1)C_j(z_2) = :: z_1^{\frac{2}{r^*}+\frac{2}{k}} \frac{(x^{-2+2k}z_2/z_1; x^{2r^*})_\infty (x^{-2}z_2/z_1; x^{2k})_\infty}{(x^{2+2k}z_2/z_1; x^{2r^*})_\infty (x^{2}z_2/z_1; x^{2k})_\infty}, \quad (j = 1, 2), \tag{A.1}$$

$$C_1(z_1)C_2(z_2) = :: z_1^{-\frac{2}{r^*}-\frac{2}{k}} \frac{(x^{s}z_2/z_1; x^{2r^*})_\infty (x^{2-s}z_2/z_1; x^{2r^*})_\infty}{(x^{-s}z_2/z_1; x^{2r^*})_\infty (x^{s-2}z_2/z_1; x^{2r^*})_\infty}$$

$$
\begin{aligned}
& \times \quad \frac{(x^{s+2k}z_2/z_1;x^{2k})_\infty (x^{s-2+2k}z_2/z_1;x^{2k})_\infty}{(x^{-s+2k}z_2/z_1;x^{2k})_\infty (x^{2-s+2k}z_2/z_1;x^{2k})_\infty}, && \text{(A.2)}\\
C_2(z_1)C_1(z_2) &= :: z_1^{-\frac{2}{r^*}-\frac{2}{k}} \frac{(x^{s+2r^*}z_2/z_1;x^{2r^*})_\infty (x^{2-s+2r^*}z_2/z_1;x^{2r^*})_\infty}{(x^{-s+2r^*}z_2/z_1;x^{2r^*})_\infty (x^{s-2+2r^*}z_2/z_1;x^{2r^*})_\infty} \\
& \times \quad \frac{(x^{s}z_2/z_1;x^{2k})_\infty (x^{s-2}z_2/z_1;x^{2k})_\infty}{(x^{-s}z_2/z_1;x^{2k})_\infty (x^{2-s}z_2/z_1;x^{2k})_\infty}, && \text{(A.3)}\\
C_j^\dagger(z_1)C_j^\dagger(z_2) &= :: z_1^{-\frac{2}{r}+\frac{2}{k}} \frac{(x^{-2+2k}z_1/z_2;x^{2k})_\infty (x^{2+2r}z_1/z_2;x^{2r})_\infty}{(x^{2+2k}z_2/z_1;x^{2k})_\infty (x^{-2+2r}z_2/z_1;x^{2r})_\infty},\ (j=1,2), && \text{(A.4)}\\
C_1^\dagger(z_1)C_2^\dagger(z_2) &= :: z_1^{\frac{2}{r}-\frac{2}{k}} \frac{(x^{s+2k}z_2/z_1;x^{2k})_\infty (x^{s-2+2k}z_2/z_1;x^{2k})_\infty}{(x^{-s+2k}z_2/z_1;x^{2k})_\infty (x^{2-s+2k}z_2/z_1;x^{2k})_\infty} \\
& \times \quad \frac{(x^{-s+2r}z_2/z_1;x^{2r})_\infty (x^{s-2+2r}z_2/z_1;x^{2r})_\infty}{(x^{s+2r}z_2/z_1;x^{2r})_\infty (x^{2-s+2r}z_2/z_1;x^{2r})_\infty}, && \text{(A.5)}\\
C_2^\dagger(z_1)C_1^\dagger(z_2) &= :: z_1^{\frac{2}{r}-\frac{2}{k}} \frac{(x^{s}z_2/z_1;x^{2k})_\infty (x^{s-2}z_2/z_1;x^{2k})_\infty}{(x^{-s}z_2/z_1;x^{2k})_\infty (x^{2-s}z_2/z_1;x^{2k})_\infty} \\
& \times \quad \frac{(x^{-s}z_2/z_1;x^{2r})_\infty (x^{s-2}z_2/z_1;x^{2r})_\infty}{(x^{s}z_2/z_1;x^{2r})_\infty (x^{2-s}z_2/z_1;x^{2r})_\infty}, && \text{(A.6)}\\
C_j(z_1)C_j^\dagger(z_2) &= :: z_1^{-\frac{2}{k}} \frac{(x^{2+k}z_2/z_1;x^{2k})_\infty}{(x^{-2+k}z_2/z_1;x^{2k})_\infty},\ (j=1,2), && \text{(A.7)}\\
C_j^\dagger(z_1)C_j(z_2) &= :: z_1^{-\frac{2}{k}} \frac{(x^{2+k}z_2/z_1;x^{2k})_\infty}{(x^{-2+k}z_2/z_1;x^{2k})_\infty},\ (j=1,2), && \text{(A.8)}\\
\widetilde{\Psi}_{1,I}(z_1)\widetilde{\Psi}_{2,I}(z_2) &= :: \frac{(x^{-s}z_2/z_1;x^{2k})_\infty (x^{2-s+2k}z_2/z_1;x^{2k})_\infty}{(x^{s+2k}z_2/z_1;x^{2k})_\infty (x^{s-2}z_2/z_1;x^{2k})_\infty}, && \text{(A.9)}\\
\widetilde{\Psi}_{2,I}(z_1)\widetilde{\Psi}_{1,I}(z_2) &= :: \frac{(x^{-s}z_2/z_1;x^{2k})_\infty (x^{2-s+2k}z_2/z_1;x^{2k})_\infty}{(x^{s+2k}z_2/z_1;x^{2k})_\infty (x^{s-2}z_2/z_1;x^{2k})_\infty}, && \text{(A.10)}\\
\widetilde{\Psi}_{1,II}(z_1)\widetilde{\Psi}_{2,II}(z_2) &= :: \frac{(x^{-s}z_2/z_1;x^{2k})_\infty (x^{2-s+2k}z_2/z_1;x^{2k})_\infty}{(x^{s+2k}z_2/z_1;x^{2k})_\infty (x^{s-2}z_2/z_1;x^{2k})_\infty}, && \text{(A.11)}\\
\widetilde{\Psi}_{2,II}(z_1)\widetilde{\Psi}_{1,II}(z_2) &= :: \frac{(x^{-s}z_2/z_1;x^{2k})_\infty (x^{2-s+2k}z_2/z_1;x^{2k})_\infty}{(x^{s+2k}z_2/z_1;x^{2k})_\infty (x^{s-2}z_2/z_1;x^{2k})_\infty}, && \text{(A.12)}\\
\widetilde{\Psi}_{1,I}^\dagger(z_1)\widetilde{\Psi}_{2,I}^\dagger(z_2) &= :: \frac{(x^{-s}z_2/z_1;x^{2k})_\infty (x^{2-s+2k}z_2/z_1;x^{2k})_\infty}{(x^{s+2k}z_2/z_1;x^{2k})_\infty (x^{s-2}z_2/z_1;x^{2k})_\infty}, && \text{(A.13)}\\
\widetilde{\Psi}_{2,I}^\dagger(z_1)\widetilde{\Psi}_{1,I}^\dagger(z_2) &= :: \frac{(x^{-s}z_2/z_1;x^{2k})_\infty (x^{2-s+2k}z_2/z_1;x^{2k})_\infty}{(x^{s+2k}z_2/z_1;x^{2k})_\infty (x^{s-2}z_2/z_1;x^{2k})_\infty}, && \text{(A.14)}\\
\widetilde{\Psi}_{1,II}^\dagger(z_1)\widetilde{\Psi}_{2,II}^\dagger(z_2) &= :: \frac{(x^{-s}z_2/z_1;x^{2k})_\infty (x^{2-s+2k}z_2/z_1;x^{2k})_\infty}{(x^{s+2k}z_2/z_1;x^{2k})_\infty (x^{s-2}z_2/z_1;x^{2k})_\infty}, && \text{(A.15)}\\
\widetilde{\Psi}_{2,II}^\dagger(z_1)\widetilde{\Psi}_{1,II}^\dagger(z_2) &= :: \frac{(x^{-s}z_2/z_1;x^{2k})_\infty (x^{2-s+2k}z_2/z_1;x^{2k})_\infty}{(x^{s+2k}z_2/z_1;x^{2k})_\infty (x^{s-2}z_2/z_1;x^{2k})_\infty}, && \text{(A.16)}\\
\widetilde{\Psi}_{1,I}(z_1)\widetilde{\Psi}_{2,II}(z_2) &= :: \frac{(x^{-s+2k}z_2/z_1;x^{2k})_\infty (x^{2-s+2k}z_2/z_1;x^{2k})_\infty}{(x^{s+2k}z_2/z_1;x^{2k})_\infty (x^{s-2+2k}z_2/z_1;x^{2k})_\infty}, && \text{(A.17)}\\
\widetilde{\Psi}_{2,II}(z_1)\widetilde{\Psi}_{1,I}(z_2) &= :: \frac{(x^{-s}z_2/z_1;x^{2k})_\infty (x^{2-s}z_2/z_1;x^{2k})_\infty}{(x^{s}z_2/z_1;x^{2k})_\infty (x^{s-2}z_2/z_1;x^{2k})_\infty}, && \text{(A.18)}\\
\widetilde{\Psi}_{1,II}(z_1)\widetilde{\Psi}_{2,I}(z_2) &= :: \frac{(x^{-s}z_2/z_1;x^{2k})_\infty (x^{2-s}z_2/z_1;x^{2k})_\infty}{(x^{s}z_2/z_1;x^{2k})_\infty (x^{s-2}z_2/z_1;x^{2k})_\infty}, && \text{(A.19)}
\end{aligned}
$$

$$\widetilde{\Psi}_{2,I}(z_1)\widetilde{\Psi}_{1,II}(z_2) = ::\frac{(x^{-s+2k}z_2/z_1;x^{2k})_\infty(x^{2-s+2k}z_2/z_1;x^{2k})_\infty}{(x^{s+2k}z_2/z_1;x^{2k})_\infty(x^{s-2+2k}z_2/z_1;x^{2k})_\infty}, \tag{A.20}$$

$$\widetilde{\Psi}^\dagger_{1,I}(z_1)\widetilde{\Psi}^\dagger_{2,II}(z_2) = ::\frac{(x^{-s}z_2/z_1;x^{2k})_\infty(x^{2-s}z_2/z_1;x^{2k})_\infty}{(x^{s}z_2/z_1;x^{2k})_\infty(x^{s-2}z_2/z_1;x^{2k})_\infty}, \tag{A.21}$$

$$\widetilde{\Psi}^\dagger_{2,II}(z_1)\widetilde{\Psi}^\dagger_{1,I}(z_2) = ::\frac{(x^{-s+2k}z_2/z_1;x^{2k})_\infty(x^{2-s+2k}z_2/z_1;x^{2k})_\infty}{(x^{s+2k}z_2/z_1;x^{2k})_\infty(x^{s-2+2k}z_2/z_1;x^{2k})_\infty}, \tag{A.22}$$

$$\widetilde{\Psi}^\dagger_{1,II}(z_1)\widetilde{\Psi}^\dagger_{2,I}(z_2) = ::\frac{(x^{-s+2k}z_2/z_1;x^{2k})_\infty(x^{2-s+2k}z_2/z_1;x^{2k})_\infty}{(x^{s+2k}z_2/z_1;x^{2k})_\infty(x^{s-2+2k}z_2/z_1;x^{2k})_\infty}, \tag{A.23}$$

$$\widetilde{\Psi}^\dagger_{2,I}(z_1)\widetilde{\Psi}^\dagger_{1,II}(z_2) = ::\frac{(x^{-s}z_2/z_1;x^{2k})_\infty(x^{2-s}z_2/z_1;x^{2k})_\infty}{(x^{s}z_2/z_1;x^{2k})_\infty(x^{s-2}z_2/z_1;x^{2k})_\infty}, \tag{A.24}$$

$$\widetilde{\Psi}_{j,I}(z_1)\widetilde{\Psi}_{j,I}(z_2) = ::(1-z_2/z_1)\frac{(x^{2+2k}z_2/z_1;x^{2k})_\infty}{(x^{-2}z_2/z_1;x^{2k})_\infty},\ (j=1,2), \tag{A.25}$$

$$\widetilde{\Psi}_{j,II}(z_1)\widetilde{\Psi}_{j,II}(z_2) = ::(1-z_2/z_1)\frac{(x^{2+2k}z_2/z_1;x^{2k})_\infty}{(x^{-2}z_2/z_1;x^{2k})_\infty},\ (j=1,2), \tag{A.26}$$

$$\widetilde{\Psi}_{j,I}(z_1)\widetilde{\Psi}_{j,II}(z_2) = ::\frac{(x^{2+2k}z_2/z_1;x^{2k})_\infty}{(x^{-2+2k}z_2/z_1;x^{2k})_\infty},\ (j=1,2), \tag{A.27}$$

$$\widetilde{\Psi}_{j,II}(z_1)\widetilde{\Psi}_{j,I}(z_2) = ::\frac{(x^{2}z_2/z_1;x^{2k})_\infty}{(x^{-2}z_2/z_1;x^{2k})_\infty},\ (j=1,2), \tag{A.28}$$

$$\widetilde{\Psi}^\dagger_{j,I}(z_1)\widetilde{\Psi}^\dagger_{j,I}(z_2) = ::(1-z_2/z_1)\frac{(x^{2+2k}z_2/z_1;x^{2k})_\infty}{(x^{-2}z_2/z_1;x^{2k})_\infty},\ (j=1,2), \tag{A.29}$$

$$\widetilde{\Psi}^\dagger_{j,II}(z_1)\widetilde{\Psi}^\dagger_{j,II}(z_2) = ::(1-z_2/z_1)\frac{(x^{2+2k}z_2/z_1;x^{2k})_\infty}{(x^{-2}z_2/z_1;x^{2k})_\infty},\ (j=1,2), \tag{A.30}$$

$$\widetilde{\Psi}^\dagger_{j,I}(z_1)\widetilde{\Psi}^\dagger_{j,II}(z_2) = ::\frac{(x^{2}z_2/z_1;x^{2k})_\infty}{(x^{-2}z_2/z_1;x^{2k})_\infty},\ (j=1,2), \tag{A.31}$$

$$\widetilde{\Psi}^\dagger_{j,II}(z_1)\widetilde{\Psi}^\dagger_{j,I}(z_2) = ::\frac{(x^{2+2k}z_2/z_1;x^{2k})_\infty}{(x^{-2+2k}z_2/z_1;x^{2k})_\infty},\ (j=1,2), \tag{A.32}$$

$$\widetilde{\Psi}_{j,I}(z_1)\widetilde{\Psi}^\dagger_{j,I}(z_2) = ::\frac{1}{(1-x^{k}z_2/z_1)}\frac{(x^{k-2}z_2/z_1;x^{2k})_\infty}{(x^{3k+2}z_2/z_1;x^{2k})_\infty},\ (j=1,2), \tag{A.33}$$

$$\widetilde{\Psi}^\dagger_{j,I}(z_1)\widetilde{\Psi}_{j,I}(z_2) = ::\frac{1}{(1-x^{-k}z_2/z_1)}\frac{(x^{-k-2}z_2/z_1;x^{2k})_\infty}{(x^{k+2}z_2/z_1;x^{2k})_\infty},\ (j=1,2), \tag{A.34}$$

$$\widetilde{\Psi}_{j,I}(z_1)\widetilde{\Psi}^\dagger_{j,II}(z_2) = ::\frac{(x^{-2+k}z_2/z_1;x^{2k})_\infty}{(x^{2+k}z_2/z_1;x^{2k})_\infty},\quad (j=1,2), \tag{A.35}$$

$$\widetilde{\Psi}^\dagger_{j,II}(z_1)\widetilde{\Psi}_{j,I}(z_2) = ::\frac{(x^{-2+k}z_2/z_1;x^{2k})_\infty}{(x^{2+k}z_2/z_1;x^{2k})_\infty},\quad (j=1,2), \tag{A.36}$$

$$\widetilde{\Psi}_{j,II}(z_1)\widetilde{\Psi}^\dagger_{j,I}(z_2) = ::\frac{(x^{-2+k}z_2/z_1;x^{2k})_\infty}{(x^{2+k}z_2/z_1;x^{2k})_\infty},\quad (j=1,2), \tag{A.37}$$

$$\widetilde{\Psi}^\dagger_{j,I}(z_1)\widetilde{\Psi}_{j,II}(z_2) = ::\frac{(x^{-2+k}z_2/z_1;x^{2k})_\infty}{(x^{2+k}z_2/z_1;x^{2k})_\infty},\quad (j=1,2), \tag{A.38}$$

$$\widetilde{\Psi}_{j,II}(z_1)\widetilde{\Psi}^\dagger_{j,II}(z_2) = ::\frac{1}{(1-x^{k}z_2/z_1)}\frac{(x^{-k-2}z_2/z_1;x^{2k})_\infty}{(x^{k+2}z_2/z_1;x^{2k})_\infty},\ (j=1,2), \tag{A.39}$$

$$\widetilde{\Psi}^\dagger_{j,II}(z_1)\widetilde{\Psi}_{j,II}(z_2) = ::\frac{1}{(1-x^{k}z_2/z_1)}\frac{(x^{k-2}z_2/z_1;x^{2k})_\infty}{(x^{3k+2}z_2/z_1;x^{2k})_\infty},\ (j=1,2). \tag{A.40}$$

References

[1] Feigin B., Kojima T., Shiraishi J. and Watanabe H., The integrals of motion for the deformed Virasoro algebra, Preprint 54 pages (2007), submitted for publication, arXiv:0705.0427.

[2] Feigin B., Kojima T., Shiraishi J. and Watanabe H., The integrals of motion for the deformed W-algebra $W_{q,t}(\widehat{sl_N})$, *Proc.for Representation Theory 2006 (Atami, Japan)*, 102-114 (2006), ISBN 4-9902328-2-8.

[3] Kojima T. and Shiraishi J., The integrals of motion for the deformed W-algebra $W_{q,t}(\widehat{gl_N})$ II: Proof of the commutation relations, *Commun.Math.Phys.*(2008) at press, arXiv:0709.2305.

[4] Bazhanov V., Lukyanov S.and Zamolodchikov Al., Integrable structure of conformal field theory :quantum KdV theory and thermodynamic Bethe ansatz, *Commun.Math.Phys.***177**, 381-398, (1976).

[5] Bazhanov V., Hibberd A. and Khoroshkin S., Integrable structure of W_3 conformal field theory, quantum Boussinesq theory and boundary affine Toda theory, *Nucl.Phys.***B622**,475-547, (2002).

[6] Kojima T., Baxter's Q-operator for the W-algebra W_N, *J.Phys* **A41**: Math.Theor.(2008) at press, arXiv.0803.3505.

[7] Konno H., An elliptic algebra $U_{q,p}(\widehat{sl_2})$ and the fusion RSOS model, *Commun.Math.Phys.* **195**, 373-403, (1998).

[8] Matsuo A., A q-deformation of Wakimoto modules, Primary fields and screening operators, *Commun.Math.Phys.***161**,33-48,(1994).

[9] Jimbo M., Konno H., Odake S. and Shiraishi J., Elliptic algebra $U_{q,p}(\widehat{sl_2})$: Drinfeld current and Vertex operators, *Commun.Math.Phys.***199**,605-647, (1999).

[10] Jimbo M., Konno H., Odake S. and Shiraishi J., Quasi-Hopf twistors for elliptic quantum groups, *Transformation Group* **4**, 303-327, (1999).

[11] Feigin B. and Odesskii A.,A family of elliptic algebras, *Internat.Math.Res.Notices* **11**,531-539, (1997).