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The Integrals of Motion for the Deformed W-Algebra $W_{q,t}(\widehat{sl_N})$ II

Proof of the commutation relations

Dedicated to Professor Tetsuji Miwa on the occasion of the 60th birthday

Takeo Kojima[1] **and Jun'ichi Shiraishi**[2]

[1] Department of Mathematics, College of Science and Technology, Nihon University, Surugadai, Chiyoda-ku, Tokyo 101-0062, JAPAN
[2] Graduate School of Mathematical Science, University of Tokyo, Komaba, Meguro-ku, Tokyo, 153-8914, JAPAN

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Abstract: We explicitly construct two classes of infinitly many commutative operators in terms of the deformed W-algebra $W_{q,t}(\widehat{sl_N})$, and give proofs of the commutation relations of these operators. We call one of them local integrals of motion and the other nonlocal one, since they can be regarded as elliptic deformations of local and nonlocal integrals of motion for the Virasoro algebra and the W_3 algebra [1,2].

1. Introduction

This is a continuation of the papers [3], [4], hereafter referred to as Part 1 [3] and Part 2 [4]. In Part 1 we constructed two classes of infinitly many commutative operators, in terms of the deformed Virasoro algebra. In Part 2 we announced conjecturous formulae of two classes of infinitly many commutative operators, in terms of the deformed W algebra $W_{q,t}(\widehat{sl_N})$, which is the higher-rank generalization of Part 1 [3]. We call one of them local integrals of motion and the other nonlocal one, since they can be regarded as elliptic deformations of local and nonlocal integrals of motion for the Virasoro algebra and the W_3 algebra [1], [2]. In this paper we give proofs of the commutation relations of the integrals of motion for the deformed W algebra $W_{q,t}(\widehat{sl_N})$.

Let us recall some facts about soliton equation and its quantization. B.Feigin and E.Frenkel [5] considered the so-called local integrals of motion $I^{(cl)}$ for the Toda field theory associated with the root system of finite and affine type $\{I^{(cl)}, H^{(cl)}\}_{P.B.} = 0$, where $H^{(cl)} = \frac{1}{2}\int (e^{\phi(t)} + e^{-\phi(t)})dt$ is the Hamiltonian of the Toda field theory. They showed the existence of infinitly many commutative integrals of motion by a cohomological argumemnt, and showed that they can be regarded as the conservation laws for the generalized KdV equation. In [5] they constructed the quantum deformtion of the local integrals of motion, too.

In other words they showed the existence of quantum deformation of the conservation laws of the generalized KdV equation. After quantization Gel'fand-Dickij bracket $\{,\}_{P.B.}$ for the second Hamiltonian structure of the generalized KdV, gives rise to the W_N algebra. V.Bazhanov et.al [1], [2] constructed field theoretical analogue of the commuting transfer matrix $\mathbf{T}(z)$, acting on the irreducible highest weight module of the Virasoro algebra and the W_3 algebra. They constructed this commuting transfer matrix $\mathbf{T}(z)$ as the trace of the monodromy matrix associated with the quantum affine symmetry $U_q(\widehat{sl_2})$ and $U_q(\widehat{sl_3})$, and showed that the commutatin relation $[\mathbf{T}(z),\mathbf{T}(w)]=0$ is a direct consequence of the Yang-Baxter relation. The coefficients of the asymptotic expansion of the operator $\log \mathbf{T}(z)$ at $z\to\infty$, produce the local integrals of motion for the Virasoro algebra and the W_3 algebra, which reproduce the conservation laws of the generalized KdV equation in the classical limit $c_{CFT}\to\infty$. They call the coefficients of the Taylor expansion of the operator $\mathbf{T}(z)$ at $z=0$, the nonlocal integrals of motion for the Virasoro algebra and the W_3 algebra. They have explicit integral representation of the nonlocal integrals in terms of the screening currents.

The purpose of this paper is to construct the elliptic version of the integrals of motion given by Bazhanov et.al [1], [2] and to construct its higher rank generalization. Bazhanov et.al's construction is based on the free field realization of the Borel subalgebra $\mathcal{B}_\pm$ of $U_q(\widehat{sl_2})$ and $U_q(\widehat{sl_3})$. By using this realization they construct the monodromy matrix as the image of the universal R-matrix $\bar{R}\in\mathcal{B}_+\otimes\mathcal{B}_-$, and make the transfer matrix $\mathbf{T}(z)$ as the trace of the monodromy matrix. The universal R-matrix $\bar{R}$ of the elliptic quantum group does not exist in $\mathcal{B}_+\otimes\mathcal{B}_-$. Hence it is impossible to construct the elliptic deformation of the transfer matris $\mathbf{T}(z)$ as the same manner as [1]. Our method of construction should be completely different from those of [1], [2]. Instead of considering the transfer mtrix $\mathbf{T}(z)$, we directly give the integral representations of the integrals of motion $\mathcal{I}_n,\mathcal{G}_n$ for the deformed W algebra $W_{q,t}(\widehat{sl_N})$. The commutativity of our integrals of motion are not understood as a direct consequence of the Yang-Baxter equation. They are understood as a consequence of the commutative subalgebra of the Feigin-Odesskii algebra [10].

The organization of this paper is as follows. In Section 2, we review the deformed W algebra, including free field realization, screening currents [6], [8]. In Section 3, we give integral representations for the local integrals of motion $\mathcal{I}_n$, and show the commutation relations :

$$[\mathcal{I}_m,\mathcal{I}_n]=[\mathcal{I}_m^*,\mathcal{I}_n^*]=0.$$

Very precisely, in Part 2 [4], we only give the Laurent series representation of the local integrals of motion, which is useful for proofs of the commutation relation and Dynkin-automorphism invariance. In this section we show the integral representations and the Laurent series representation give the same local integrals of motion. In Section 4, we give explicit formulae for the nonlocal integrals of motion $\mathcal{G}_n$, and show the commutation relations :

$$[\mathcal{G}_m,\mathcal{G}_n]=[\mathcal{G}_m^*,\mathcal{G}_n^*]=[\mathcal{G}_m,\mathcal{G}_n^*]=0,$$

$$[\mathcal{I}_m,\mathcal{G}_n]=[\mathcal{I}_m^*,\mathcal{G}_n]=[\mathcal{I}_m,\mathcal{G}_n^*]=[\mathcal{I}_m^*,\mathcal{G}_n^*]=0.$$

We show the commutation relation $[\mathcal{I}_m, \mathcal{G}_n] = 0$ using Dynkin-automorphism invariance $\eta(\mathcal{I}_n) = \mathcal{I}_n$ and $\eta(\mathcal{G}_n) = \mathcal{G}_n$, which will be shown in the next section. In Section 5, we give proofs of Dynkin-automorphis invariance :

$$\eta(\mathcal{I}_n) = \mathcal{I}_n, \quad \eta(\mathcal{I}_n^*) = \mathcal{I}_n^*, \quad \eta(\mathcal{G}_n) = \mathcal{G}_n, \quad \eta(\mathcal{G}_n^*) = \mathcal{G}_n^*.$$

In Appendix we summarize the normal ordering of the basic operators. We would like to point out a differnt point between the case of the deformed Virasoro $Vir_{q,t} = W_{q,t}(\widehat{sl_2})$ and its higher-rank generalization, $W_{q,t}(\widehat{sl_N})$, $(N \geq 3)$. Basically situations of $W_{q,t}(\widehat{sl_N})$, $(N \geqq 3)$ are more complicated than those of $Vir_{q,t} = W_{q,t}(\widehat{sl_2})$. However, one thing of $W_{q,t}(\widehat{sl_N})$, $(N \geqq 3)$ is simpler than those of $Vir_{q,t} = W_{q,t}(\widehat{sl_2})$. In the case of $Vir_{q,t} = W_{q,t}(\widehat{sl_2})$, the integrals of motions $\mathcal{I}_n$, $\mathcal{G}_n$ have singularity at $s = N = 2$. Hence we considered the renormalized limits for the integral of motions $\mathcal{I}_n$, $\mathcal{G}_n$ in the lase section of the paper [3]. In the case of $W_{q,t}(\widehat{sl_N})$, $(N \geqq 3)$, the integrals of motions $\mathcal{I}_n$, $\mathcal{G}_n$ do not have singularity at $s = N \geqq 3$.

At the end of Introduction, we would like to mention about two important degenerating limits of the deformed W algebra. One is the CFT-limit[1], [2] and the other is the classical limit[14]. In the CFT-limit V.Bazhanov et.al. [1], [2] constructed infinitly many integrals of motion for the Virasoro algebra, as we mentioned above. We give a comment on the CFT-limit in Section 4. In the classical limit, the deformed Virasoro algebra degenerates to the Poisson-Virasoro algebra introduced by E.Frenkel and N.Reshetikhin [14].

2. The Deformed W-Algebra $W_{q,t}(\widehat{sl_N})$

In this section we review the deformed W-algebra and its screening currents. We prepare the notations to be used in this paper. Throughout this paper, we fix generic three parameters $0 < x < 1$, $r \in \mathbb{C}$ and $s \in \mathbb{C}$. Let us set $z = x^{2u}$. Let us set $r^* = r - 1$. The symbol $[u]_r$ for $\mathrm{Re}(r) > 0$ stands for the Jacobi theta function

$$[u]_r = x^{\frac{u^2}{r} - u} \frac{\Theta_{x^{2r}}(x^{2u})}{(x^{2r}; x^{2r})^3}, \quad \Theta_q(z) = (z; q)_\infty (qz^{-1}; q)_\infty (q; q)_\infty, \tag{2.1}$$

where we have used the standard notation

$$(z; q)_\infty = \prod_{j=0}^{\infty} (1 - q^j z). \tag{2.2}$$

We set the parametrizations τ, τ^*

$$x = e^{-\pi\sqrt{-1}/r\tau} = e^{-\pi\sqrt{-1}/r^*\tau^*}. \tag{2.3}$$

The theta function $[u]_r$ enjoys the quasi-periodicity property

$$[u + r]_r = -[u]_r, \quad [u + r\tau]_r = -e^{-\pi\sqrt{-1}\tau - \frac{2\pi\sqrt{-1}}{r}u}[u]_r. \tag{2.4}$$

The symbol $[a]$ stands for

$$[a] = \frac{x^a - x^{-a}}{x - x^{-1}}. \tag{2.5}$$

2.1. Free Field Realization. Let $\epsilon_i (1 \leqq i \leqq N)$ be an orthonormal basis in $\mathbb{R}^N$ relative to the standard basis in $\mathbb{R}^N$ relative to the standard inner product $(,)$. Let us set $\bar{\epsilon}_i = \epsilon_i - \epsilon, \epsilon = \frac{1}{N}\sum_{j=1}^N \epsilon_j$. We idetify $\epsilon_{N+1} = \epsilon_1$. Let $P = \sum_{i=1}^N \mathbb{Z}\bar{\epsilon}_i$ the weight lattice. Let us set $\alpha_i = \bar{\epsilon}_i - \bar{\epsilon}_{i+1} \in P$.

Let β_m^j be the oscillators $(1 \leqq j \leqq N, m \in \mathbb{Z} - \{0\})$ with the commutation relations

$$[\beta_m^i, \beta_n^j] = \begin{cases} m\frac{[(r-1)m]}{[rm]}\frac{[(s-1)m]}{[sm]}\delta_{n+m,0} & (1 \leqq i = j \leqq N) \\ -m\frac{[(r-1)m]}{[rm]}\frac{[m]}{[sm]}x^{sm\ \mathrm{sgn}(i-j)}\delta_{n+m,0} & (1 \leqq i \neq j \leqq N) \end{cases} \tag{2.6}$$

We also introduce the zero mode operator P_λ, $(\lambda \in P)$. They are $\mathbb{Z}$-linear in P and satisfy

$$[iP_\lambda, Q_\mu] = (\lambda, \mu), \quad (\lambda, \mu \in P). \tag{2.7}$$

Let us intrduce the bosonic Fock space $\mathcal{F}_{l,k}(l, k \in P)$ generated by $\beta_{-m}^j (m > 0)$ over the vacuum vector $|l, k\rangle$:

$$\mathcal{F}_{l,k} = \mathbb{C}[\{\beta_{-1}^j, \beta_{-2}^j, \cdots\}_{1 \leqq j \leqq N}]|l, k\rangle, \tag{2.8}$$

where

$$\beta_m^j|l, k\rangle = 0, (m > 0), \tag{2.9}$$

$$P_\alpha|l, k\rangle = \left(\alpha, \sqrt{\frac{r}{r-1}}l - \sqrt{\frac{r-1}{r}}k\right)|l, k\rangle, \tag{2.10}$$

$$|l, k\rangle = e^{i\sqrt{\frac{r}{r-1}}Q_l - i\sqrt{\frac{r-1}{r}}Q_k}|0, 0\rangle. \tag{2.11}$$

Let us set the Dynkin-diagram automorphism η by

$$\eta(\beta_m^1) = x^{-\frac{2s}{N}m}\beta_m^2, \cdots, \eta(\beta_m^{N-1}) = x^{-\frac{2s}{N}m}\beta_m^N,\ \eta(\beta_m^N) = x^{\frac{2s}{N}(N-1)m}\beta_m^1 \tag{2.12}$$

and $\eta(\epsilon_i) = \epsilon_{i+1},\ (1 \leqq i \leqq N)$.

2.2. The Deformed W-Algebra. In this section we give short review of the deformed W-algebra $W_{q,t}(\widehat{sl_N})$ [7], [8], [9].

Definition 1. *We set the fundamental operator $\Lambda_j(z), (1 \leqq j \leqq N)$ by*

$$\Lambda_j(z) = x^{-2\sqrt{r(r-1)}P_{\bar{\epsilon}_j}} : \exp\left(\sum_{m \neq 0}\frac{x^{rm} - x^{-rm}}{m}\beta_m^j z^{-m}\right) : \quad (1 \leqq j \leqq N). \tag{2.13}$$

Definition 2. *Let us set the operator $T_j(z)$, $(1 \leqq j \leqq N)$ by*

$$T_j(z) = \sum_{1 \leqq s_1 < s_2 < \cdots < s_j \leqq N} : \Lambda_{s_1}(x^{-j+1}z)\Lambda_{s_2}(x^{-j+3}z)\cdots\Lambda_{s_j}(x^{j-1}z) : . \tag{2.14}$$

Proposition 1. *The actions of η on the fundamental operators $\Lambda_j(z)$, $(1 \leqq j \leqq N)$ are given by*

$$\eta(\Lambda_j(z)) = \Lambda_{j+1}(x^{\frac{2s}{N}}z), \quad (1 \leqq j \leqq N-1), \quad \eta(\Lambda_N(z)) = \Lambda_1(x^{\frac{2s}{N}-2s}z) \tag{2.15}$$

Proposition 2. *The operators $T_j(z)$, $(1 \leqq j \leqq N)$ satisfy the following relations.*

$$\begin{aligned}
&f_{i,j}(z_2/z_1)T_i(z_1)T_j(z_2) - f_{j,i}(z_1/z_2)T_j(z_2)T_i(z_1) \\
&= c\sum_{k=1}^{i}\prod_{l=1}^{k-1}\Delta(x^{2l+1}) \times \left(\delta\left(\frac{x^{j-i+2k}z_2}{z_1}\right) f_{i-k,j+k}(x^{-j+i})T_{i-k}(x^{-k}z_1)T_{j+k}(x^k z_2)\right. \\
&\left. - \delta\left(\frac{x^{-j+i-2k}z_2}{z_1}\right) f_{i-k,j+k}(x^{j-i})T_{i-k}(x^k z_1)T_{j+k}(x^{-k}z_2)\right), \quad (1 \leqq i \leqq j \leqq N),
\end{aligned} \tag{2.16}$$

where we have used the delta-function $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$. Here we set the constnt c and the auxiliary function $\Delta(z)$ by

$$c = -\frac{(1-x^{2r})(1-x^{-2r+2})}{(1-x^2)}, \quad \Delta(z) = \frac{(1-x^{2r-1}z)(1-x^{1-2r}z)}{(1-xz)(1-x^{-1}z)}. \tag{2.17}$$

Here we set the structure functions,

$$\begin{aligned}
&f_{i,j}(z) \\
&= \exp\left(\sum_{m=1}^{\infty}\frac{1}{m}\frac{(1-x^{2rm})(1-x^{-2(r-1)m})(1-x^{2mMin(i,j)})(1-x^{2m(s-Max(i,j))})}{(1-x^{2m})(1-x^{2sm})}x^{|i-j|m}z^m\right).
\end{aligned} \tag{2.18}$$

Above proposition is one parameter "s" generalization of [9]. The proof is given by the same manner.

Example For $N = 2$ the operators $T_1(z), T_2(z)$ satisfy

$$\begin{aligned}
&f_{1,1}(z_2/z_1)T_1(z_1)T_1(z_2) - f_{1,1}(z_1/z_2)T_1(z_2)T_1(z_1) \\
&= c(\delta(x^2z_2/z_1)T_2(xz_2) - \delta(x^2z_1/z_2)T_2(x^{-1}z_2)), &(2.19)\\
&f_{1,2}(z_2/z_1)T_1(z_1)T_2(z_2) = f_{2,1}(z_1/z_2)T_2(z_2)T_1(z_1), &(2.20)\\
&f_{2,2}(z_2/z_1)T_2(z_1)T_2(z_2) = f_{2,2}(z_1/z_2)T_2(z_2)T_2(z_1). &(2.21)
\end{aligned}$$

Example For $N = 3$ the operators $T_1(z), T_2(z), T_3(z)$ satisfy

$$\begin{aligned}
&f_{1,1}(z_2/z_1)T_1(z_1)T_1(z_2) - f_{1,1}(z_1/z_2)T_1(z_2)T_1(z_1) \\
&= c(\delta(x^2z_2/z_1)T_2(xz_2) - \delta(x^2z_1/z_2)T_2(x^{-1}z_2)), &(2.22)\\
&f_{1,2}(z_2/z_1)T_1(z_1)T_2(z_2) - f_{2,1}(z_1/z_2)T_2(z_2)T_1(z_1) \\
&= c(\delta(x^3z_2/z_1)T_3(xz_2) - \delta(x^3z_1/z_2)T_3(x^{-1}z_2)), &(2.23)\\
&f_{2,2}(z_2/z_1)T_2(z_1)T_2(z_2) - f_{2,2}(z_1/z_2)T_2(z_2)T_2(z_1) &(2.24)\\
&= cf_{1,3}(1)(\delta(x^2z_2/z_1)T_1(xz_2)T_3(xz_2) - \delta(x^2z_1/z_2)T_1(x^{-1}z_2)T_3(x^{-1}z_2)), \\
&f_{1,3}(z_2/z_1)T_1(z_1)T_3(z_2) = f_{3,1}(z_1/z_2)T_3(z_2)T_1(z_1), &(2.25)\\
&f_{2,3}(z_2/z_1)T_2(z_1)T_3(z_2) = f_{3,2}(z_1/z_2)T_3(z_2)T_2(z_1), &(2.26)\\
&f_{3,3}(z_2/z_1)T_3(z_1)T_3(z_2) = f_{3,3}(z_1/z_2)T_3(z_2)T_3(z_1). &(2.27)
\end{aligned}$$

Definition 3. *The deformed W-algebra is defined by the generators $\widehat{T}_m^{(j)}$, $(1 \leqq j \leqq N, m \in \mathbb{Z})$ with the defining relations (2.16). Here we should understand $\widehat{T}_m^{(j)}$ as the Fourier coefficients of the operators $\widehat{T}_j(z) = \sum_{m \in \mathbb{Z}} \widehat{T}_m^{(j)} z^{-m}$, $(1 \leqq j \leqq N)$.*

2.3. Screening Currents. In this section we introduce the screening currents $E_j(z)$ and $F_j(z)$.

Definition 4. *We set the screening currents $F_j(z), (1 \leqq j \leqq N)$ by*

$$\begin{aligned} F_j(z) &= e^{i\sqrt{\frac{r-1}{r}}Q_{\alpha_j}} (x^{(\frac{2s}{N}-1)j} z)^{\sqrt{\frac{r-1}{r}}P_{\alpha_j} + \frac{r-1}{r}} \\ &\quad \times : \exp\left(\sum_{m \neq 0} \frac{1}{m} B_m^j z^{-m} \right) :, \ (1 \leqq j \leqq N-1) \qquad (2.28) \\ F_N(z) &= e^{i\sqrt{\frac{r-1}{r}}Q_{\alpha_N}} (x^{2s-N} z)^{\sqrt{\frac{r-1}{r}}P_{\bar{\epsilon}_N} + \frac{r-1}{2r}} z^{-\sqrt{\frac{r-1}{r}}P_{\bar{\epsilon}_1} + \frac{r-1}{2r}} \\ &\quad \times : \exp\left(\sum_{m \neq 0} \frac{1}{m} B_m^N z^{-m} \right) :, \qquad (2.29) \end{aligned}$$

We set the screening currents $E_j(z), (1 \leqq j \leqq N)$ by

$$\begin{aligned} E_j(z) &= e^{-i\sqrt{\frac{r}{r-1}}Q_{\alpha_j}} (x^{(\frac{2s}{N}-1)j} z)^{-\sqrt{\frac{r}{r-1}}P_{\alpha_j} + \frac{r}{r-1}} \\ &\quad \times : \exp\left(-\sum_{m \neq 0} \frac{1}{m} \frac{[rm]}{[(r-1)m]} B_m^j z^{-m} \right) :, \ (1 \leqq j \leqq N-1) \qquad (2.30) \\ E_N(z) &= e^{-i\sqrt{\frac{r}{r-1}}Q_{\alpha_N}} (x^{2s-N} z)^{-\sqrt{\frac{r}{r-1}}P_{\bar{\epsilon}_N} + \frac{r}{2(r-1)}} z^{\sqrt{\frac{r}{r-1}}P_{\bar{\epsilon}_1} + \frac{r}{2(r-1)}} \\ &\quad \times : \exp\left(-\sum_{m \neq 0} \frac{1}{m} \frac{[rm]}{[(r-1)m]} B_m^N z^{-m} \right) : . \qquad (2.31) \end{aligned}$$

Here we have set

$$\begin{aligned} B_m^j &= (\beta_m^j - \beta_m^{j+1}) x^{-\frac{2s}{N}jm}, \ (1 \leqq j \leqq N-1), \qquad (2.32) \\ B_m^N &= (x^{-2sm} \beta_m^N - \beta_m^1). \qquad (2.33) \end{aligned}$$

The screening currents $F_j(z), E_j(z)$ $(1 \leqq j \leqq N-1)$ have already been studied in [11], [12], [13]. We introduce new screening current $F_N(z), E_N(z)$, which can be regarded as "affinization" of screening currents $F_j(z), E_j(z)$ $(1 \leqq j \leqq N-1)$.

The following commutation relations are convenient for calculations.

$$[\beta^j_m, B^j_n] = m\delta_{m+n,0}\frac{[r^*m]}{[rm]}x^{(-1+\frac{2s}{N}j)m}, \quad (1 \leqq j \leqq N) \tag{2.34}$$

$$[\beta^{j+1}_m, B^j_n] = -m\delta_{m+n,0}\frac{[r^*m]}{[rm]}x^{(1+\frac{2s}{N}j)m}, \quad (1 \leqq j \leqq N-1) \tag{2.35}$$

$$[\beta^1_m, B^N_n] = -m\delta_{m+n,0}\frac{[r^*m]}{[rm]}x^{m}, \tag{2.36}$$

$$[B^j_m, B^j_n] = m\delta_{m+n,0}\frac{[r^*m]}{[rm]}\frac{[2m]}{[m]}, \quad (1 \leqq j \leqq N) \tag{2.37}$$

$$[B^j_m, B^{j+1}_n] = -m\delta_{m+n,0}\frac{[r^*m]}{[rm]}x^{(-1+\frac{2s}{N})m}, \quad (1 \leqq j \leqq N). \tag{2.38}$$

Here we read $B^{N+1}_m = B^1_m$. We summarize the commutation relations of the screening currents for $N \geqq 3$.

Proposition 3. *The screening currents* $F_j(z)$, $(1 \leqq j \leqq N; N \geqq 3)$ *satisfy the following commutation relations for* $\mathrm{Re}(r) > 0$

$$\left[u_1 - u_2 - \frac{s}{N}\right]_r F_j(z_1)F_{j+1}(z_2) = \left[u_2 - u_1 + \frac{s}{N} - 1\right]_r F_{j+1}(z_2)F_j(z_1), \quad (1 \leqq j \leqq N), \tag{2.39}$$

$$[u_1 - u_2]_r[u_1 - u_2 + 1]_r F_j(z_1)F_j(z_2) = [u_2 - u_1]_r[u_2 - u_1 + 1]_r F_j(z_2)F_j(z_1), \quad (1 \leqq j \leqq N), \tag{2.40}$$

$$F_i(z_1)F_j(z_2) = F_j(z_2)F_i(z_1), \quad (|i - j| \geqq 2). \tag{2.41}$$

We read $F_{N+1}(z) = F_1(z)$. *The screening currents* $F_j(z)$, $(1 \leqq j \leqq N; N \geqq 3)$ *satisfy the following commutation relations for* $\mathrm{Re}(r) < 0$.

$$\left[u_1 - u_2 + 1 - \frac{s}{N}\right]_{-r} F_j(z_1)F_{j+1}(z_2) = \left[u_2 - u_1 + \frac{s}{N}\right]_{-r} F_{j+1}(z_2)F_j(z_1), \quad (1 \leqq j \leqq N), \tag{2.42}$$

$$[u_1 - u_2]_{-r}[u_1 - u_2 - 1]_{-r} F_j(z_1)F_j(z_2) = [u_2 - u_1]_{-r}[u_2 - u_1 - 1]_{-r} F_j(z_2)F_j(z_1), \quad (1 \leqq j \leqq N), \tag{2.43}$$

$$F_i(z_1)F_j(z_2) = F_j(z_2)F_i(z_1), \quad (|i - j| \geqq 2). \tag{2.44}$$

We read $F_{N+1}(z) = F_1(z)$.
The screening currents $E_j(z)$, $(1 \leqq j \leqq N; N \geqq 3)$ *satisfy the following commutation relations for* $\mathrm{Re}(r^*) > 0$

$$\left[u_1 - u_2 + 1 - \frac{s}{N}\right]_{r^*} E_j(z_1)E_{j+1}(z_2) = \left[u_2 - u_1 + \frac{s}{N}\right]_{r^*} E_{j+1}(z_2)E_j(z_1), \quad (1 \leqq j \leqq N), \tag{2.45}$$

$$[u_1 - u_2]_{r^*}[u_1 - u_2 - 1]_{r^*} E_j(z_1)E_j(z_2) = [u_2 - u_1]_{r^*}[u_2 - u_1 - 1]_{r^*} E_j(z_2)E_j(z_1), \quad (1 \leqq j \leqq N), \tag{2.46}$$

$$E_i(z_1)E_j(z_2)=E_j(z_2)E_i(z_1), \quad (|i-j|\geqq 2). \tag{2.47}$$

We read $E_{N+1}(z)=E_1(z)$.
The screening currents $E_j(z)$, $(1\leqq j\leqq N; N\geqq 3)$ *satisfy the following commutation relations for* $\mathrm{Re}(r^*)<0$.

$$\left[u_1-u_2-\frac{s}{N}\right]_{-r^*}E_j(z_1)E_{j+1}(z_2)=\left[u_2-u_1+\frac{s}{N}-1\right]_{-r^*}E_{j+1}(z_2)E_j(z_1), \quad (1\leqq j\leqq N), \tag{2.48}$$

$$[u_1-u_2]_{-r^*}[u_1-u_2+1]_{-r^*}E_j(z_1)E_j(z_2)=[u_2-u_1]_{-r^*}[u_2-u_1+1]_{-r^*}E_j(z_2)E_j(z_1), \quad (1\leqq j\leqq N), \tag{2.49}$$

$$E_i(z_1)E_j(z_2)=E_j(z_2)E_i(z_1), \quad (|i-j|\geqq 2). \tag{2.50}$$

Proposition 4. *The screening currents* $E_j(z), F_j(z)$, $(1\leqq j\leqq N; N\geqq 3)$ *satisfy the following commutation relation* $\mathrm{Re}(r)<0$.

$$[E_j(z_1),F_j(z_2)]=\frac{1}{x-x^{-1}}(\delta(xz_2/z_1)H_j(x^r z_2)-\delta(xz_1/z_2)H_j(x^{-r}z_2)), (1\leqq j\leqq N), \tag{2.51}$$

$$E_i(z_1)F_j(z_2)=F_j(z_2)E_i(z_1), \ (1\leqq i\neq j\leqq N). \tag{2.52}$$

Here we have set

$$\begin{aligned}H_j(z)&=x^{(1-\frac{2s}{N})2j}e^{-\frac{i}{\sqrt{rr^*}}Q_{\alpha_j}}(x^{(\frac{2s}{N}-1)j}z)^{-\frac{1}{\sqrt{rr^*}}P_{\alpha_j}+\frac{1}{rr^*}}\\&\times:\exp\left(-\sum_{m\neq 0}\frac{1}{m}\frac{[m]}{[r^*m]}B_m^j z^{-m}\right):, \quad (1\leqq j\leqq N-1),\end{aligned} \tag{2.53}$$

$$\begin{aligned}H_N(z)&=x^{2(N-2s)}e^{-\frac{i}{\sqrt{rr^*}}Q_{\alpha_N}}(x^{2s-N}z)^{-\frac{1}{\sqrt{rr^*}}P_{\bar{\epsilon}_N}+\frac{1}{2rr^*}}z^{-\frac{1}{\sqrt{rr^*}}P_{\bar{\epsilon}_1}+\frac{1}{2rr^*}}\\&\times:\exp\left(-\sum_{m\neq 0}\frac{1}{m}\frac{[m]}{[r^*m]}B_m^N z^{-m}\right):.\end{aligned} \tag{2.54}$$

Proposition 5. *The actions of* η *on the screenings* $F_j(z)$, $(1\leqq j\leqq N; N\geqq 3)$ *are given by*

$$\eta(F_j(z))=F_{j+1}(z)(x^{\frac{2s}{N}-1})^{-\sqrt{\frac{r^*}{r}}P_{\alpha_{j+1}}-\frac{r^*}{r}}, \quad (1\leqq j\leqq N-2), \tag{2.55}$$

$$\eta(F_{N-1}(z))=F_N(z)(x^{1-\frac{2s}{N}})^{\sqrt{\frac{r^*}{r}}P_{\bar{\epsilon}_N}+\frac{r^*}{2r}}(x^{1-\frac{2s}{N}})^{-\sqrt{\frac{r^*}{r}}P_{\bar{\epsilon}_1}+\frac{r^*}{2r}}, \tag{2.56}$$

$$\eta(F_N(z))=F_1(z)(x^{(1-\frac{2s}{N})(N-1)})^{\sqrt{\frac{r^*}{r}}P_{\bar{\epsilon}_1}+\frac{r^*}{2r}}(x^{1-\frac{2s}{N}})^{-\sqrt{\frac{r^*}{r}}P_{\bar{\epsilon}_2}+\frac{r^*}{2r}}. \tag{2.57}$$

Especially we have

$$\eta(F_1(z_1)F_2(z_2)\cdots F_N(z_N))=F_N(z_1)F_1(z_2)\cdots F_1(z_N). \tag{2.58}$$

The actions of η *on the screenings* $E_j(z)$, $(1\leqq j\leqq N; N\geqq 3)$ *are given by*

$$\eta(E_j(z))=E_{j+1}(z)(x^{\frac{2s}{N}-1})^{\sqrt{\frac{r}{r^*}}P_{\alpha_{j+1}}-\frac{r}{r^*}}, \quad (1\leqq j\leqq N-2), \tag{2.59}$$

$$\eta(E_{N-1}(z))=E_N(z)(x^{1-\frac{2s}{N}})^{-\sqrt{\frac{r}{r^*}}P_{\bar{\epsilon}_N}+\frac{r}{2r^*}}(x^{1-\frac{2s}{N}})^{\sqrt{\frac{r}{r^*}}P_{\bar{\epsilon}_1}+\frac{r}{2r^*}}, \tag{2.60}$$

$$\eta(E_N(z))=E_1(z)(x^{(1-\frac{2s}{N})(N-1)})^{-\sqrt{\frac{r}{r^*}}P_{\bar{\epsilon}_1}+\frac{r}{2r^*}}(x^{1-\frac{2s}{N}})^{\sqrt{\frac{r}{r^*}}P_{\bar{\epsilon}_2}+\frac{r}{2r^*}}. \tag{2.61}$$

Especially we have

$$\eta(E_1(z_1)E_2(z_2)\cdots E_N(z_N))=E_2(z_1)\cdots E_N(z_{N-1})E_1(z_N). \tag{2.62}$$

Proposition 6. *The screening currents* $F_j(z)$, $(1\leqq j\leqq N; N\geqq 3)$ *and the fundamental operators* $\Lambda_j(z)$, $(1\leqq j\leqq N; N\geqq 3)$ *commute up to delta-function* $\delta(z)=\sum_{n\in\mathbb{Z}}z^m$.

$$[\Lambda_j(z_1),F_j(z_2)]=(-x^{r^*}+x^{-r^*})\delta\left(x^{\frac{2s}{N}j-r}\frac{z_2}{z_1}\right)\mathcal{A}_j(x^{\frac{2s}{N}j-r}z_1),\quad (1\leqq j\leqq N-1), \tag{2.63}$$

$$[\Lambda_{j+1}(z_1),F_j(z_2)]=(x^{r^*}-x^{-r^*})\delta\left(x^{\frac{2s}{N}j+r}\frac{z_2}{z_1}\right)\mathcal{A}_j(x^{\frac{2s}{N}j+r}z_2),\quad (1\leqq j\leqq N-1), \tag{2.64}$$

$$[\Lambda_N(z_1),F_N(z_2)]=(-x^{r^*}+x^{-r^*})\delta\left(x^{-r+2s}\frac{z_2}{z_1}\right)\mathcal{A}_N(x^{-r}z_2), \tag{2.65}$$

$$[\Lambda_1(z_1),F_N(z_2)]=(x^{r^*}-x^{-r^*})\delta\left(x^{r}\frac{z_2}{z_1}\right)\mathcal{A}_N(x^{r}z_2). \tag{2.66}$$

Here we have set

$$\begin{aligned}\mathcal{A}_j(z)&=e^{i\sqrt{\frac{r^*}{r}}Q_{\alpha_j}}x^{-\sqrt{rr^*}(P_{\bar{\epsilon}_j}+P_{\bar{\epsilon}_{j+1}})}(zx^{-j})^{\sqrt{\frac{r^*}{r}}P_{\alpha_j}+\frac{r^*}{r}}\\&\quad\times:\exp\left(\sum_{m\neq 0}\frac{1}{m}(x^{rm}\beta_m^j-x^{-rm}\beta_m^{j+1})z^{-m}\right):,\quad (1\leqq j\leqq N-1),\end{aligned} \tag{2.67}$$

$$\begin{aligned}\mathcal{A}_N(z)&=e^{i\sqrt{\frac{r^*}{r}}Q_{\alpha_N}}x^{-\sqrt{rr^*}(P_{\bar{\epsilon}_N}+P_{\bar{\epsilon}_1})}(zx^{2s-N})^{\sqrt{\frac{r^*}{r}}P_{\bar{\epsilon}_N}+\frac{r^*}{2r}}z^{-\sqrt{\frac{r^*}{r}}P_{\bar{\epsilon}_1}+\frac{r^*}{2r}}\\&\quad\times:\exp\left(\sum_{m\neq 0}\frac{1}{m}(x^{(r-2s)m}\beta_m^N-x^{-rm}\beta_m^1)z^{-m}\right):.\end{aligned} \tag{2.68}$$

$$[\Lambda_j(z_1),E_j(z_2)]=(-x^{r}+x^{-r})\delta\left(x^{\frac{2s}{N}j+r^*}\frac{z_2}{z_1}\right)\mathcal{B}_j(x^{\frac{2s}{N}j+r^*}z_2),\quad (1\leqq j\leqq N-1), \tag{2.69}$$

$$[\Lambda_{j+1}(z_1),E_j(z_2)]=(x^{r}-x^{-r})\delta\left(x^{\frac{2s}{N}j-r^*}\frac{z_2}{z_1}\right)\mathcal{B}_j(x^{\frac{2s}{N}j-r^*}z_1),\quad (1\leqq j\leqq N-1), \tag{2.70}$$

$$[\Lambda_N(z_1),E_N(z_2)]=(-x^{r}+x^{-r})\delta\left(x^{r^*+2s}\frac{z_2}{z_1}\right)\mathcal{B}_N(x^{r^*}z_1), \tag{2.71}$$

$$[\Lambda_1(z_1),E_N(z_2)]=(x^{r}-x^{-r})\delta\left(x^{-r^*}\frac{z_2}{z_1}\right)\mathcal{B}_N(x^{-r^*}z_2). \tag{2.72}$$

Here we have set

$$\mathcal{B}_j(z)=e^{-i\sqrt{\frac{r}{r^*}}Q_{\alpha_j}}x^{-\sqrt{rr^*}(P_{\bar{\epsilon}_j}+P_{\bar{\epsilon}_{j+1}})}(zx^{-j})^{-\sqrt{\frac{r}{r^*}}P_{\alpha_j}+\frac{r}{r^*}}$$
$$\times :\exp\left(-\sum_{m\neq 0}\frac{1}{m}\frac{[rm]}{[r^*m]}(x^{-r^*m}\beta_m^j-x^{r^*m}\beta_m^{j+1})z^{-m}\right):,\ (1\leqq j\leqq N-1), \tag{2.73}$$

$$\mathcal{B}_N(z)=e^{-i\sqrt{\frac{r^*}{r}}Q_{\alpha_N}}x^{-\sqrt{rr^*}(P_{\bar{\epsilon}_N}+P_{\bar{\epsilon}_1})}(zx^{2s-N})^{-\sqrt{\frac{r}{r^*}}P_{\bar{\epsilon}_N}+\frac{r}{2r^*}}z^{\sqrt{\frac{r}{r^*}}P_{\bar{\epsilon}_1}+\frac{r}{2r^*}}$$
$$\times :\exp\left(\sum_{m\neq 0}\frac{1}{m}\frac{[rm]}{[r^*m]}(x^{(-r^*-2s)m}\beta_m^N-x^{r^*m}\beta_m^1)z^{-m}\right):. \tag{2.74}$$

2.4. Comparsion with another definition. At first glance, our definition of the deformed W-algebra is different from those in [7], [8], [9]. In this section we show they are essentially the same thing. Let us set the element $\mathcal{C}_m$ by

$$\mathcal{C}_m=\sum_{j=1}^N x^{(N-2j+1)m}\beta_m^j. \tag{2.75}$$

This element $\mathcal{C}_m$ is η-invariant, $\eta(\mathcal{C}_m)=\mathcal{C}_m$. Let us divide $\Lambda_j(z)$ into $\Lambda_j^{DWA}(z)$ and $\mathcal{Z}(z)$.

$$\Lambda_j(z)=\Lambda_j^{DWA}(z)\mathcal{Z}(z),\quad (1\leqq j\leqq N), \tag{2.76}$$

where we set

$$\Lambda_j^{DWA}(z)=x^{-2\sqrt{r(r-1)}P_{\bar{\epsilon}_j}}:\exp\left(\sum_{m\neq 0}\frac{x^{rm}-x^{-rm}}{m}\left(\beta_m^j-\frac{[m]_x}{[Nm]_x}\mathcal{C}_m\right)z^{-m}\right):, \tag{2.77}$$

$$\mathcal{Z}(z)=:\exp\left(\sum_{m\neq 0}\frac{x^{rm}-x^{-rm}}{m}\frac{[m]_x}{[Nm]_x}\mathcal{C}_m z^{-m}\right):. \tag{2.78}$$

Let us set

$$T_j^{DWA}(z)=\sum_{1\leqq s_1<s_2<\cdots<s_j\leqq N}:\Lambda_{s_1}^{DWA}(x^{-j+1}z)\Lambda_{s_2}^{DWA}(x^{-j+3}z)\cdots\Lambda_{s_j}^{DWA}(x^{j-1}z):. \tag{2.79}$$

Proposition 7. *The bosonic operators* $T_j^{DWA}(z)$, $(1 \leqq j \leqq N-1)$ *satisfy the following relations.*

$$\begin{aligned} & f_{i,j}^{DWA}(z_2/z_1)T_i^{DWA}(z_1)T_j^{DWA}(z_2) - f_{j,i}^{DWA}(z_1/z_2)T_j^{DWA}(z_2)T_i^{DWA}(z_1) \\ & = c\sum_{k=1}^{i}\prod_{l=1}^{k-1} \Delta(x^{2l+1}) \times \left(\delta\left(\frac{x^{j-i+2k}z_2}{z_1}\right) f_{i-k,j+k}^{DWA}(x^{-j+i})T_{i-k}^{DWA}(x^{-k}z_1)T_{j+k}^{DWA}(x^k z_2) \right. \\ & \left. - \delta\left(\frac{x^{-j+i-2k}z_2}{z_1}\right) f_{i-k,j+k}^{DWA}(x^{j-i})T_{i-k}^{DWA}(x^k z_1)T_{j+k}^{DWA}(x^{-k}z_2) \right), \quad (1 \leqq i \leqq j \leqq N-1), \end{aligned} \tag{2.80}$$

where $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$. *We should understand* $T_N^{DWA}(z) = 1, T_j^{DWA}(z) = 0$, $(j > N)$.
Here we set the constant c *and the auxiliary function* $\Delta(z)$ *in (2.17). Here we set the structure functions,*

$$\begin{aligned} & f_{i,j}^{DWA}(z) = f_{i,j}(z)|_{s=N} \\ & = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} \frac{(1-x^{2rm})(1-x^{-2(r-1)m})(1-x^{2mMin(i,j)})(1-x^{2m(N-Max(i,j))})}{(1-x^{2m})(1-x^{2Nm})} x^{|i-j|m} z^m \right). \end{aligned} \tag{2.81}$$

Proposition 8. *The operators* $T_j^{DWA}(z)$ *and* $\mathcal{Z}(z)$ *commutes with each other.*

$$T_j^{DWA}(z_1)\mathcal{Z}(z_2) = \mathcal{Z}(z_2)T_j^{DWA}(z_1), \quad (1 \leqq j \leqq N-1). \tag{2.82}$$

Therefore three parameter deformed W-algebra $T_j(z)$ is realized as an extension of two parameter deformed W-algebra $T_j^{DWA}(z)$ in [7], [8], [9]. Note that upon the specialization $s = N$ we have

$$[B_n^N, B_m^N] = 0, \ [B_m^j, B_n^N] = 0 \text{ for } j \neq N. \tag{2.83}$$

Hence we can regard $B_m^N = 0$ and $T_j(z) = T_j^{DWA}(z)$, $T_N^{DWA}(z) = 1$.

3. Local Integrals of Motion

In this section we construct the local integrals of motion $\mathcal{I}_n$. We study the generic case : $0 < x < 1$, $r \in \mathbb{C}$ and $\mathrm{Re}(s) > 0$.

3.1. Local Integrals of Motion for $W_{q,t}(\widehat{sl_N})$.

Let us set the function $h(u)$ and $h^*(u)$ by

$$h(u) = \frac{[u]_s[u+r]_s}{[u+1]_s[u+r^*]_s}, \quad h^*(u) = \frac{[u]_s[u-r^*]_s}{[u+1]_s[u-r]_s}, \tag{3.84}$$

where we have set $z = x^{2u}$.

Definition 5.

- *We define* $\mathcal{I}_n$ *for regime* $\mathrm{Re}(s)>2$ *and* $\mathrm{Re}(r^*)<0$ *by*

$$\mathcal{I}_n=\int\cdots\int_C\prod_{j=1}^n\frac{dz_j}{2\pi\sqrt{-1}z_j}\prod_{1\leqq j<k\leqq n}h(u_k-u_j)T_1(z_1)\cdots T_1(z_n)\quad(n=1,2,\cdots).\tag{3.85}$$

Here, the contour C *encircles* $z_j=0$ *in such a way that* $z_j=x^{-2+2sl}z_k, x^{-2r^*+2sl}z_k$ $(l=0,1,2,\cdots)$ *is inside and* $z_j=x^{2-2sl}z_k, x^{2r^*-2sl}z_k$ $(l=0,1,2,\cdots)$ *is outside for* $1\leqq j<k\leqq n$. *We call* $\mathcal{I}_n$ *the local integrals of motion for the deformed* W*-algebra. The definitions of* $\mathcal{I}_n$ *for generic* $\mathrm{Re}(s)>0$ *and* $r\in\mathbb{C}$ *should be understood as analytic continuation.*

- *We define* $\mathcal{I}_n^*$ *for regime* $\mathrm{Re}(s)>2$ *and* $\mathrm{Re}(r)>0$ *by*

$$\mathcal{I}_n=\int\cdots\int_C\prod_{j=1}^n\frac{dz_j}{2\pi\sqrt{-1}z_j}\prod_{1\leqq j<k\leqq n}h^*(u_k-u_j)T_1(z_1)\cdots T_1(z_n)\quad(n=1,2,\cdots).\tag{3.86}$$

Here, the contour C *encircles* $z_j=0$ *in such a way that* $z_j=x^{-2+2sl}z_k, x^{2r+2sl}z_k$ $(l=0,1,2,\cdots)$ *is inside and* $z_j=x^{2-2sl}z_k, x^{-2r-2sl}z_k$ $(l=0,1,2,\cdots)$ *is outside for* $1\leqq j<k\leqq n$. *We call* $\mathcal{I}_n^*$ *the local integrals of motion for the deformed* W*-algebra. The definitions of* $\mathcal{I}_n^*$ *for generic* $\mathrm{Re}(s)>0$ *and* $r\in\mathbb{C}$ *should be understood as analytic continuation.*

The following is one of **Main Results** of this paper.

Theorem 1. *The local integrals of motion* $\mathcal{I}_n$ *commute with each other*

$$[\mathcal{I}_n,\mathcal{I}_m]=0\quad(m,n=1,2,\cdots).\tag{3.87}$$

The local integrals of motion $\mathcal{I}_n^*$ *commute with each other*

$$[\mathcal{I}_n^*,\mathcal{I}_m^*]=0\quad(m,n=1,2,\cdots).\tag{3.88}$$

3.2. Laurent-Series Formulae. In this subsection we prepare another formulae of the local integrals of motion $\mathcal{I}_n$. Because the integral contour of the definition of the local integrals of motion $\mathcal{I}_n$ is not annulus. *i.e.* $|x^{-p}z_k|<|z_j|<|x^pz_k|$, the defining relations of the deformed W-algebra (2.16) should be used carefully. Hence, in order to show the commutation relations $[\mathcal{I}_m,\mathcal{I}_n]=0$, it is better for us to deform the integral representations of the local integrals of motion $\mathcal{I}_n$ to another formulae, in which the defining relations of the deformed W-algebra (2.16) can be used safely.

Let us set the auxiliary function $s(z), s^*(z)$ by $h(u)=s(z)f_{11}(z)$, $h^*(u)=s^*(z)f_{11}(z)$, $(z=x^{2u})$ where $h(u),h^*(u)$ and $f_{11}(z)$ are given in the previous section. We have explicitly

$$s(z)=x^{-2r^*}\frac{(z;x^{2s})_\infty(x^{2s-2r}z;x^{2s})_\infty}{(x^{2s-2}z;x^{2s})_\infty(x^{-2r^*}z;x^{2s})_\infty}\times\frac{(1/z;x^{2s})_\infty(x^{2s-2r}/z;x^{2s})_\infty}{(x^{2s-2}/z;x^{2s})_\infty(x^{-2r^*}/z;x^{2s})_\infty},\tag{3.89}$$

$$s^*(z)=x^{-2r^*}\frac{(z;x^{2s})_\infty(x^{2s+2r^*}z;x^{2s})_\infty}{(x^{2s-2}z;x^{2s})_\infty(x^{2r}z;x^{2s})_\infty}\times\frac{(1/z;x^{2s})_\infty(x^{2s+2r^*}/z;x^{2s})_\infty}{(x^{2s-2}/z;x^{2s})_\infty(x^{2r}/z;x^{2s})_\infty}\tag{3.90}$$

Let us set the auxiliary functions $g_{i,j}(z)$ by fusion procedure

$$\begin{aligned} g_{i,1}(z) &= g_{1,1}(x^{-i+1}z)g_{1,1}(x^{-i+3}z)\cdots g_{1,1}(x^{i-1}z), \\ g_{i,j}(z) &= g_{i,1}(x^{-j+1}z)g_{i,1}(x^{-j+3}z)\cdots g_{i,1}(x^{j-1}z). \end{aligned} \tag{3.91}$$

where $g_{11}(z) = f_{11}(z)$ is the structure function of the deformed W-algebra defined in (2.16).

$$f_{1,1}(z) = \frac{1}{1-z}\frac{(x^{2s-2}z;x^{2s})_\infty(x^{2r}z;x^{2s})_\infty(x^{-2r^*}z;x^{2s})_\infty}{(x^2z;x^{2s})_\infty(x^{2r^*+2s}z;x^{2s})_\infty(x^{2s-2r}z;x^{2s})_\infty}. \tag{3.92}$$

The structure functions $f_{1,j}(z)$ and $g_{1,j}(z)$ have the following relations

$$g_{1,j}(z) = \Delta(x^{-j+2}z)\Delta(x^{-j+4}z)\cdots\Delta(x^{j-2}z)f_{1,j}(z). \tag{3.93}$$

Here $\Delta(z)$ is given by $\Delta(z) = \frac{(1-x^{r+r^*}z)(1-x^{-r-r^*}z)}{(1-xz)(1-x^{-1}z)}$.

Let us set the formal power series $\mathcal{A}(z_1, z_2, \cdots, z_n)$ by

$$\mathcal{A}(z_1, z_2, \cdots, z_n) = \sum_{k_1,\cdots,k_n\in\mathbb{Z}} a_{k_1,\cdots,k_n} z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}. \tag{3.94}$$

We define the symbol $[\cdots]_{1,z_1\cdots z_n}$ by

$$[\mathcal{A}(z_1, z_2, \cdots, z_n)]_{1,z_1\cdots z_n} = a_{0,0,\cdots,0}. \tag{3.95}$$

Let us set $D = \{(z_1, \cdots, z_n) \in \mathbb{C}^n | \sum_{k_1,\cdots,k_n\in\mathbb{Z}} |a_{k_1,\cdots,k_n} z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}| < +\infty\}$. When we assume closed curve J is contained in D, we have

$$[\mathcal{A}(z_1, z_2, \cdots, z_n)]_{1,z_1\cdots z_n} = \int\cdots\int_J \prod_{j=1}^n \frac{dz_j}{2\pi\sqrt{-1}z_j}\mathcal{A}(z_1, z_2, \cdots, z_n). \tag{3.96}$$

Let us set the auxiliary functions, $s_{11}(z) = s(z),\ h_{11}(z) = h(u)$,$(z = x^{2u})$ and

$$\begin{aligned} s_{i,1}(z) &= s_{1,1}(x^{-i+1}z)s_{1,1}(x^{-i+3}z)\cdots s_{1,1}(x^{i-1}z), \\ s_{i,j}(z) &= s_{i,1}(x^{-j+1}z)s_{i,1}(x^{-j+3}z)\cdots s_{i,1}(x^{j-1}z), \end{aligned} \tag{3.97}$$

$$\begin{aligned} h_{i,1}(z) &= h_{1,1}(x^{-i+1}z)h_{1,1}(x^{-i+3}z)\cdots h_{1,1}(x^{i-1}z), \\ h_{i,j}(z) &= h_{i,1}(x^{-j+1}z)h_{i,1}(x^{-j+3}z)\cdots h_{i,1}(x^{j-1}z), \end{aligned} \tag{3.98}$$

and

$$\begin{aligned} s^*_{i,1}(z) &= s^*_{1,1}(x^{-i+1}z)s^*_{1,1}(x^{-i+3}z)\cdots s^*_{1,1}(x^{i-1}z), \\ s^*_{i,j}(z) &= s^*_{i,1}(x^{-j+1}z)s^*_{i,1}(x^{-j+3}z)\cdots s^*_{i,1}(x^{j-1}z), \end{aligned} \tag{3.99}$$

$$\begin{aligned} h^*_{i,1}(z) &= h^*_{1,1}(x^{-i+1}z)h^*_{1,1}(x^{-i+3}z)\cdots h^*_{1,1}(x^{i-1}z), \\ h^*_{i,j}(z) &= h^*_{i,1}(x^{-j+1}z)h^*_{i,1}(x^{-j+3}z)\cdots h^*_{i,1}(x^{j-1}z). \end{aligned} \tag{3.100}$$

In what follows we use the notation of the ordered product

$$\prod_{\overrightarrow{l\in L}} T_1(z_l) = T_1(z_{l_1})T_1(z_{l_2})\cdots T_1(z_{l_n}), \quad (L = \{l_1, \cdots, l_n | l_1 < l_2 < \cdots < l_n\}). \tag{3.101}$$

Theorem 2. *For* $\mathrm{Re}(s) > N$ *and* $\mathrm{Re}(r^*) < 0$*, the local integrals of motion* $\mathcal{I}_n$ *are written as*

$$\mathcal{I}_n = \left[\prod_{1 \leqq j < k \leqq n} s(z_k/z_j)\mathcal{O}_n(z_1, z_2, \cdots, z_n)\right]_{1, z_1 \cdots z_n}. \tag{3.102}$$

For $\mathrm{Re}(s) > N$ *and* $\mathrm{Re}(r) > 0$*, the local integrals of motion* $\mathcal{I}_n^*$ *are written as*

$$\mathcal{I}_n^* = \left[\prod_{1 \leqq j < k \leqq n} s^*(z_k/z_j)\mathcal{O}_n(z_1, z_2, \cdots, z_n)\right]_{1, z_1 \cdots z_n}. \tag{3.103}$$

Here we set the operator $\mathcal{O}_n(z_1, z_2, \cdots, z_n)$ *by*

$$\begin{aligned}
\mathcal{O}_n(z_1, z_2, \cdots, z_n) = & \sum_{\substack{\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_N \geqq 0 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + N\alpha_N = n}} \sum_{\substack{\left\{A_j^{(s)}\right\}_{\substack{s=1,\cdots,N \\ j=1,\cdots,\alpha_s}} \\ A_j^{(s)} \subset \{1,2,\cdots,n\}, \ |A_j^{(s)}| = s, \ \cup_{s=1}^N \cup_{j=1}^{\alpha_s} A_j^{(s)} = \{1,2,\cdots,n\} \\ Min(A_1^{(s)}) < Min(A_2^{(s)}) < \cdots < Min(A_{\alpha_s}^{(s)})}} \\
& \times \prod_{\overrightarrow{j \in A_{Min}^{(1)}}} T_1(z_j) \prod_{\overrightarrow{j \in A_{Min}^{(2)}}} T_2(x^{-1}z_j) \cdots \prod_{\overrightarrow{j \in A_{Min}^{(t)}}} T_t(x^{-1+t-2[\frac{t}{2}]}z_j) \cdots \prod_{\overrightarrow{j \in A_{Min}^{(N)}}} T_N(x^{-1+N-2[\frac{N}{2}]}z_j) \\
& \times \prod_{t=1}^N \left((-c)^{t-1} \prod_{u=1}^{t-1} \Delta(x^{2u+1})^{t-u-1}\right)^{\alpha_t} \prod_{t=1}^N \prod_{\substack{j=1 \\ j_1 = A_{j,1}^{(t)} \\ \cdots \\ j_t = A_{j,t}^{(t)}}}^{\alpha_t} \sum_{\substack{\sigma \in S_t \\ \sigma(1)=1}} \prod_{\substack{u=1 \\ u \neq [\frac{t}{2}]+1}}^{t} \delta\left(\frac{x^2 z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}}\right) \\
& \times \prod_{t=1}^N \prod_{\substack{j<k \\ j,k \in A_{Min}^{(t)}}} g_{t,t}\left(\frac{z_k}{z_j}\right) \prod_{1 \leqq t < u \leqq N} \prod_{\substack{j \in A_{Min}^{(t)} \\ k \in A_{Min}^{(u)}}} g_{t,u}\left(x^{u-t-2[\frac{u}{2}]+2[\frac{t}{2}]} \frac{z_k}{z_j}\right).
\end{aligned} \tag{3.104}$$

Here we have set the constant c *and the function* $\Delta(z)$ *in (2.17). When the index set* $A_j^{(t)} = \{j_1, j_2, \cdots, j_t | j_1 < j_2 < \cdots < j_t\}$*,* $(1 \leqq t \leqq N, 1 \leqq j \leqq \alpha_t)$*, we set* $A_{j,k}^{(t)} = j_k$*, and* $A_{Min}^{(t)} = \{A_{1,1}^{(t)}, A_{2,1}^{(t)}, \cdots, A_{\alpha_t,1}^{(t)}\}$*. Here we should understand* $z_{j_{\sigma(t+1)}} = z_{j_{\sigma(1)}}$ *in the delta-function* $\delta\left(\frac{x^2 z_{j_{\sigma(t+1)}}}{z_{j_{\sigma(t)}}}\right)$*.*

Example We summarize the operators $\mathcal{O}_n$ very explicitly.

$$\mathcal{O}_1(z) = T_1(z), \tag{3.105}$$

$$\mathcal{O}_2(z_1, z_2) = g_{1,1}(z_2/z_1)T_1(z_1)T_1(z_2) - c\delta(x^2z_2/z_1)T_2(x^{-1}z_1), \tag{3.106}$$

$$\begin{aligned}
\mathcal{O}_3(z_1, z_2, z_3) = & \, g_{11}(z_2/z_1)g_{1,1}(z_3/z_1)g_{1,1}(z_3/z_2)T_1(z_1)T_1(z_2)T_1(z_3) \\
& - cg_{1,2}(x^{-1}z_2/z_1)T_1(z_1)\delta(x^2z_3/z_2)T_2(x^{-1}z_2) \\
& - cg_{1,2}(x^{-1}z_1/z_2)T_1(z_2)\delta(x^2z_3/z_1)T_2(x^{-1}z_1) \\
& - cg_{1,2}(x^{-1}z_1/z_3)T_1(z_3)\delta(x^2z_2/z_1)T_2(x^{-1}z_1) \\
& + c^2\Delta(x^3)(\delta(x^2z_2/z_1)\delta(x^2z_1/z_3) + \delta(x^2z_1/z_2)\delta(x^2z_3/z_1))T_3(z_1).
\end{aligned} \tag{3.107}$$

We should understand above as $T_j(z) = 0$ for $j > N$.

3.3. Weakly sense equality. In order to show thorem, we introduce a "weak sense" equality

Definition 6. *We say the operators $\mathcal{P}(z_1,z_2,\cdots,z_n)$ and $\mathcal{Q}(z_1,z_2,\cdots,z_n)$ are equal in the "weak sense" if*

$$\prod_{1\leqq i<j\leqq n} s(z_j/z_i)\mathcal{P}(z_1,z_2,\cdots,z_n)=\prod_{1\leqq i<j\leqq n} s(z_j/z_i)\mathcal{Q}(z_1,z_2,\cdots,z_n). \tag{3.108}$$

We write $\mathcal{P}(z_1,z_2,\cdots,z_n)\sim\mathcal{Q}(z_1,z_2,\cdots,z_n)$, showing the weak equality.

For example $\delta(z_1/z_2)\sim 0$ and $\frac{1}{z_1-z_2}\delta(z_1/z_2)\sim 0$.

Proposition 9. *The following relations hold in the weak sense for $1\leqq j\leqq N$*

$$\begin{aligned}&\left(g_{1,j}\left(\frac{x^{-1+j-2[\frac{j}{2}]}w_1}{z_1}\right)T_1(z_1)T_j(x^{-1+j-2[\frac{j}{2}]}w_1)-g_{j,1}\left(\frac{x^{1-j+2[\frac{j}{2}]}z_1}{w_1}\right)T_j(x^{-1+j-2[\frac{j}{2}]}w_1)T_1(z_1)\right)\\&\times\sum_{\substack{\sigma\in S_j\\ \sigma(1)=1}}\prod_{\substack{t=1\\ t\neq[\frac{j}{2}]+1}}^{j}\delta\left(\frac{x^2w_{\sigma(t+1)}}{w_{\sigma(t)}}\right)\sim c\prod_{t=1}^{j-1}\varDelta(x^{2t+1})\sum_{\substack{\sigma\in S_j\\ \sigma(1)=1}}\prod_{\substack{t=1\\ t\neq[\frac{j}{2}]+1}}^{j}\delta\left(\frac{x^2w_{\sigma(t+1)}}{w_{\sigma(t)}}\right)\\&\times\left(\delta\left(\frac{x^{2j-2[\frac{j}{2}]}w_1}{z_1}\right)T_{j+1}(x^{j-2[\frac{j}{2}]}w_1)-\delta\left(\frac{x^{-2-2[\frac{j}{2}]}w_1}{z_1}\right)T_{j+1}(x^{j-2-2[\frac{j}{2}]}w_1)\right).\end{aligned}\tag{3.109}$$

We should understand $T_{N+1}(z)=0$ and $w_{\sigma(j+1)}=w_{\sigma(1)}$ in the delta-function $\delta\left(\frac{x^2w_{\sigma(j+1)}}{w_{\sigma(j)}}\right)$.

Proof. We explain the mechanism by the simplest case for $N\geqq 3$.

$$\begin{aligned}&\left(g_{1,2}(x^{-1}z_2/z_1)T_1(z_1)T_2(x^{-1}z_2)-g_{2,1}(xz_1/z_2)T_2(x^{-1}z_2)T_1(z_1)\right)\delta(x^2z_3/z_2)\\&=g_{1,2}(x^{-1}z_2/z_1)c(\delta(z_2/z_1)-\delta(x^{-2}z_2/z_1))\delta(x^2z_3/z_2)T_1(z_1)T_2(x^{-1}z_2)\qquad(3.110)\\&+\varDelta(xz_1/z_2)\delta(x^2z_3/z_2)(f_{1,2}(x^{-1}z_2/z_1)T_1(z_1)T_2(x^{-1}z_2)-f_{2,1}(xz_1/z_2)T_2(x^{-1}z_2)T_1(z_1)).\end{aligned}$$

Here we have used $g_{1,2}(z)=\varDelta(z)f_{1,2}(z)$ and

$$\varDelta(z)-\varDelta(z^{-1})=c(\delta(xz)-\delta(x^{-1}z)),\quad \varDelta(z)=\frac{(1-x^{2r-1}z)(1-x^{-2r+1}z)}{(1-xz)(1-x^{-1}z)}. \tag{3.111}$$

Using $\delta(z_1/z_2)\sim 0$ and $\delta(x^2z_1/z_2)\delta(x^2z_3/z_2)\sim 0$, $\varDelta(x^3)=\varDelta(x^{-3})$, and the defining relation of the deformed W-algebra (2.16), we get this proposition. Q.E.D.

As the same manner as above, we have the following proposition.

Proposition 10. *The following relations hold in the weak sense for* $i, j \geqq 2$

$$\begin{aligned} & g_{i,j}\left(\frac{x^{j-i-2[\frac{j-i}{2}]}w_1}{z_1}\right) T_i(x^{-1+i-2[\frac{i}{2}]}z_1)T_j(x^{-1+j-2[\frac{j}{2}]}w_1) \\ & \times \sum_{\substack{\sigma\in S_j \\ \sigma(1)=1}} \prod_{\substack{t=1 \\ t\neq[\frac{i}{2}]+1}}^{i} \delta\left(\frac{x^2 z_{\sigma(t+1)}}{z_{\sigma(t)}}\right) \sum_{\substack{\sigma\in S_j \\ \sigma(1)=1}} \prod_{\substack{t=1 \\ t\neq[\frac{j}{2}]+1}}^{j} \delta\left(\frac{x^2 w_{\sigma(t+1)}}{w_{\sigma(t)}}\right) \\ & \sim g_{j,i}\left(\frac{x^{i-j-2[\frac{i-j}{2}]}z_1}{w_1}\right) T_j(x^{-1+j-2[\frac{j}{2}]}w_1)T_i(x^{-1+i-2[\frac{i}{2}]}z_1) \\ & \times \sum_{\substack{\sigma\in S_j \\ \sigma(1)=1}} \prod_{\substack{t=1 \\ t\neq[\frac{i}{2}]+1}}^{i} \delta\left(\frac{x^2 z_{\sigma(t+1)}}{z_{\sigma(t)}}\right) \sum_{\substack{\sigma\in S_j \\ \sigma(1)=1}} \prod_{\substack{t=1 \\ t\neq[\frac{j}{2}]+1}}^{j} \delta\left(\frac{x^2 w_{\sigma(t+1)}}{w_{\sigma(t)}}\right). \end{aligned} \tag{3.112}$$

We should understand $T_j(z) = 0$ for $j > N$.

Let us introduce S_n-invariance in the "weak sense".

Definition 7. *We call the operator* $\mathcal{P}(z_1, z_2, \cdots, z_n)$ *is* S_n*-invariant in the "weak sense" if*

$$\mathcal{P}(z_1, z_2, \cdots, z_n) \sim \mathcal{P}(z_{\sigma(1)}, z_{\sigma(2)}, \cdots, z_{\sigma(n)}), \quad (\sigma \in S_n). \tag{3.113}$$

Example The operator $\mathcal{O}_2(z_1, z_2) = g_{11}(z_2/z_1)T_1(z_1)T_1(z_2) - c\delta(x^2z_2/z_1)T_2(x^{-1}z_1)$ is S_2-invariant.

Theorem 3. *The operator* $\mathcal{O}_n$ *defined in Theorem 2 is* S_n*-invariant in the weak sense.*

$$\mathcal{O}_n(z_1, z_2, \cdots, z_n) \sim \mathcal{O}_n(z_{\sigma(1)}, z_{\sigma(2)}, \cdots, z_{\sigma(n)}) \quad (\sigma \in S_n). \tag{3.114}$$

This theorem plays an important role in proof of the main theorem 1. We will show above theorem in the next section.

3.4. Proof of S_n*-Invariance for* $\mathcal{O}_n(z_1, \cdots, z_n)$. In this section we give proof of theorem 3. Proof for special case $\widehat{sl_2}$ is summarized in [3]. By straightforward but tedious calculations we have the following proposition.

Proposition 11. *The following relation holds in weakly sense.*

$$\begin{aligned}
&\prod_{1\leqq j<k\leqq M} g_{1,1}(z_k/z_j) \prod_{\overrightarrow{1\leqq j\leqq M}} T_1(z_j)-(z_1\leftrightarrow z_2)\\
&\sim\sum_{t=0}^{M}\sum_{3\leqq j_3<j_4<\cdots<j_{t+2}\leqq M}(-1)^t c^{t+1}\prod_{u=1}^{t}\Delta(x^{2u+1})^{t+1-u}\\
&\times\prod_{\substack{3\leqq j<k\leqq M\\ j,k\neq j_3,\cdots,j_{t+2}}} g_{1,1}\left(\frac{z_k}{z_j}\right)\prod_{\substack{3\leqq j\leqq M\\ j\neq j_3,\cdots,j_{t+2}}} g_{1,t+2}\left(x^{-1+t-2[\frac{t}{2}]}\frac{z_1}{z_j}\right)\prod_{\substack{\overrightarrow{3\leqq j\leqq M}\\ j\neq j_3,\cdots,j_{t+2}}} T_1(z_j)\\
&\times T_{t+2}(x^{-1+t-2[\frac{t}{2}]}z_1)\sum_{\substack{\sigma\in S_{t+2}\\ \sigma(1)=1}}\prod_{\substack{u=1\\ u\neq[\frac{t}{2}]+2\\ j_1=1,j_2=2}}^{t+2}\delta\left(\frac{x^2 z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}}\right)-(z_1\leftrightarrow z_2). \qquad (3.115)
\end{aligned}$$

We should understand $T_j(z)=0$ $(j>N)$.

Proof of Theorem 3. In order to show S_n-invariance, it is enough to show the case of the permutations $\sigma=(i,i+1)$ for $1\leqq i\leqq n-1$. Because of the cancellations, the differnce $\mathcal{O}_n(\cdots,z_i,z_{i+1},\cdots)-\mathcal{O}_n(\cdots,z_{i+1},z_i,\cdots)$ has simplification. We don't have to consider every summation $\sum_{\left\{A_j^{(s)}\right\}_{\substack{s=1,\cdots,N\\ j=1,\cdots,\alpha_s}}}$ in the definition of $\mathcal{O}_n$. We consider the following N-cases for $\sigma=(i,i+1)$

(1) $\{i,i+1\}\subset\cup_{j=1}^{\alpha_1}A_j^{(1)}$,
(2) $A_J^{(2)}=\{i,i+1\}$ for some J,
(3) $A_J^{(3)}=\{i,i+1,j_3|i+1<j_3\}$ for some J,
$\cdots\cdots\cdots$
(s) $A_J^{(s)}=\{i,i+1,j_3,\cdots,j_s|i+1<j_3<\cdots<j_s\}$ for some J,
$\cdots\cdots\cdots$
(N) $A_J^{(N)}=\{i,i+1,j_3,j_4,\cdots,j_N|i+1<j_3<j_4<\cdots<j_N\}$ for some J.

We have

$$\begin{aligned}
&\mathcal{O}_n(z_1,\cdots,z_i,z_{i+1},\cdots,z_n)-\mathcal{O}_n(z_1,\cdots,z_{i+1},z_i,\cdots,z_n)\\
&=\widetilde{\mathcal{O}}_n(z_1,\cdots,z_i,z_{i+1},\cdots,z_n)-\widetilde{\mathcal{O}}_n(z_1,\cdots,z_{i+1},z_i,\cdots,z_n), \qquad (3.116)
\end{aligned}$$

Here we have set

$$
\begin{aligned}
&\widetilde{\mathcal{O}}_n(z_1,\cdots,z_i,z_{i+1},\cdots,z_n)=\sum_{\substack{\alpha_1,\alpha_2,\cdots,\alpha_N\geqq 0\\ \alpha_1+2\alpha_2+\cdots+N\alpha_N=n}}\prod_{t=1}^{N}\left((-c)^{t-1}\prod_{u=1}^{t-2}\Delta(x^{2u+1})^{t-u-1}\right)^{\alpha_t}\\
&\times\left(\sum_{\substack{\{A_j^{(s)}\}_{\substack{s=1,\cdots,N\\ j=1,\cdots,\alpha_s}}\\ \{i,i+1\}\subset\cup_{j=1}^{\alpha_s}A_j^{(1)}}}+\sum_{t=2}^{N}\sum_{\substack{\{A_j^{(s)}\}_{\substack{s=1,\cdots,N\\ j=1,\cdots,\alpha_s}}\\ A_J^{(t)}=\{i,i+1,j_3,\cdots,j_t|i+1<j_3<\cdots<j_t\}\ \text{for some}\ J}}\right)\prod_{\substack{\longrightarrow\\ 1\leqq s\leqq N}}\prod_{\substack{\longrightarrow\\ j\in A_{Min}^{(s)}}}T_s(x^{-1+s+[\frac{s}{2}]}z_j)\\
&\times\prod_{t=1}^{N}\prod_{\substack{j=1\\ j_1=A_{j,1}^{(t)}\\ \cdots\\ j_t=A_{j,t}^{(t)}}}^{\alpha_t}\sum_{\substack{\sigma\in S_t\\ \sigma(1)=1}}\prod_{\substack{u=1\\ u\neq[\frac{t}{2}]+1}}^{t}\delta\left(\frac{x^2z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}}\right)\\
&\times\prod_{t=1}^{N}\prod_{\substack{1\leqq j<k\leqq n\\ j,k\in A_{Min}^{(t)}}}g_{t,t}\left(\frac{z_k}{z_j}\right)\prod_{1\leqq t<u\leqq N}\prod_{\substack{j\in A_{Min}^{(t)}\\ k\in A_{Min}^{(u)}}}g_{t,u}\left(x^{u-t-2[\frac{u-t}{2}]}\frac{z_k}{z_j}\right).
\end{aligned}
\tag{3.117}
$$

Let us consider the formulae relating to the first term in $\widetilde{\mathcal{O}}_n(\cdots,z_i,z_{i+1},\cdots)-\widetilde{\mathcal{O}}_n(\cdots,z_{i+1},z_i,\cdots)$. Let us start from

$$
\begin{aligned}
&\sum_{\substack{\alpha_1\geqq 2\ \text{and}\ \alpha_2,\cdots,\alpha_N\geqq 0\\ \alpha_1+2\alpha_2+\cdots+N\alpha_N=n}}\prod_{t=1}^{N}\left((-c)^{t-1}\prod_{u=1}^{t-2}\Delta(x^{2u+1})^{t-u-1}\right)^{\alpha_t}\sum_{\substack{\{A_j^{(s)}\}_{\substack{s=1,\cdots,N\\ j=1,\cdots,\alpha_s}}\\ \{i,i+1\}\subset\cup_{j=1}^{\alpha_s}A_j^{(1)}}}\\
&\times\prod_{\substack{\longrightarrow\\ j\in A_{Min}^{(1)}\\ j<i}}T_1(z_j)\cdot T_1(z_i)T_1(z_{i+1})\cdot\prod_{\substack{\longrightarrow\\ j\in A_{Min}^{(1)}\\ i+1<j}}T_1(z_j)\prod_{\substack{\longrightarrow\\ 2\leqq s\leqq N}}\prod_{\substack{\longrightarrow\\ j\in A_{Min}^{(s)}}}T_s(x^{-1+s+[\frac{s}{2}]}z_j)\\
&\times\prod_{t=2}^{N}\prod_{\substack{j=1\\ j_1=A_{j,1}^{(t)}\\ \cdots\\ j_t=A_{j,t}^{(t)}}}^{\alpha_t}\sum_{\substack{\sigma\in S_t\\ \sigma(1)=1}}\prod_{\substack{u=1\\ u\neq[\frac{t}{2}]+1}}^{t}\delta\left(\frac{x^2z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}}\right)\\
&\times\prod_{t=1}^{N}\prod_{\substack{1\leqq j<k\leqq n\\ j,k\in A_{Min}^{(t)}}}g_{t,t}\left(\frac{z_k}{z_j}\right)\prod_{1\leqq t<u\leqq N}\prod_{\substack{j\in A_{Min}^{(t)}\\ k\in A_{Min}^{(u)}}}g_{t,u}\left(x^{u-t-2[\frac{u-t}{2}]}\frac{z_k}{z_j}\right)-(z_i\leftrightarrow z_{i+1}).
\end{aligned}
\tag{3.118}
$$

By using the weakly sense relations in Proposition 9 we change the ordering of $T_1(z_i)T_1(z_{i+1})$ and $\prod_{\substack{\longrightarrow\\ j\in A_{Min}^{(1)}\\ i+1<j}}T_1(z_j)$. We have

$$\begin{aligned}
&-\sum_{\substack{\alpha_1+2\alpha_2+\cdots+N\alpha_N=n \\ \alpha_1\geqq 2 \text{ and } \alpha_2,\cdots,\alpha_N\geqq 0}} \sum_{t=2}^{N} \sum_{\substack{\{A_j^{(s)}\} \\ \{i,i+1\}\subset A_{Min}^{(1)}}} \sum_{\substack{i+1<j_3<\cdots<j_t \\ j_3,\cdots,j_t\in A_{Min}^{(1)}}} \\
&\times \prod_{\substack{s=1 \\ s\neq t}}^{N} \left((-c)^{s-1}\prod_{u=1}^{s-2}\varDelta(x^{2u+1})^{s-u-1}\right)^{\alpha_s} \left((-c)^{t-1}\prod_{u=1}^{t-2}\varDelta(x^{2u+1})^{t-u-1}\right)^{\alpha_t+1} \\
&\times \prod_{\substack{\longrightarrow \\ j\in A_{Min}^{(1)}-\{i,i+1\}}} T_1(z_j)T_t(x^{-1+t-2[\frac{t}{2}]}z_i) \prod_{\substack{\longrightarrow \\ 2\leqq s\leqq N}} \prod_{\substack{\longrightarrow \\ j\in A_{Min}^{(s)}}} T_s(x^{-1+s-2[\frac{s}{2}]}z_j) \\
&\times \sum_{\substack{\sigma\in S_t \\ \sigma(1)=1}} \prod_{\substack{u=1 \\ u\neq[\frac{t}{2}]+1 \\ j_1=1,j_2=2}}^{t} \delta\left(\frac{x^2 z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}}\right) \prod_{s=2}^{N} \prod_{\substack{j=1 \\ j_1=A_{j,1}^{(s)},\cdots,j_s=A_{j,s}^{(s)}}}^{\alpha_s} \sum_{\substack{\sigma\in S_s \\ \sigma(1)=1}} \prod_{\substack{u=1 \\ u\neq[\frac{s}{2}]+1}}^{s} \delta\left(\frac{x^2 z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}}\right) \\
&\times \prod_{\substack{j<k \\ j,k\in A_{Min}^{(1)}-\{i,i+1\}}} g_{1,1}(z_k/z_j) \prod_{s=2}^{N} \prod_{\substack{j<k \\ j,k\in A_{Min}^{(s)}}} g_{s,s}(z_k/z_j) \prod_{2\leqq t<u\leqq N} \prod_{\substack{j\in A_{Min}^{(t)} \\ k\in A_{Min}^{(u)}}} g_{t,u}\left(x^{u-t-2[\frac{u-t}{2}]}z_k/z_j\right) \\
&\times \prod_{s=2}^{N} \prod_{\substack{j\in A_{Min}^{(1)}-\{i,i+1\} \\ k\in A_{Min}^{(s)}}} g_{1,s}\left(x^{-1+s-2[\frac{s}{2}]}\frac{z_k}{z_j}\right) \prod_{j\in A_{Min}^{(1)}-\{i,i+1\}} g_{1,t}\left(x^{-1+t-2[\frac{t}{2}]}\frac{z_i}{z_j}\right) \\
&\times \prod_{s=2}^{N} \prod_{j\in A_{Min}^{(s)}} g_{t,s}\left(x^{s-t-2[\frac{s-t}{2}]}\frac{z_j}{z_i}\right) - (z_i \leftrightarrow z_{i+1}). \qquad (3.119)
\end{aligned}$$

We change the summation variables $\{A_j^{(s)}\}$ in

$$\sum_{0\leqq t\leqq N-2} \sum_{\substack{\alpha_1+2\alpha_2+\cdots+N\alpha_N=n \\ \alpha_1\geqq 2 \text{ and } \alpha_2,\cdots,\alpha_N\geqq 0}} \sum_{\substack{\{A_j^{(s)}\}_{\substack{s=1,\cdots,N \\ j=1,\cdots,\alpha_s}} \\ \{i,i+1\}\subset A_{Min}^{(1)}}} \sum_{\substack{i+1<j_3<\cdots<j_{t+2} \\ j_3,\cdots,j_{t+2}\in A_{Min}^{(1)}}} \qquad (3.120)$$

to the following $\{B_j^{(s)}\}$,

$$\sum_{0\leqq t\leqq N-2} \sum_{\substack{\beta_1+2\beta_2+\cdots+N\beta_N=n \\ \beta_1,\beta_2,\cdots,\beta_N\geqq 0}} \sum_{\substack{\{B_j^{(s)}\}_{\substack{s=1,\cdots,N \\ j=1,\cdots,\beta_s}} \\ B_J^{(t)}=\{i,i+1,j_3,\cdots,j_{t+2}|i+1<j_3<\cdots<j_{t+2}\} \text{ for some } J}} \qquad .(3.121)$$

Similtaneously let us use the weakly sense relations in Proposition 10 on the commutation relations between $T_i(z)$ and $T_j(w)$ for $i,j\geqq 2$ and make the ordering

$$\prod_{\substack{\longrightarrow \\ 1\leqq s\leqq N}} \prod_{\substack{\longrightarrow \\ j\in B_{Min}^{(s)}}} T_j(x^{-1+s-2[\frac{s}{2}]}z_j),$$

where $B^{(s)}_{Min} = \{Min(B^{(s)}_1), \cdots, Min(B^{(s)}_{\alpha_s})\}$. We have exactly the same summation of the second to N-th terms of $\widetilde{\mathcal{O}}_n(\cdots, z_i, z_{i+1}, \cdots) - \widetilde{\mathcal{O}}_n(\cdots, z_{i+1}, z_i, \cdots)$ up to signature. Now we have shown theorem for $\widehat{sl_N}$ case. Q.E.D.

3.5. Derivation of Laurent-Series Formulae. In this section we give proof of Theorem 2.

Proof of Theorem 2 At first we give proof for $\widehat{sl_3}$ case. We start from the integral representation $\mathcal{I}_n$ in (3.85). Let us pay attention to the poles $z_{J_1} = x^{-2}z_1, (2 \leqq J_1 \leqq n)$. We have

$$\begin{aligned} \mathcal{I}_n = & \int\cdots\int_{C(1)} \prod_{j=1}^{n} \frac{dz_j}{2\pi\sqrt{-1}z_j} \prod_{1\leqq j<k\leqq n} h(u_k-u_j) \prod_{\overrightarrow{1\leqq j\leqq n}} T_1(z_j) \\ & - \sum_{J_1=2}^{n} \int\cdots\int_{\widehat{C}(J_1)} \prod_{\substack{j=1\\ j\neq J_1}}^{n} \frac{dz_j}{2\pi\sqrt{-1}z_j} \int_{C_{x^{-2}z_1}} \frac{dz_{J_1}}{2\pi\sqrt{-1}z_{J_1}} \prod_{1\leqq j<k\leqq n} h(u_k-u_j) \prod_{\overrightarrow{1\leqq j\leqq n}} T_1(z_j). \end{aligned} \tag{3.122}$$

Here we have set

$$\begin{aligned} C(1) : & |x^{-2}z_k| < |z_1| < |x^{2-2s}z_k|, \ (2 \leqq k \leqq n) \\ & |x^{-2}z_k| < |z_j| < |x^{2}z_k|, \ (2 \leqq j < k \leqq n), \end{aligned} \tag{3.123}$$

$$\begin{aligned} \widehat{C}(J_1) : & |x^{-2}z_k| < |z_1| < |x^{2-2s}z_k|, \ (2 \leqq k \leqq J_1-1) \\ & |x^{-2}z_k| < |z_1| < |x^{2}z_k|, \ (J_1+1 \leqq k \leqq n), \\ & |x^{-2}z_k| < |z_j| < |x^{2}z_k|, \ (2 \leqq j < k \leqq n; j,k \neq J_1). \end{aligned} \tag{3.124}$$

Here $C_{x^{-2}z_1}$ is a small circle which encircle $x^{-2}z_1$ anticlockwise. The region $\{(z_1, z_k) \in \mathbb{C}^2 | |x^{-2}z_k| < |z_1| < |x^{2-2s}z_k|\}$ for $2 \leqq k \leqq J_1$, are annulus. Hence the defining relations of the deformed W-algebra can be used. Let us change the ordering of $T_1(z_1)$ and $T_1(z_k)$ for $2 \leqq k \leqq J_1 - 1$, and take the residue of $T_1(z_1)T_1(z_{J_1})$ at $z_{J_1} = x^{-2}z_1$. We have

$$\begin{aligned} & \int\cdots\int_{\widehat{C}(J_1)} \prod_{\substack{j=1\\ j\neq J_1}}^{n} \frac{dz_j}{2\pi\sqrt{-1}z_j} \int_{C_{x^{-2}z_1}} \frac{dz_{J_1}}{2\pi\sqrt{-1}z_{J_1}} \prod_{1\leqq j<k\leqq n} h(u_k-u_j) \prod_{\overrightarrow{1\leqq j\leqq n}} T_1(z_j) \\ = & \ c \int\cdots\int_{C(J_1)} \prod_{\substack{j=1\\ j\neq J_1}}^{n} \frac{dz_j}{2\pi\sqrt{-1}z_j} \prod_{\overrightarrow{2\leqq j\leqq J_1-1}} T_1(z_j) \cdot T_2(x^{-1}z_1) \cdot \prod_{\overrightarrow{J_1+1\leqq j\leqq n}} T_1(z_j) \\ \times & \prod_{\substack{2\leqq j<k\leqq n\\ j,k\neq J_1}} h_{11}(u_k-u_j) \prod_{j=2}^{J_1-1} h_{12}\left(u_1-u_j-\frac{1}{2}\right) \prod_{j=J_1+1}^{n} h_{21}\left(u_j-u_1+\frac{1}{2}\right) \end{aligned} \tag{3.125}$$

Here we have set

$$\begin{aligned} C(J_1) : & |x^{-2}z_j| < |z_1| < |x^{4}z_j|, \ (2 \leqq j \leqq n; j \neq J_1), \\ & |x^{-2}z_k| < |z_j| < |x^{2}z_k|, \ (2 \leqq j < k \leqq n; j,k \neq J_1). \end{aligned} \tag{3.126}$$

Let us pay attention to the poles at $z_{J_2}=x^2z_1$, $(2\leqq J_2\leqq n; J_2\neq J_1)$. We deform the RHS of (3.125) to the following.

$$\begin{aligned}
&c\int\cdots\int_{C(J_1)(J_1)}\prod_{\substack{j=1\\ j\neq J_1}}^{n}\frac{dz_j}{2\pi\sqrt{-1}z_j}\prod_{\overrightarrow{2\leqq j\leqq J_1-1}}T_1(z_j)\cdot T_2(x^{-1}z_1)\cdot\prod_{\overrightarrow{J_1+1\leqq j\leqq n}}T_1(z_j)\\
&\times\prod_{\substack{2\leqq j<k\leqq n\\ j,k\neq J_1}}h_{11}(u_k-u_j)\prod_{j=2}^{J_1-1}h_{12}\left(u_1-u_j-\frac{1}{2}\right)\prod_{j=J_1+1}^{n}h_{21}\left(u_j-u_1+\frac{1}{2}\right)\\
&-c\sum_{\substack{J_2=2\\ J_2\neq J_1}}^{n}\int\cdots\int_{\widehat{C}(J_1)(J_2)}\prod_{\substack{j=1\\ j\neq J_1,J_2}}^{n}\frac{dz_j}{2\pi\sqrt{-1}z_j}\int_{C_{x^2z_1}}\frac{dz_{J_2}}{2\pi\sqrt{-1}z_{J_2}}\\
&\times\prod_{\overrightarrow{2\leqq j\leqq J_1-1}}T_1(z_j)\cdot T_2(x^{-1}z_1)\cdot\prod_{\overrightarrow{J_1+1\leqq j\leqq n}}T_1(z_j)\\
&\times\prod_{\substack{2\leqq j<k\leqq n\\ j,k\neq J_1}}h_{11}(u_k-u_j)\prod_{j=2}^{J_1-1}h_{12}\left(u_1-u_j-\frac{1}{2}\right)\prod_{j=J_1+1}^{n}h_{21}\left(u_j-u_1+\frac{1}{2}\right).
\end{aligned}\tag{3.127}$$

Here we have set

$$\begin{aligned}
C(J_1)(J_1):\ &|x^{-2+2s}z_j|<|z_1|<|x^4z_j|,\ (2\leqq j\leqq n; j\neq J_1),\\
&|x^{-2}z_k|<|z_j|<|x^2z_k|,\ (2\leqq j<k\leqq n; j,k\neq J_1).
\end{aligned}\tag{3.128}$$

For $2\leqq J_2<J_1\leqq n$ we set

$$\begin{aligned}
\widehat{C}(J_1)(J_2):\ &|x^{-2+2s}z_j|<|z_1|<|x^4z_j|,\ (J_2\leqq j\leqq J_1-1),\\
&|x^{-2}z_j|<|z_1|<|x^4z_j|,\ (2\leqq j\leqq J_2-1 \text{ or } J_1+1\leqq j\leqq n),\\
&|x^{-2}z_k|<|z_j|<|x^2z_k|,\ (2\leqq j<k\leqq n; j,k\neq J_1).
\end{aligned}\tag{3.129}$$

For $2\leqq J_1<J_2\leqq n$ we set

$$\begin{aligned}
\widehat{C}(J_1)(J_2):\ &|x^{-2+2s}z_j|<|z_1|<|x^4z_j|,\ (2\leqq j\leqq J_2; j\neq J_1),\\
&|x^{-2}z_j|<|z_1|<|x^4z_j|,\ (J_2+1\leqq j\leqq n),\\
&|x^{-2}z_k|<|z_j|<|x^2z_k|,\ (2\leqq j<k\leqq n; j,k\neq J_1).
\end{aligned}\tag{3.130}$$

The above formulae for this integrand $C(J_1)(J_2)$ holds for $\mathrm{Re}(s)>N\geqq 3$. For $N=2$ another treatment should be done. Let us study the first term

$$c\int\cdots\int_{C(J_1)(J_1)}\prod_{j\neq J_1}\frac{dz_j}{2\pi\sqrt{-1}}.$$

See the integral contour $C(J_1)(J_1)$. The region $\{(z_1,z_j)\in\mathbb{C}^2||x^{-2+2s}z_j|<|z_1|<|x^4z_j|\}$ for $j\neq J_1$ are annulus. Hence the defining relations of the deformed W-algebra can be used. By using the weakly sense relation in Proposition 9, we deform the first term to the following.

$$c\int\cdots\int_{C(J_1)(J_1)}\prod_{\substack{j=1\\ j\neq J_1}}^{n}\frac{dz_j}{2\pi\sqrt{-1}z_j}\prod_{\substack{\overrightarrow{2\leqq j\leqq n}\\ j\neq J_1}}T_1(z_j)\cdot T_2(x^{-1}z_1)$$
$$\times\prod_{\substack{2\leqq j<k\leqq n\\ j,k\neq J_1}}h_{11}(u_k-u_j)\prod_{\substack{j=2\\ j\neq J_1}}^{n}h_{12}\left(u_1-u_j-\frac{1}{2}\right). \tag{3.131}$$

Let us study the second term

$$-c\sum_{J_2\neq 2,J_1}\int\cdots\int_{\widehat{C}(J_1)(J_2)}\prod_{j\neq J_1,J_2}\frac{dz_j}{2\pi\sqrt{-1}z_j}\int_{C_{x^2z_1}}\frac{dz_{J_2}}{2\pi\sqrt{-1}z_{J_2}}.$$

See the integral contour $\widehat{C}(J_1)(J_2)$. The region $\{(z_1,z_j)\in\mathbb{C}^2||x^{-2+2s}z_j|<|z_1|<|x^4z_j|\}$ for $2\leqq j\leqq J_1$, are annulus for $\mathrm{Re}(s)>N=3$. Hence the defining relations of the deformed W-algebra can be used. Let us change the ordering of $T_1(z_{J_2})$ and $T_1(z_k)$ and make the product of the operators $T_1(z_{J_2})T_2(x^{-1}z_1)$ or $T_2(x^{-1}z_1)T_1(z_{J_2})$. Let us take the residue of $T_1(z_{J_2})T_2(x^{-1}z_1)$ and $T_2(x^{-1}z_1)T_1(z_{J_2})$ at $z_{J_2}=x^2z_1$ by regarding the weakly sense equation in Proposition 9. We have

$$c^2\varDelta(x^3)\sum_{\substack{J_2=2\\ J_2\neq J_1}}^{n}\int\cdots\int_{C(J_1)(J_2)}\prod_{\substack{j=1\\ j\neq J_1,J_2}}^{n}\frac{dz_j}{2\pi\sqrt{-1}z_j}\prod_{\substack{\overrightarrow{2\leqq j\leqq J_1-1}\\ j\neq J_2}}T_1(z_j)\cdot T_3(z_1)\prod_{\substack{\overrightarrow{J_1+1\leqq j\leqq n}\\ j\neq J_2}}T_1(z_j)$$
$$\times\prod_{\substack{2\leqq j<k\leqq n\\ j,k\neq J_1,J_2}}h_{11}(u_k-u_j)\prod_{\substack{j=2\\ j\neq J_2}}^{J_1-1}h_{13}\left(u_1-u_j\right)\prod_{\substack{j=J_1+1\\ j\neq J_2}}^{n}h_{31}(u_j-u_1). \tag{3.132}$$

Here we have set

$$\begin{aligned}C(J_1)(J_2):&|x^{-4+2s}z_j|<|z_1|<|x^{4-2s}z_j|,\ (2\leqq j\leqq n;j\neq J_1,J_2),\\ &|x^{-2}z_k|<|z_j|<|x^2z_k|,\ (2\leqq j<k\leqq n;j\neq J_1,J_2).\end{aligned} \tag{3.133}$$

This integral contur $C(J_1)(J_2)$ holds only for $N=3$ case. For $N\geq 4$ case another treatment should be done. The region $\{(z_1,z_j)\in\mathbb{C}^2||x^{-4+2s}z_j|<|z_1|<|x^4z_j|\}$ are annulus. We move $T_3(z_j)$ to the right, and get

$$c^2\varDelta(x^3)\sum_{\substack{J_2=2\\ J_2\neq J_1}}^{n}\int\cdots\int_{C(J_1)(J_2)}\prod_{\substack{j=1\\ j\neq J_1,J_2}}^{n}\frac{dz_j}{2\pi\sqrt{-1}z_j}\prod_{\substack{\overrightarrow{2\leqq j\leqq n}\\ j\neq J_1,J_2}}T_1(z_j)\cdot T_3(z_1)$$
$$\times\prod_{\substack{2\leqq j<k\leqq n\\ j,k\neq J_1,J_2}}h_{11}(u_k-u_j)\prod_{\substack{j=2\\ j\neq J_1,J_2}}^{n}h_{13}\left(u_1-u_j\right). \tag{3.134}$$

Summing up every terms, we have

$$\begin{aligned}\mathcal{I}_n = &\left(\sum_{\substack{A^{(1)}=\{1\} \\ A^{(2)}=A^{(3)}=\phi}} -c \sum_{\substack{A^{(2)}=\{1,j\} \\ A^{(1)}=A^{(3)}=\phi}} +2!c^2 \Delta(x^3) \sum_{\substack{A^{(3)}=\{1,j,k\} \\ A^{(1)}=A^{(2)}=\phi}} \right) \\ &\times \int \cdots \int_{C\{A^{(1)},A^{(2)},A^{(3)},A_c\}} \prod_{j \in A_c \cup A^{(1)}_{Min} \cup A^{(2)}_{Min} \cup A^{(3)}_{Min}} \frac{dz_j}{2\pi\sqrt{-1}z_j} \\ &\times \prod_{\substack{\longrightarrow \\ j \in A_c \cup A^{(1)}_{Min}}} T_1(z_j) \prod_{\substack{\longrightarrow \\ j \in A^{(2)}_{Min}}} T_2(x^{-1}z_j) \prod_{\substack{\longrightarrow \\ j \in A^{(3)}_{MIn}}} T_3(z_j) \prod_{\substack{j<k \\ j,k \in A_c \cup A^{(1)}_{Min}}} h_{11}(u_k - u_j) \\ &\times \prod_{k \in A^{(2)}_{Min}} \prod_{j \in A^{(1)}_{Min} \cup A_c} h_{12}\left(u_k - u_j - \frac{1}{2}\right) \prod_{k \in A^{(3)}_{Min}} \prod_{j \in A^{(1)}_{Min} \cup A_c} h_{13}\left(u_k - u_j\right). \end{aligned} \tag{3.135}$$

Here we have set $A_c = \{1, 2, \cdots, n\} - A^{(1)} \cup A^{(2)} \cup A^{(3)}$. We have set $A^{(t)}_{Min} = \{j_1\}$ for $A^{(t)} = \{j_1 < j_2 < \cdots < j_t\}$. Here we have set $C\{A^{(1)}, A^{(2)}, A^{(3)}, A_c\}$ by

$$\begin{aligned} &|x^{-2}z_k| < |z_1| < |x^{2-2s}z_k|, \ (k \in A_c \text{ for } A^{(1)} \neq \phi), \\ &|x^{-2+2s}z_k| < |z_1| < |x^4 z_k|, \ (k \in A_c \text{ for } A^{(2)} \neq \phi), \\ &|x^{-4+2s}z_k| < |z_1| < |x^{4-2s}z_k|, \ (k \in A_c \text{ for } A^{(3)} \neq \phi), \\ &|x^{-2}z_k| < |z_j| < |x^2 z_k|, \ (j < k; j, k \in A_c). \end{aligned} \tag{3.136}$$

Next we deform the part $\prod_{\substack{\longrightarrow \\ j \in A_c}} T_1(z_j)$. Let us take the residue at $z_J = x^{-2}z_2$, and continue similar calculations as above. We use the weakly sense equations in Proposition 10 and change the ordering of $T_2(x^{-1}w)$ and $T_3(w)$, without taking residues. Now we have shown theorem for $\widehat{sl_3}$. Proof for $\widehat{sl_N}$ is similar. Q.E.D.

3.6. Proof of $[\mathcal{I}_m, \mathcal{I}_n] = 0$. In this section we show the commutation relation $[\mathcal{I}_m, \mathcal{I}_n] = 0$.

Proposition 12. *The folloing theta identity holds.*

$$\sum_{\sigma \in S_{m+n}} \prod_{j=1}^{n} \prod_{k=n+1}^{n+m} \frac{1}{h(u_{\sigma(k)} - u_{\sigma(j)})} = \sum_{\sigma \in S_{m+n}} \prod_{j=1}^{m} \prod_{k=m+1}^{n+m} \frac{1}{h(u_{\sigma(k)} - u_{\sigma(j)})}. \tag{3.137}$$

$$\sum_{\sigma \in S_{m+n}} \prod_{j=1}^{n} \prod_{k=n+1}^{n+m} \frac{1}{h^*(u_{\sigma(k)} - u_{\sigma(j)})} = \sum_{\sigma \in S_{m+n}} \prod_{j=1}^{m} \prod_{k=m+1}^{n+m} \frac{1}{h^*(u_{\sigma(k)} - u_{\sigma(j)})}. \tag{3.138}$$

Here $h(u)$ *and* $h^*(u)$ *are given in (3.84).*

This theta identity was written in [10] without proof. We have shown this theta identity by induction, in [3]. Hence we omit details here.

Proposition 13. *The following weakly sense equation holds.*

$$\mathcal{O}_n(z_1,\cdots,z_n)\mathcal{O}_m(z_{n+1},\cdots,z_{n+m}) \sim \prod_{\substack{1\leqq j\leqq n\\ n+1\leqq k\leqq n+m}} \frac{1}{g_{11}(z_k/z_j)}\mathcal{O}_{m+n}(z_1,\cdots,z_{n+m}). \tag{3.139}$$

Proof. This is direct consequence of the following explicit folrmulae

$$\begin{aligned}\mathcal{O}_{n+m}(z_1,\cdots,z_{n+m}) &\sim \prod_{\substack{1\leqq j\leqq n\\ n+1\leqq k\leqq n+m}} g_{11}(z_k/z_j)\mathcal{O}_n(z_1,\cdots,z_n)\mathcal{O}_m(z_{n+1},\cdots,z_{n+m})\\
&+ \sum_{\substack{\alpha,\alpha_2,\cdots,\alpha_N\geqq 0\\ \alpha_1+2\alpha_2+\cdots+N\alpha_N=n\\ 2\leqq 2\alpha_2+\cdots+N\alpha_N\leqq n}} \prod_{t=1}^{N}\left((-c)^{t-1}\prod_{u=1}^{t-1}\mathit{\Delta}(x^{2u+1})^{t-u-1}\right)^{\alpha_t}\\
&\times \sum_{\substack{\{L_j^{(s)}\}_{\substack{s=1,\cdots,N\\ j=1\cdots,\alpha_s}},\{R_j^{(s)}\}_{\substack{s=1,\cdots,N\\ j=1\cdots,\alpha_s}}\\ \sum_{s=2}^N |L_1^{(s)}||R_1^{(s)}|\geqq 1}} \prod_{t=1}^{N}\prod_{\substack{j=1\\ j_1=A_{j,1}^{(t)}\\ \cdots\\ j_t=A_{j,t}^{(t)}}}^{\alpha_t} \sum_{\substack{\sigma\in S_t\\ \sigma(1)=1}} \prod_{\substack{u=1\\ u\neq[\frac{t}{2}]+1}}^{t} \delta\left(\frac{x^2 z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}}\right)\\
&\times \prod_{t=1}^{N}\prod_{\substack{j<k\\ j,k\in A_{Min}^{(t)}}} g_{t,t}\left(\frac{z_k}{z_j}\right) \prod_{1\leqq t<u\leqq N}\prod_{\substack{j\in A_{Min}^{(t)}\\ k\in A_{Min}^{(u)}}} g_{t,u}\left(x^{u-t-2[\frac{u}{2}]+2[\frac{t}{2}]}\frac{z_k}{z_j}\right).\end{aligned} \tag{3.140}$$

Here the summation $\sum_{\substack{\{L_j^{(s)}\}_{\substack{s=1,\cdots,N\\ j=1\cdots,\alpha_s}},\{R_j^{(s)}\}_{\substack{s=1,\cdots,N\\ j=1\cdots,\alpha_s}}\\ \sum_{s=2}^N |L_1^{(s)}||R_1^{(s)}|\geqq 1}}$ is taken over the conditions that

$$\begin{aligned}&\cup_{s=1}^{N}\cup_{j=1}^{\alpha_s} L_j^{(s)}=\{1,2,\cdots,m\}, L_i^{(s)}\cap L_j^{(s)}=\phi,(i\neq j),\\
&Min(L_1^{(s)})<Min(L_2^{(s)})<\cdots<Min(L_{\alpha_s}^{(s)}),\\
&\cup_{s=1}^{N}\cup_{j=1}^{\alpha_s} R_j^{(s)}=\{m+1,m+2,\cdots,m+n\}, R_i^{(s)}\cap R_j^{(s)}=\phi,(i\neq j),\\
&Min(R_1^{(s)})<Min(R_2^{(s)})<\cdots<Min(R_{\alpha_s}^{(s)}).\end{aligned} \tag{3.141}$$

Here we have set $A_j^{(s)}=L_j^{(s)}\cup R_j^{(s)}$. We have set $A_{j,k}^{(s)}=j_k$ for $A_j^{(s)}=\{j_1<j_2<\cdots<j_s\}$, and $A_{Min}^{(s)}=\{A_{1,1}^{(s)},A_{2,1}^{(s)},\cdot,A_{\alpha_s,1}^{(s)}\}$. We want to point out that every term of the summation $\sum_{\{L_j^{(s)}\}_{\substack{s=1,\cdots,N\\ j=1\cdots,\alpha_s}},\{R_j^{(s)}\}_{\substack{s=1,\cdots,N\\ j=1\cdots,\alpha_s}}}$ has the delta-function $\delta(x^2z_k/z_j)$, $(1\leqq j\leqq m, m+1\leqq k\leqq m+n)$. Dividing $\prod_{\substack{1\leqq j\leqq n\\ n+1\leqq k\leqq n+m}} g_{11}(z_k/z_j)$ to both sides and using $1/g_{11}(x^{-2})=0$, we have shown this proposition . Q.E.D.

Proof of Theorem 1. At first we restrict ourself to the regime, $\mathrm{Re}(s) > N$ and $\mathrm{Re}(r) < 0$, in order to use the power series formulae of the local integrals of motion, $\mathcal{I}_n$. In Proposition 3 we have shown for $\sigma \in S_n$

$$\prod_{1\leqq j<k\leqq n} s(z_k/z_j)\mathcal{O}_n(z_1,\cdots,z_n)=\prod_{1\leqq j<k\leqq n} s(z_{\sigma(k)}/z_{\sigma(j)})\mathcal{O}_n(z_{\sigma(1)},\cdots,z_{\sigma(n)}). \tag{3.142}$$

Hence we have

$$\begin{aligned}
&\mathcal{I}_n\cdot\mathcal{I}_m\\
&=\left[\prod_{1\leqq j<k\leqq n} s(z_k/z_j)\mathcal{O}_n(z_1,\cdots,z_n)\prod_{n+1\leqq j<k\leqq n+m} s(z_k/z_j)\mathcal{O}_m(z_{n+1},\cdots,z_{n+m})\right]_{1,z_1\cdots z_{n+m}}\\
&=\Bigg[\frac{1}{(n+m)!}\sum_{\sigma\in S_{n+m}}\prod_{j=1}^{n}\prod_{k=n+1}^{n+m}\frac{1}{h(u_{\sigma(k)}-u_{\sigma(j)})}\prod_{1\leqq j<k\leqq n+m} s(z_k/z_j)\\
&\times\ \mathcal{O}_{n+m}(z_1,\cdots,z_{n+m})\Bigg]_{1,z_1\cdots z_{n+m}}.
\end{aligned} \tag{3.143}$$

Hence the commutation relation $\mathcal{I}_n\cdot\mathcal{I}_m=\mathcal{I}_m\cdot\mathcal{I}_n$ is reduced to the theta identity in Proposition 12.

$$\sum_{\sigma\in S_{m+n}}\prod_{j=1}^{n}\prod_{k=n+1}^{n+m}\frac{1}{h(u_{\sigma(k)}-u_{\sigma(j)})}=\sum_{\sigma\in S_{m+n}}\prod_{j=1}^{m}\prod_{k=m+1}^{n+m}\frac{1}{h(u_{\sigma(k)}-u_{\sigma(j)})}. \tag{3.144}$$

Proof of the commutation relation $[\mathcal{I}_m^*,\mathcal{I}_n^*]=0$ is given as similar way. Here we omit details for $\mathcal{I}_n^*$. Q.E.D.

4. Nonlocal Integrals of Motion

In this section we give explicit formulae of the nonlocal integrals of motion. We study generic case : $0<x<1, \mathrm{Re}(r)\neq 0$ and $s\in\mathbb{C}$ (resp. $0<x<1, \mathrm{Re}(r^*)\neq 0$ and $s\in\mathbb{C}$).

4.1. Nonlocal Integrals of Motion. We explicitly construct the nonlocal integrals of motion and state the main results for $N\geqq 3$. The results for $N=2$ is summarized in [3].

Definition 8.
- *For the regime* $\mathrm{Re}(r)>0$ *and* $0<\mathrm{Re}(s)<N$, *we define a family of operators* $\mathcal{G}_m$, $(m=1,2,\cdots)$ *by*

$$\begin{aligned}\mathcal{G}_m=&\prod_{t=1}^{N}\prod_{j=1}^{m}\oint_C\frac{dz_j^{(t)}}{2\pi\sqrt{-1}z_j^{(t)}}\\ &\times F_1(z_1^{(1)})\cdots F_1(z_m^{(1)})F_2(z_1^{(2)})\cdots F_2(z_m^{(2)})\cdots F_N(z_1^{(N)})\cdots F_N(z_m^{(N)})\\ &\times\frac{\displaystyle\prod_{t=1}^{N}\prod_{1\leqq i<j\leqq m}\left[u_i^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_i^{(t)}-1\right]_r}{\displaystyle\prod_{t=1}^{N-1}\prod_{i,j=1}^{m}\left[u_i^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_r\prod_{i,j=1}^{m}\left[u_i^{(1)}-u_j^{(N)}+\frac{s}{N}\right]_r}\\ &\times\vartheta\left(\sum_{j=1}^{m}u_j^{(1)}\middle|\sum_{j=1}^{m}u_j^{(2)}\middle|\cdots\middle|\sum_{j=1}^{m}u_j^{(N)}\right).\end{aligned}\tag{4.145}$$

Here we have set the theta function $\vartheta(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$ *by*

$$\vartheta(u^{(1)}|\cdots|u^{(t)}+r|\cdots|u^{(N)})=\vartheta(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}),\quad(1\leqq t\leqq N),\tag{4.146}$$

$$\begin{aligned}&\vartheta(u^{(1)}|\cdots|u^{(t)}+r\tau|\cdots|u^{(N)})\\ =&e^{-2\pi i\tau+\frac{2\pi i}{r}(u_{t-1}-2u_t+u_{t+1}+\sqrt{r(r-1)}P_{\alpha_t})}\vartheta(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}),\quad(1\leqq t\leqq N),\end{aligned}\tag{4.147}$$

$$\vartheta(u^{(1)}+k|\cdots|u^{(N)}+k)=\vartheta(u^{(1)}|\cdots|u^{(N)}),\quad(k\in\mathbb{C}),\tag{4.148}$$

$$\eta(\vartheta(u^{(1)}|\cdots|u^{(N)}))=\vartheta(u^{(N)}|u^{(1)}|\cdots|u^{(N-1)}).\tag{4.149}$$

Here the integral contour C *is given by*

$$|x^{\frac{2s}{N}}z_j^{(t+1)}|<|z_i^{(t)}|<|x^{-2+\frac{2s}{N}}z_j^{(t+1)}|,\quad(1\leqq t\leqq N-1,1\leqq i,j\leqq m),\tag{4.150}$$

$$|x^{2-\frac{2s}{N}}z_j^{(1)}|<|z_i^{(N)}|<|x^{-\frac{2s}{N}}z_j^{(1)}|,\quad(1\leqq i,j\leqq m).\tag{4.151}$$

For generic $s\in\mathbb{C}$, *the definition of* $\mathcal{G}_n$ *should be understood as analytic continuation. We call the operator* $\mathcal{G}_n$ *the nonlocal integrals of motion for the deformed* W*-algebra* $W_{q,t}(\widehat{sl_N})$.

- *For the regime* $\mathrm{Re}(r)<0$ *and* $0<\mathrm{Re}(s)<N$, *we define a family of operators*

$\mathcal{G}_m$, $(m=1,2,\cdots)$ by

$$\begin{aligned}\mathcal{G}_m=&\prod_{t=1}^N\prod_{j=1}^m\oint_C\frac{dz_j^{(t)}}{2\pi\sqrt{-1}z_j^{(t)}}\\&\times F_1(z_1^{(1)})\cdots F_1(z_m^{(1)})F_2(z_1^{(2)})\cdots F_2(z_m^{(2)})\cdots F_N(z_1^{(N)})\cdots F_N(z_m^{(N)})\\&\times\frac{\displaystyle\prod_{t=1}^N\prod_{1\leqq i<j\leqq m}\left[u_i^{(t)}-u_j^{(t)}\right]_{-r}\left[u_j^{(t)}-u_i^{(t)}+1\right]_{-r}}{\displaystyle\prod_{t=1}^{N-1}\prod_{i,j=1}^m\left[u_i^{(t)}-u_j^{(t+1)}-\frac{s}{N}\right]_{-r}\prod_{i,j=1}^m\left[u_i^{(1)}-u_j^{(N)}-1+\frac{s}{N}\right]_{-r}}\\&\times\vartheta\left(\sum_{j=1}^m u_j^{(1)}\middle|\sum_{j=1}^m u_j^{(2)}\middle|\cdots\middle|\sum_{j=1}^m u_j^{(N)}\right).\end{aligned}\tag{4.152}$$

Here we have set the theta function $\vartheta(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$ by

$$\vartheta(u^{(1)}|\cdots|u^{(t)}+r|\cdots|u^{(N)})=\vartheta(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}),\quad(1\leqq t\leqq N)\tag{4.153}$$

$$\begin{aligned}&\vartheta(u^{(1)}|\cdots|u^{(t)}-r\tau|\cdots|u^{(N)})\\=\;&e^{-2\pi i\tau-\frac{2\pi i}{r}(u_{t-1}-2u_t+u_{t+1}+\sqrt{r(r-1)}P_{\alpha_t})}\vartheta(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}),\quad(1\leqq t\leqq N),\end{aligned}\tag{4.154}$$

$$\vartheta(u^{(1)}+k|\cdots|u^{(N)}+k)=\vartheta(u^{(1)}|\cdots|u^{(N)}),\quad(k\in\mathbb{C}),\tag{4.155}$$

$$\eta(\vartheta(u^{(1)}|\cdots|u^{(N)}))=\vartheta(u^{(N)}|u^{(1)}|\cdots|u^{(N-1)}).\tag{4.156}$$

Here the integral contour C is given by

$$|x^{-2+\frac{2s}{N}}z_j^{(t+1)}|<|z_i^{(t)}|<|x^{\frac{2s}{N}}z_j^{(t+1)}|,\quad(1\leqq t\leqq N-1,1\leqq i,j\leqq m),\tag{4.157}$$

$$|x^{-\frac{2s}{N}}z_j^{(1)}|<|z_i^{(N)}|<|x^{2-\frac{2s}{N}}z_j^{(1)}|,\quad(1\leqq i,j\leqq m).\tag{4.158}$$

For generic $s\in\mathbb{C}$, the definition of $\mathcal{G}_n$ should be understood as analytic continuation. We call the operator $\mathcal{G}_n$ the nonlocal integrals of motion for the deformed W-algebra $W_{q,t}(\widehat{sl_N})$.

- *For* $\mathrm{Re}(r^*)>0$ *and* $0<\mathrm{Re}(s)<N$, *we define a family of operators*

$\mathcal{G}_m^*$, $(m=1,2,\cdots)$ by

$$\begin{aligned}\mathcal{G}_m^* = &\prod_{t=1}^N \prod_{j=1}^m \oint_{C^*} \frac{dz_j^{(t)}}{2\pi\sqrt{-1}z_j^{(t)}}\\ &\times E_1(z_1^{(1)})\cdots E_1(z_m^{(1)})E_2(z_1^{(2)})\cdots E_2(z_m^{(2)})\cdots E_N(z_1^{(N)})\cdots E_N(z_m^{(N)})\\ &\times \frac{\displaystyle\prod_{t=1}^N \prod_{1\leqq i<j\leqq m} \left[u_i^{(t)}-u_j^{(t)}\right]_{r^*}\left[u_j^{(t)}-u_i^{(t)}+1\right]_{r^*}}{\displaystyle\prod_{t=1}^{N-1}\prod_{i,j=1}^m \left[u_i^{(t)}-u_j^{(t+1)}-\frac{s}{N}\right]_{r^*}\prod_{i,j=1}^m \left[u_i^{(1)}-u_j^{(N)}-1+\frac{s}{N}\right]_{r^*}}\\ &\times \vartheta^*\left(\sum_{j=1}^m u_j^{(1)}\left|\sum_{j=1}^m u_j^{(2)}\right|\cdots\left|\sum_{j=1}^m u_j^{(N)}\right.\right). \end{aligned} \tag{4.159}$$

Here we have set the theta function $\vartheta^*(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$ by

$$\vartheta^*(u^{(1)}|\cdots|u^{(t)}+r|\cdots|u^{(N)})=\vartheta^*(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}),\quad (1\leqq t\leqq N) \tag{4.160}$$

$$\begin{aligned}&\vartheta^*(u^{(1)}|\cdots|u^{(t)}+r^*\tau|\cdots|u^{(N)})\\ &= e^{-2\pi i\tau+\frac{2\pi i}{r^*}(u_{t-1}-2u_t+u_{t+1}+\sqrt{r(r-1)}P_{\alpha_t})}\vartheta^*(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}),\quad (1\leqq t\leqq N),\end{aligned} \tag{4.161}$$

$$\vartheta^*(u^{(1)}+k|\cdots|u^{(N)}+k)=\vartheta^*(u^{(1)}|\cdots|u^{(N)}),\quad (k\in\mathbb{C}), \tag{4.162}$$

$$\eta(\vartheta^*(u^{(1)}|\cdots|u^{(N)}))=\vartheta^*(u^{(N)}|u^{(1)}|\cdots|u^{(N-1)}). \tag{4.163}$$

Here the integral contour C^* is given by

$$|x^{-2+\frac{2s}{N}}z_j^{(t+1)}|<|z_i^{(t)}|<|x^{\frac{2s}{N}}z_j^{(t+1)}|,\quad (1\leqq t\leqq N-1, 1\leqq i,j\leqq m), \tag{4.164}$$

$$|x^{-\frac{2s}{N}}z_j^{(1)}|<|z_i^{(N)}|<|x^{2-\frac{2s}{N}}z_j^{(1)}|,\quad (1\leqq i,j\leqq m). \tag{4.165}$$

For generic $s\in\mathbb{C}$, the definition of $\mathcal{G}_n$ should be understood as analytic continuation. We call the operator $\mathcal{G}_n$ the nonlocal integrals of motion for the deformed W-algebra $W_{q,t}(\widehat{sl_N})$.

- *For $\mathrm{Re}(r^*)<0$ and $0<\mathrm{Re}(s)<N$, we define a family of operators*

$\mathcal{G}_m^*$, $(m=1,2,\cdots)$ by

$$\begin{aligned}\mathcal{G}_m^* = & \prod_{t=1}^{N}\prod_{j=1}^{m}\oint_{C^*}\frac{dz_j^{(t)}}{2\pi\sqrt{-1}z_j^{(t)}} \\ & \times E_1(z_1^{(1)})\cdots E_1(z_m^{(1)})E_2(z_1^{(2)})\cdots E_2(z_m^{(2)})\cdots E_N(z_1^{(N)})\cdots E_N(z_m^{(N)}) \\ & \times \frac{\displaystyle\prod_{t=1}^{N}\prod_{1\leqq i<j\leqq m}\left[u_i^{(t)}-u_j^{(t)}\right]_{-r^*}\left[u_j^{(t)}-u_i^{(t)}-1\right]_{-r^*}}{\displaystyle\prod_{t=1}^{N-1}\prod_{i,j=1}^{m}\left[u_i^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_{-r^*}\prod_{i,j=1}^{m}\left[u_i^{(1)}-u_j^{(N)}+\frac{s}{N}\right]_{-r^*}} \\ & \times \vartheta^*\left(\sum_{j=1}^{m}u_j^{(1)}\left|\sum_{j=1}^{m}u_j^{(2)}\right|\cdots\left|\sum_{j=1}^{m}u_j^{(N)}\right.\right). \end{aligned} \tag{4.166}$$

Here we have set the theta function $\vartheta^*(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$ by

$$\vartheta^*(u^{(1)}|\cdots|u^{(t)}+r|\cdots|u^{(N)})=\vartheta^*(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}),\quad (1\leqq t\leqq N) \tag{4.167}$$

$$\vartheta^*(u^{(1)}|\cdots|u^{(t)}r^*\tau|\cdots|u^{(N)}) \tag{4.168}$$
$$=e^{-2\pi i\tau-\frac{2\pi i}{r^*}(u_{t-1}-2u_t+u_{t+1}+\sqrt{r(r-1)}P_{\alpha_t})}\vartheta^*(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}),\quad (1\leqq t\leqq N),$$

$$\vartheta^*(u^{(1)}+k|\cdots|u^{(N)}+k)=\vartheta^*(u^{(1)}|\cdots|u^{(N)}),\quad (k\in\mathbb{C}), \tag{4.169}$$

$$\eta(\vartheta^*(u^{(1)}|\cdots|u^{(N)}))=\vartheta^*(u^{(N)}|u^{(1)}|\cdots|u^{(N-1)}). \tag{4.170}$$

Here the integral contour C^* is given by

$$|x^{\frac{2s}{N}}z_j^{(t+1)}|<|z_i^{(t)}|<|x^{-2+\frac{2s}{N}}z_j^{(t+1)}|,\quad (1\leqq t\leqq N-1, 1\leqq i,j\leqq m), \tag{4.171}$$

$$|x^{2-\frac{2s}{N}}z_j^{(1)}|<|z_i^{(N)}|<|x^{-\frac{2s}{N}}z_j^{(1)}|,\quad (1\leqq i,j\leqq m). \tag{4.172}$$

For generic $s\in\mathbb{C}$, the definition of $\mathcal{G}_n$ should be understood as analytic continuation. We call the operator $\mathcal{G}_n$ the nonlocal integrals of motion for the deformed W-algebra $W_{q,t}(\widehat{sl_N})$.

We summarize explicit formulae for the integrand function $\vartheta(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$.

Proposition 14. *For $\alpha_1,\alpha_2,\cdots,\alpha_N\in\mathbb{C}$ and $\mathrm{Re}(r)>0$, we set*

$$\begin{aligned}\vartheta_\alpha(u^{(1)}|u^{(2)}|\cdots|u^{(N)}) = & [u^{(1)}-u^{(2)}-\sqrt{rr^*}P_{\bar{\epsilon}_2}+\alpha_1P_{\bar{\epsilon}_1}+\alpha_2P_{\bar{\epsilon}_2}+\cdots+\alpha_NP_{\bar{\epsilon}_N}]_r \\ & \times[u^{(2)}-u^{(3)}-\sqrt{rr^*}P_{\bar{\epsilon}_3}+\alpha_1P_{\bar{\epsilon}_1}+\alpha_2P_{\bar{\epsilon}_2}+\cdots+\alpha_NP_{\bar{\epsilon}_N}]_r \\ & \times\cdots \\ & \times[u^{(N)}-u^{(1)}-\sqrt{rr^*}P_{\bar{\epsilon}_1}+\alpha_1P_{\bar{\epsilon}_1}+\alpha_2P_{\bar{\epsilon}_2}+\cdots+\alpha_NP_{\bar{\epsilon}_N}]_r.\end{aligned} \tag{4.173}$$

This theta function $\vartheta_\alpha(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$ *satisfies the conditions (4.146), (4.147), (4.148) and (4.149).*

The followings are some of **Main Results**.

Theorem 4. *The nonlocal integrals of motion* $\mathcal{G}_n$ *commute with each other.*

$$[\mathcal{G}_m,\mathcal{G}_n]=0,\quad (m,n=1,2,\cdots). \tag{4.174}$$

The nonlocal integrals of motion $\mathcal{G}_n^*$ *commute with each other.*

$$[\mathcal{G}_m^*,\mathcal{G}_n^*]=0,\quad (m,n=1,2,\cdots). \tag{4.175}$$

Theorem 5. *The nonlocal integrals of motion* $\mathcal{G}_n$ *and* $\mathcal{G}_n^*$ *commute with each other for regime* $0<\mathrm{Re}(r)$ *and* $\mathrm{Re}(r^*)<0$.

$$[\mathcal{G}_m,\mathcal{G}_n^*]=0,\quad (m,n=1,2,\cdots). \tag{4.176}$$

Theorem 6. *The local integrals of motion* $\mathcal{I}_n$, $\mathcal{I}_n^*$ *and nonlocal integrals of motion* $\mathcal{G}_m$, $\mathcal{G}_m^*$ *commute with each other.*

$$[\mathcal{I}_n,\mathcal{G}_m]=0,\quad [\mathcal{I}_n,\mathcal{G}_m^*]=0, \tag{4.177}$$

$$[\mathcal{I}_n^*,\mathcal{G}_m]=0,\quad [\mathcal{I}_n^*,\mathcal{G}_m^*]=0,\quad (m,n=1,2,\cdots). \tag{4.178}$$

Comment (CFT-limit)

We would like to give some comments on relations between the elliptic integrals of motion and those of CFT. Our integrals of motion can be regarded as elliptic deformation of those for the Virasoro algebra and the W_3 algebra. At the first, we would like to comment the nonlocal integrals of motion $\mathcal{G}_m$. We demand that together with $x\to 1$, the parameter u tends to a limiting value in such a way that $z=x^{2u}$ is fixed. We call this limit CFT-limit. The elliptic screening currents $F_j(z),E_j(z)$ of this paper becomes those of the CFT [1], [2], in the CFT-limit. The theta function in the integrand of the nonlocal integrals of motion $\mathcal{G}_m,\mathcal{G}_m^*$ degenerates trivial scalars. Hence the nonlocal integrals of motion $\mathcal{G}_m$ of this paper becomes those of CFT in the CFT limit. We have checked this relation for $N=2,3$ cases. However there does not exist any paper on integrals of motion of CFT for $N\geqq 4$ case, we have already obtained conjecturous formulae of **T**-**Q**-operators for general W_N-algebra. We checked that this degeneration holds for general W_N-algebra, in the CFT-limit. In the future we would like to report this **T**-**Q**-operators of CFT at another place. For the second we would like to comment for the local integrals of motion $\mathcal{I}_m$. The limit of this case is more complicated. See details in [3]. However we have already known the general formulae for elliptic version of the local integrals of motion $\mathcal{I}_m$, unfortunately, only few leading temrs of the local integrals of motion for the Virasoro algebra and the W_3-algebra are written in [1], [2]. We have checked that few leading terms of our local integrals of motion $\mathcal{I}_1$ and $\mathcal{I}_2$ for $W_{q,t}(\widehat{sl_2})$ degenerates to the known results for the Virasoro algebra in the special limit [3].

4.2. Proof of $[\mathcal{G}_m, \mathcal{G}_n] = 0$. In this section we study the commutation relations $[\mathcal{G}_m, \mathcal{G}_n] = 0$ for $\mathrm{Re}(r) > 0$. We omit details for other cases, because they are similar.

Proposition 15. *For* $\mathrm{Re}(r) > 0$ *we have*

$$\begin{aligned}
&\sum_{\sigma_1 \in S_{m+n}} \sum_{\sigma_2 \in S_{m+n}} \cdots \sum_{\sigma_N \in S_{m+n}} \widehat{\vartheta}_\alpha \left(\sum_{j=1}^{m} u^{(1)}_{\sigma_1(j)} \middle| \sum_{j=1}^{m} u^{(2)}_{\sigma_2(j)} \middle| \cdots \middle| \sum_{j=1}^{m} u^{(N)}_{\sigma_N(j)} \right) \\
&\times \widehat{\vartheta}_\beta \left(\sum_{j=m+1}^{m+n} u^{(1)}_{\sigma_1(j)} \middle| \sum_{j=m+1}^{m+n} u^{(2)}_{\sigma_2(j)} \middle| \cdots \middle| \sum_{j=m+1}^{m+n} u^{(N)}_{\sigma_N(j)} \right) \\
&\times \prod_{t=1}^{N} \prod_{i=1}^{m} \prod_{j=m+1}^{m+n} \frac{\left[u^{(t)}_{\sigma_t(i)} - u^{(t+1)}_{\sigma_{t+1}(j)} - \frac{s}{N}\right]_r \left[u^{(t+1)}_{\sigma_{t+1}(i)} - u^{(t)}_{\sigma_t(j)} + 1 - \frac{s}{N}\right]_r}{\left[u^{(t)}_{\sigma_t(i)} - u^{(t)}_{\sigma_t(j)}\right]_r \left[u^{(t)}_{\sigma_t(j)} - u^{(t)}_{\sigma_t(i)} - 1\right]_r} \\
&= \sum_{\sigma_1 \in S_{m+n}} \sum_{\sigma_2 \in S_{m+n}} \cdots \sum_{\sigma_N \in S_{m+n}} \widehat{\vartheta}_\beta \left(\sum_{j=1}^{n} u^{(1)}_{\sigma_1(j)} \middle| \sum_{j=1}^{n} u^{(2)}_{\sigma_2(j)} \middle| \cdots \middle| \sum_{j=1}^{n} u^{(N)}_{\sigma_N(j)} \right) \\
&\times \widehat{\vartheta}_\alpha \left(\sum_{j=n+1}^{m+n} u^{(1)}_{\sigma_1(j)} \middle| \sum_{j=n+1}^{m+n} u^{(2)}_{\sigma_2(j)} \middle| \cdots \middle| \sum_{j=n+1}^{m+n} u^{(N)}_{\sigma_N(j)} \right) \\
&\times \prod_{t=1}^{N} \prod_{i=1}^{n} \prod_{j=n+1}^{m+n} \frac{\left[u^{(t)}_{\sigma_t(i)} - u^{(t+1)}_{\sigma_{t+1}(j)} - \frac{s}{N}\right]_r \left[u^{(t+1)}_{\sigma_{t+1}(i)} - u^{(t)}_{\sigma_t(j)} + 1 - \frac{s}{N}\right]_r}{\left[u^{(t)}_{\sigma_t(i)} - u^{(t)}_{\sigma_t(j)}\right]_r \left[u^{(t)}_{\sigma_t(j)} - u^{(t)}_{\sigma_t(i)} - 1\right]_r}. \qquad (4.179)
\end{aligned}$$

Here $\widehat{\vartheta}_\alpha(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$ *and* $\widehat{\vartheta}_\beta(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$ *are given by*

$$\widehat{\vartheta}_\alpha(u^{(1)}|\cdots|u^{(t)}+r|\cdots|u^{(N)}) = \widehat{\vartheta}_\alpha(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}), \quad (1 \leqq t \leqq N) \qquad (4.180)$$

$$\begin{aligned}
&\widehat{\vartheta}_\alpha(u^{(1)}|\cdots|u^{(t)}+r\tau|\cdots|u^{(N)}) \qquad (4.181) \\
&= e^{-2\pi i\tau + \frac{2\pi i}{r}(u_{t-1} - 2u_t + u_{t+1} + \sqrt{r(r-1)}P_{\alpha_t}) + \nu_{\alpha,t}} \widehat{\vartheta}_\alpha(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}), \quad (1 \leqq t \leqq N),
\end{aligned}$$

$$\widehat{\vartheta}_\beta(u^{(1)}|\cdots|u^{(t)}+r|\cdots|u^{(N)}) = \widehat{\vartheta}_\beta(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}), \quad (1 \leqq t \leqq N) \qquad (4.182)$$

$$\begin{aligned}
&\widehat{\vartheta}_\beta(u^{(1)}|\cdots|u^{(t)}+r\tau|\cdots|u^{(N)}) \qquad (4.183) \\
&= e^{-2\pi i\tau + \frac{2\pi i}{r}(u_{t-1} - 2u_t + u_{t+1} + \sqrt{r(r-1)}P_{\alpha_t}) + \nu_{\beta,t}} \widehat{\vartheta}_\beta(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}), \quad (1 \leqq t \leqq N).
\end{aligned}$$

Here $\nu_{\alpha,t}, \nu_{\beta,t} \in \mathbb{C}$, $(1 \leqq t \leqq N)$.

Proof. In order to consider elliptic function, we divide the above theta identity by $\widehat{\vartheta}_\gamma$ with $\nu_{\gamma,t}\in\mathbb{C}$, $(1\leqq t\leqq N)$:

$$\widehat{\vartheta}_\gamma(u^{(1)}|\cdots|u^{(t)}+r|\cdots|u^{(N)})=\widehat{\vartheta}_\gamma(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}),\quad(1\leqq t\leqq N)\tag{4.184}$$

$$\begin{aligned}&\widehat{\vartheta}_\gamma(u^{(1)}|\cdots|u^{(t)}+r\tau|\cdots|u^{(N)})\\&=e^{-2\pi i\tau+\frac{2\pi i}{r}(u_{t-1}-2u_t+u_{t+1}+\sqrt{r(r-1)}P_{\alpha_t})+\nu_{\gamma,t}}\widehat{\vartheta}_\gamma(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}),\quad(1\leqq t\leqq N).\end{aligned}\tag{4.185}$$

Let us set

$$\begin{aligned}\mathrm{LHS}(m,n)=&\sum_{\substack{K_1\cup K_1^c=\{1,2,\cdots,n+m\}\\|K_1|=m,|K_1^c|=n}}\cdots\sum_{\substack{K_N\cup K_N^c=\{1,2,\cdots,n+m\}\\|K_N|=m,|K_N^c|=n}}\\&\times\frac{\widehat{\vartheta}_\alpha\left(\sum_{j\in K_1}u_j^{(1)}\middle|\cdots\middle|\sum_{j\in K_N}u_j^{(N)}\right)\widehat{\vartheta}_\beta\left(\sum_{j\in K_1^c}u_j^{(1)}\middle|\cdots\middle|\sum_{j\in K_N^c}u_j^{(N)}\right)}{\widehat{\vartheta}_\gamma\left(\sum_{j=1}^{m+n}u_j^{(1)}\middle|\cdots\middle|\sum_{j=1}^{m+n}u_j^{(N)}\right)}\\&\times\prod_{t=1}^N\frac{\prod_{i\in K_t}\prod_{j\in K_{t+1}^c}\left[u_i^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_r\prod_{i\in K_{t+1}}\prod_{j\in K_t^c}\left[u_i^{(t+1)}-u_j^{(t)}+\frac{s}{N}\right]_r}{\prod_{i\in K_t}\prod_{j\in K_t^c}\left[u_i^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_i^{(t)}-1\right]_r},\end{aligned}\tag{4.186}$$

$$\begin{aligned}\mathrm{RHS}(m,n)=&\sum_{\substack{K_1\cup K_1^c=\{1,2,\cdots,n+m\}\\|K_1|=n,|K_1^c|=m}}\cdots\sum_{\substack{K_N\cup K_N^c=\{1,2,\cdots,n+m\}\\|K_N|=n,|K_N^c|=m}}\\&\times\frac{\widehat{\vartheta}_\beta\left(\sum_{j\in K_1}u_j^{(1)}\middle|\cdots\middle|\sum_{j\in K_N}u_j^{(N)}\right)\widehat{\vartheta}_\alpha\left(\sum_{j\in K_1^c}u_j^{(1)}\middle|\cdots\middle|\sum_{j\in K_N^c}u_j^{(N)}\right)}{\widehat{\vartheta}_\gamma\left(\sum_{j=1}^{m+n}u_j^{(1)}\middle|\cdots\middle|\sum_{j=1}^{m+n}u_j^{(N)}\right)}\\&\times\prod_{t=1}^N\frac{\prod_{i\in K_t}\prod_{j\in K_{t+1}^c}\left[u_i^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_r\prod_{i\in K_{t+1}}\prod_{j\in K_t^c}\left[u_i^{(t+1)}-u_j^{(t)}+\frac{s}{N}\right]_r}{\prod_{i\in K_t}\prod_{j\in K_t^c}\left[u_i^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_i^{(t)}-1\right]_r},\end{aligned}\tag{4.187}$$

Candidates of poles of both LHS(m,n) and RHS(m,n) are $u_i^{(t)}=u_j^{(t)}$ and $u_i^{(t)}=u_j^{(t)}+1$ and $\vartheta_\gamma=0$. Let us show that the points $u_i^{(t)}=u_j^{(t)}$ are regular. Take the residue of the LHS(m,n) at $u_1^{(1)}=u_2^{(1)}$. We have

$$
\begin{aligned}
&\mathrm{Res}_{u_1^{(1)}=u_2^{(1)}}\left(\frac{1}{[u_1^{(1)}-u_2^{(1)}]_r[u_2^{(1)}-u_1^{(1)}-1]_r}+\frac{1}{[u_1^{(1)}-u_2^{(1)}]_r[u_1^{(1)}-u_2^{(1)}-1]_r}\right)\\
&\times\sum_{\substack{L_1\cup L_1^c=\{3,4,\cdots,n+m\}\\|L_1|=m-1,|L_1^c|=n-1}}\ \sum_{\substack{K_2\cup K_2^c=\{1,2,\cdots,n+m\}\\|K_1|=m,|K_1^c|=n}}\cdots\sum_{\substack{K_N\cup K_N^c=\{1,2,\cdots,n+m\}\\|K_N|=m,|K_N^c|=n}}\\
&\times\frac{\widehat{\vartheta}_\alpha(\sum_{j\in L_1\cup\{1\}}u_j^{(1)}|\cdots|\sum_{j\in K_N}u_j^{(N)})\widehat{\vartheta}_\beta(\sum_{j\in L_1^c\cup\{1\}}u_j^{(1)}|\cdots|\sum_{j\in K_N^c}u_j^{(N)})}{\widehat{\vartheta}_\gamma(\sum_{j=1}^{m+n}u_j^{(1)}|\cdots|\sum_{j=1}^{n+m}u_j^{(N)})}\\
&\times\frac{\prod_{j\in K_2^c}\left[u_1^{(1)}-u_j^{(2)}+1-\frac{s}{N}\right]_r\prod_{i\in K_2}\left[u_i^{(2)}-u_2^{(1)}+\frac{s}{N}\right]_r}{\prod_{j\in L_1^c}\left[u_1^{(1)}-u_j^{(1)}\right]_r\left[u_j^{(1)}-u_1^{(1)}-1\right]_r}\\
&\times\frac{\prod_{j\in K_N}\left[u_i^{(N)}-u_2^{(1)}+1-\frac{s}{N}\right]_r\prod_{j\in K_N^c}\left[u_1^{(1)}-u_j^{(N)}+\frac{s}{N}\right]_r}{\prod_{j\in L_1}\left[u_i^{(1)}-u_2^{(1)}\right]_r\left[u_2^{(1)}-u_i^{(1)}-1\right]_r}\\
&\times\frac{\prod_{i\in L_1}\prod_{j\in K_2^c}\left[u_i^{(1)}-u_j^{(2)}+1-\frac{s}{N}\right]_r\prod_{i\in K_2}\prod_{j\in L_1^c}\left[u_i^{(2)}-u_j^{(1)}+\frac{s}{N}\right]_r}{\prod_{i\in L_1}\prod_{j\in L_1^c}\left[u_i^{(1)}-u_j^{(1)}\right]_r}\\
&\times\frac{\prod_{j\in K_N}\prod_{j\in L_1^c}\left[u_i^{(N)}-u_j^{(1)}+1-\frac{s}{N}\right]_r\prod_{i\in L_1}\prod_{j\in K_N^c}\left[u_i^{(1)}-u_j^{(N)}+\frac{s}{N}\right]_r}{\prod_{i\in L_1}\prod_{j\in L_1^c}\left[u_j^{(1)}-u_i^{(1)}-1\right]_r}\qquad(4.188)\\
&\times\prod_{t=2}^{N}\frac{\prod_{i\in K_t}\prod_{j\in K_{t+1}^c}\left[u_i^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_r\prod_{i\in K_{t+1}}\prod_{j\in K_t^c}\left[u_i^{(t+1)}-u_j^{(t)}+\frac{s}{N}\right]_r}{\prod_{i\in K_t}\prod_{j\in K_t^c}\left[u_i^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_i^{(t)}-1\right]_r}=0.
\end{aligned}
$$

Because the first term : $\mathrm{Res}_{u_1^{(1)}=u_2^{(1)}}\left(\frac{1}{[u_1^{(1)}-u_2^{(1)}]_r[u_2^{(1)}-u_1^{(1)}-1]_r}+\frac{1}{[u_1^{(1)}-u_2^{(1)}]_r[u_1^{(1)}-u_2^{(1)}-1]_r}\right)=0$, we have $\mathrm{Res}_{u_1^{(1)}=u_2^{(1)}}\mathrm{LHS}(m,n)=0$. Because LHS$(m,n)$ is symmetric with respect to variables $u_1^{(1)},u_2^{(1)},\cdots,u_{m+n}^{(1)}$, we have $\mathrm{Res}_{u_i^{(1)}=u_j^{(1)}}\mathrm{LHS}(m,n)=0$ for $1\leqq i\neq j\leqq m+n$. As the same manner as above, we conclude that points $u_i^{(t)}=u_j^{(t)}$ of LHS(m,n) and RHS(m,n) for $1\leqq t\leqq N$, $1\leqq i\neq j\leqq m+n$ are regular. Let us show LHS$(m,n)=$ RHS(m,n) by induction for $m+n$. Candidates of poles are only $u_i^{(t)} = u_j^{(t)} + 1$, $1 \leqq t \leqq N$ and $1 \leqq i \neq j \leqq m+n$. We assume $1 \leqq m < n$ without loosing generality. (The case $m = n$ is trivial.) At first we show the starting point $1 = m < n$: $\mathrm{LHS}(1,n) = \mathrm{RHS}(1,n)$. By straightfoeward calculations, we have

$$\begin{aligned}
&\mathrm{Res}_{u_2^{(1)}=u_1^{(1)}+1}\cdots\mathrm{Res}_{u_2^{(N)}=u_1^{(N)}+1}\mathrm{LHS}(1,n)\\
&=\prod_{t=1}^{N}\mathrm{Res}_{u_2^{(t)}=u_1^{(t)}+1}\frac{\left[u_1^{(t)}-u_2^{(t+1)}+1-\frac{s}{N}\right]_r\left[u_1^{(t+1)}-u_2^{(t)}+\frac{s}{N}\right]_r}{\left[u_1^{(t)}-u_2^{(t)}\right]_r\left[u_2^{(t)}-u_1^{(t)}-1\right]_r}\\
&\times\prod_{t=1}^{N}\prod_{j=3}^{n+1}\frac{\left[u_1^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_r\left[u_1^{(t+1)}-u_j^{(t)}+\frac{s}{N}\right]_r}{\left[u_1^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_1^{(t)}-1\right]_r}\\
&\times\frac{\widehat{\vartheta}_\alpha(u_1^{(1)}|\cdots|u_1^{(N)})\widehat{\vartheta}_\beta\left(\sum_{j=2}^{n+1}u_j^{(1)}\middle|\cdots\middle|\sum_{j=2}^{n+1}u_j^{(N)}\right)}{\widehat{\vartheta}_\gamma\left(\sum_{j=1}^{n+1}u_j^{(1)}\middle|\cdots\middle|\sum_{j=1}^{n+1}u_j^{(N)}\right)},
\end{aligned}\tag{4.189}$$

$$\begin{aligned}
&\mathrm{Res}_{u_2^{(1)}=u_1^{(1)}+1}\cdots\mathrm{Res}_{u_2^{(N)}=u_1^{(N)}+1}\mathrm{RHS}(1,n)\\
&=\prod_{t=1}^{N}\mathrm{Res}_{u_2^{(t)}=u_1^{(t)}+1}\frac{\left[u_1^{(t)}-u_2^{(t+1)}+1-\frac{s}{N}\right]_r\left[u_1^{(t+1)}-u_2^{(t)}+\frac{s}{N}\right]_r}{\left[u_1^{(t)}-u_2^{(t)}\right]_r\left[u_2^{(t)}-u_1^{(t)}-1\right]_r}\\
&\times\prod_{t=1}^{N}\prod_{i=3}^{n+1}\frac{\left[u_i^{(t)}-u_2^{(t+1)}+1-\frac{s}{N}\right]_r\left[u_i^{(t+1)}-u_2^{(t)}+\frac{s}{N}\right]_r}{\left[u_i^{(t)}-u_2^{(t)}\right]_r\left[u_2^{(t)}-u_i^{(t)}-1\right]_r}\\
&\times\frac{\widehat{\vartheta}_\alpha(u_2^{(1)}|\cdots|u_2^{(N)})\widehat{\vartheta}_\beta\left(\sum_{\substack{j=1\\j\neq 2}}^{n+1}u_j^{(1)}\middle|\cdots\middle|\sum_{\substack{j=1\\j\neq 2}}^{n+1}u_j^{(N)}\right)}{\widehat{\vartheta}_\gamma\left(\sum_{j=1}^{n+1}u_j^{(1)}\middle|\cdots\middle|\sum_{j=1}^{n+1}u_j^{(N)}\right)}.
\end{aligned}\tag{4.190}$$

Upon specialization $u_2^{(t)} = u_1^{(t)}+1$, $(1 \leqq t \leqq N)$, we have $\frac{[u_1^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}]_r[u_1^{(t+1)}-u_j^{(t)}+\frac{s}{N}]_r}{[u_1^{(t)}-u_j^{(t)}]_r[u_j^{(t)}-u_1^{(t)}-1]_r} = \frac{[u_i^{(t)}-u_2^{(t+1)}+1-\frac{s}{N}]_r[u_i^{(t+1)}-u_2^{(t)}+\frac{s}{N}]_r}{[u_i^{(t)}-u_2^{(t)}]_r[u_2^{(t)}-u_i^{(t)}-1]_r}$. Hence we have $\mathrm{Res}_{u_2^{(1)}=u_1^{(1)}+1}\cdots\mathrm{Res}_{u_2^{(N)}=u_1^{(N)}+1}\mathrm{LHS}(1,n) = \mathrm{Res}_{u_2^{(1)}=u_1^{(1)}+1}\cdots\mathrm{Res}_{u_2^{(N)}=u_1^{(N)}+1}\mathrm{RHS}(1,n)$, using periodic condition $\widehat{\vartheta}_\alpha(u_1^{(1)}+k|\cdots|u_1^{(N)}+k) = \widehat{\vartheta}_\alpha(u_1^{(1)}|\cdots|u_1^{(N)})$. Both $\mathrm{LHS}(1,n)$ and $\mathrm{RHS}(1,n)$ are symmetric with respect to $u_1^{(t)}, u_2^{(t)}, \cdots, u_{n+1}^{(t)}$, we have

$$\mathrm{Res}_{u_{i_1}^{(1)}=u_{j_1}^{(1)}+1}\cdots\mathrm{Res}_{u_{i_N}^{(N)}=u_{j_N}^{(N)}+1}\mathrm{LHS}(1,n) = \mathrm{Res}_{u_{i_1}^{(1)}=u_{j_1}^{(1)}+1}\cdots\mathrm{Res}_{u_{i_N}^{(N)}=u_{j_N}^{(N)}+1}\mathrm{RHS}(1,n),\tag{4.191}$$

for $1 \leqq i_t \neq j_t \leqq n+1$ and $1 \leqq t \leqq N$. After taking the residues finitely many times, every residue relation which comes from $\mathrm{LHS}(1,n) = \mathrm{RHS}(1,n)$, is redued to the above (4.191). Hence we have shown the starting relations $n > m = 1$. For the second, we show the general $n > m \geqq 1$. We assume the relation $\mathrm{LHS}(m-1,n-1) = \mathrm{RHS}(m-1,n-1)$. Let us take the residue at $u_1^{(t)} = u_2^{(t)} + 1$, $(1 \leqq t \leqq N)$. We have

$$\begin{aligned}
&\mathrm{Res}_{u_2^{(1)}=u_1^{(1)}+1} \cdots \mathrm{Res}_{u_2^{(N)}=u_1^{(N)}+1} (\mathrm{LHS}(m,n) - \mathrm{RHS}(m,n)) \\
&= \prod_{t=1}^{N} \mathrm{Res}_{u_2^{(t)}=u_1^{(t)}+1} \frac{\left[u_1^{(t)} - u_2^{(t+1)} + 1 - \frac{s}{N}\right]_r \left[u_1^{(t+1)} - u_2^{(t)} + \frac{s}{N}\right]_r}{\left[u_1^{(t)} - u_2^{(t)}\right]_r \left[u_2^{(t)} - u_1^{(t)} - 1\right]_r} \\
&\times \prod_{t=1}^{N} \prod_{j=3}^{m+n} \frac{\left[u_1^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N}\right]_r \left[u_1^{(t+1)} - u_j^{(t)} + \frac{s}{N}\right]_r}{\left[u_1^{(t)} - u_j^{(t)}\right]_r \left[u_j^{(t)} - u_1^{(t)} - 1\right]_r} \\
&\times \sum_{\substack{L_1 \cup L_1^c = \{3,4,\cdots,n+m\} \\ |L_1| = m-1, |L_1^c| = n-1}} \sum_{\substack{L_2 \cup L_2^c = \{3,4,\cdots,n+m\} \\ |L_2| = m-1, |L_2^c| = n-1}} \cdots \sum_{\substack{L_N \cup L_N^c = \{3,4,\cdots,n+m\} \\ |L_N| = m-1, |L_N^c| = n-1}} \\
&\times \frac{\widehat{\vartheta}_\alpha \left(\sum_{j \in L_1 \cup \{1\}} u_j^{(1)} \middle| \cdots \middle| \sum_{j \in L_N \cup \{1\}} u_j^{(N)} \right) \widehat{\vartheta}_\beta \left(\sum_{j \in L_1^c \cup \{1\}} u_j^{(1)} \middle| \cdots \middle| \sum_{j \in L_N^c \cup \{1\}} u_j^{(N)} \right)}{\widehat{\vartheta}_\gamma \left(\sum_{j=1}^{m+n} u_j^{(1)} \middle| \cdots \middle| \sum_{j=1}^{m+n} u_j^{(N)} \right)} \\
&\times \left(\prod_{t=1}^{N} \frac{\prod_{i \in L_t} \prod_{j \in L_{t+1}^c} \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N}\right]_r \prod_{i \in L_{t+1}} \prod_{j \in L_t^c} \left[u_i^{(t+1)} - u_j^{(t)} + \frac{s}{N}\right]_r}{\prod_{i \in L_t} \prod_{j \in L_{t+1}^c} \left[u_i^{(t)} - u_j^{(t)}\right]_r \left[u_j^{(t)} - u_i^{(t)} - 1\right]_r} \right. \\
&\left. - \prod_{t=1}^{N} \frac{\prod_{i \in L_t^c} \prod_{j \in L_{t+1}} \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N}\right]_r \prod_{i \in L_{t+1}^c} \prod_{j \in L_t} \left[u_i^{(t+1)} - u_j^{(t)} + \frac{s}{N}\right]_r}{\prod_{i \in L_t^c} \prod_{j \in L_{t+1}} \left[u_i^{(t)} - u_j^{(t)}\right]_r \left[u_j^{(t)} - u_i^{(t)} - 1\right]_r} \right) = 0.
\end{aligned} \tag{4.192}$$

We have already used the hypothesis for $(m-1,n-1)$. Both $\mathrm{LHS}(m,n)$ and $\mathrm{RHS}(m,n)$ are symmetric with respect to $u_1^{(t)}, u_2^{(t)}, \cdots, u_{m+n}^{(t)}$, we have

$$\mathrm{Res}_{u_{i_1}^{(1)}=u_{j_1}^{(1)}+1} \cdots \mathrm{Res}_{u_{i_N}^{(N)}=u_{j_N}^{(N)}+1} \mathrm{LHS}(m,n) = \mathrm{Res}_{u_{i_1}^{(1)}=u_{j_1}^{(1)}+1} \cdots \mathrm{Res}_{u_{i_N}^{(N)}=u_{j_N}^{(N)}+1} \mathrm{RHS}(m,n), \tag{4.193}$$

for $1 \leqq i_t \neq j_t \leqq m+n$ and $1 \leqq t \leqq N$. After taking the residues finitely many times, every residue relation which comes from $\mathrm{LHS}(m,n) = \mathrm{RHS}(m,n)$, is redued to the above (4.193). Hence we have shown $\mathrm{LHS}(m,n) = \mathrm{RHS}(m,n)$ for $n > m \geqq 1$. Q.E.D.

Now let us show the commutation relation $[\mathcal{G}_m, \mathcal{G}_n] = 0$.

Proof of Theorem 4. We show $[\mathcal{G}_m, \mathcal{G}_n] = 0$ for $\mathrm{Re}(r) > 0$ and $0 < \mathrm{Re}(s) < N$. Others are shown by similar way. We use the integral representation of the nonlocal integrals of motion. The following operators in the integrand of the nonlocal integrals of motion satisfies the S_n-invariance. For $\sigma_1, \sigma_2, \cdots, \sigma_N \in S_{m+n}$, we have

$$\begin{aligned}
& F_1(z^{(1)}_{\sigma_1(1)}) \cdots F_1(z^{(1)}_{\sigma_1(m+n)}) F_2(z^{(2)}_{\sigma_2(1)}) \cdots F_2(z^{(2)}_{\sigma_2(m+n)}) \cdots F_N(z^{(N)}_{\sigma_N(1)}) \cdots F_N(z^{(N)}_{\sigma_N(m+n)}) \\
& \times \prod_{t=1}^{N} \prod_{1 \leqq i < j \leqq m+n} \left[u^{(t)}_{\sigma_t(i)} - u^{(t)}_{\sigma_t(j)}\right]_r \left[u^{(t)}_{\sigma_t(j)} - u^{(t)}_{\sigma_t(i)} - 1\right]_r \\
& = F_1(z^{(1)}_1) \cdots F_1(z^{(1)}_{m+n}) F_2(z^{(2)}_1) \cdots F_2(z^{(2)}_{m+n}) \cdots F_N(z^{(N)}_1) \cdots F_N(z^{(N)}_{m+n}) \\
& \times \prod_{t=1}^{N} \prod_{1 \leqq i < j \leqq m+n} \left[u^{(t)}_i - u^{(t)}_j\right]_r \left[u^{(t)}_j - u^{(t)}_i - 1\right]_r .
\end{aligned} \tag{4.194}$$

Hence we have

$$\begin{aligned}
\mathcal{G}_m \cdot \mathcal{G}_n = & \prod_{t=1}^{N} \prod_{j=1}^{m+n} \oint \frac{dz^{(t)}_j}{2\pi\sqrt{-1} z^{(t)}_j} F_1(z^{(1)}_1) \cdots F_1(z^{(1)}_{m+n}) F_2(z^{(2)}_1) \cdots F_2(z^{(2)}_{m+n}) \cdots F_N(z^{(N)}_1) \cdots F_N(z^{(N)}_{m+n}) \\
& \times \frac{\displaystyle\prod_{t=1}^{N} \prod_{1 \leqq i < j \leqq m+n} \left[u^{(t)}_i - u^{(t)}_j\right]_r \left[u^{(t)}_j - u^{(t)}_i - 1\right]_r}{\displaystyle\prod_{t=1}^{N-1} \prod_{i,j=1}^{m+n} \left[u^{(t)}_i - u^{(t+1)}_j + 1 - \frac{s}{N}\right]_r} \\
& \times \frac{1}{\displaystyle\prod_{i=1}^{m} \prod_{j=m+1}^{m+n} \left[u^{(N)}_i - u^{(1)}_j + \frac{s}{N}\right]_r \prod_{i=m+1}^{m+n} \prod_{j=1}^{m} \left[u^{(N)}_j - u^{(1)}_i - \frac{s}{N} + 1\right]_r} \\
& \times \frac{1}{\displaystyle\prod_{i,j=1}^{m} \left[u^{(1)}_i - u^{(N)}_j + \frac{s}{N}\right]_r \prod_{i,j=m+1}^{m+n} \left[u^{(1)}_i - u^{(N)}_j + \frac{s}{N}\right]_r} \\
& \times \frac{1}{((m+n)!)^N} \sum_{\sigma_1 \in S_{m+n}} \cdots \sum_{\sigma_N \in S_{m+n}} \\
& \times \vartheta_\alpha\Big(\sum_{j=1}^{m} u^{(1)}_{\sigma_1(j)} \Big| \cdots \Big| \sum_{j=1}^{m} u^{(N)}_{\sigma_N(j)}\Big) \vartheta_\beta\Big(\sum_{j=m+1}^{m+n} u^{(1)}_{\sigma_1(j)} \Big| \cdots \Big| \sum_{j=m+1}^{m+n} u^{(N)}_{\sigma_N(j)}\Big) \\
& \times \prod_{t=1}^{N} \prod_{i=1}^{m} \prod_{j=m+1}^{m+n} \frac{\left[u^{(t)}_{\sigma_t(i)} - u^{(t+1)}_{\sigma_{t+1}(j)} + 1 - \frac{s}{N}\right]_r \left[u^{(t+1)}_{\sigma_{t+1}(i)} - u^{(t)}_{\sigma_t(j)} + \frac{s}{N}\right]_r}{\left[u^{(t)}_{\sigma_t(i)} - u^{(t)}_{\sigma_t(j)}\right]_r \left[u^{(t)}_{\sigma_t(j)} - u^{(t)}_{\sigma_t(i)} - 1\right]_r} .
\end{aligned} \tag{4.195}$$

Therefore we have the following theta function identity as a sufficient condition of the commutation relations $\mathcal{G}_m \cdot \mathcal{G}_n = \mathcal{G}_n \cdot \mathcal{G}_m$.

$$\begin{aligned}
&\sum_{\sigma_1 \in S_{m+n}} \sum_{\sigma_2 \in S_{m+n}} \cdots \sum_{\sigma_N \in S_{m+n}} \vartheta_\alpha \left(\sum_{j=1}^{m} u^{(1)}_{\sigma_1(j)} \middle| \sum_{j=1}^{m} u^{(2)}_{\sigma_2(j)} \middle| \cdots \middle| \sum_{j=1}^{m} u^{(N)}_{\sigma_N(j)} \right) \\
&\times \vartheta_\beta \left(\sum_{j=m+1}^{m+n} u^{(1)}_{\sigma_1(j)} \middle| \sum_{j=m+1}^{m+n} u^{(2)}_{\sigma_2(j)} \middle| \cdots \middle| \sum_{j=m+1}^{m+n} u^{(N)}_{\sigma_N(j)} \right) \\
&\times \prod_{t=1}^{N} \prod_{i=1}^{m} \prod_{j=m+1}^{m+n} \frac{\left[u^{(t)}_{\sigma_t(i)} - u^{(t+1)}_{\sigma_{t+1}(j)} - \frac{s}{N}\right]_r \left[u^{(t+1)}_{\sigma_{t+1}(i)} - u^{(t)}_{\sigma_t(j)} + \frac{s}{N}\right]_r}{\left[u^{(t)}_{\sigma_t(i)} - u^{(t)}_{\sigma_t(j)}\right]_r \left[u^{(t)}_{\sigma_t(j)} - u^{(t)}_{\sigma_t(i)} - 1\right]_r} \\
&= \sum_{\sigma_1 \in S_{m+n}} \sum_{\sigma_2 \in S_{m+n}} \cdots \sum_{\sigma_N \in S_{m+n}} \vartheta_\beta \left(\sum_{j=1}^{n} u^{(1)}_{\sigma_1(j)} \middle| \sum_{j=1}^{n} u^{(2)}_{\sigma_2(j)} \middle| \cdots \middle| \sum_{j=1}^{n} u^{(N)}_{\sigma_N(j)} \right) \\
&\times \vartheta_\alpha \left(\sum_{j=n+1}^{m+n} u^{(1)}_{\sigma_1(j)} \middle| \sum_{j=n+1}^{m+n} u^{(2)}_{\sigma_2(j)} \middle| \cdots \middle| \sum_{j=n+1}^{m+n} u^{(N)}_{\sigma_N(j)} \right) \\
&\times \prod_{t=1}^{N} \prod_{i=1}^{n} \prod_{j=n+1}^{m+n} \frac{\left[u^{(t)}_{\sigma_t(i)} - u^{(t+1)}_{\sigma_{t+1}(j)} - \frac{s}{N}\right]_r \left[u^{(t+1)}_{\sigma_{t+1}(j)} - u^{(t)}_{\sigma_t(i)} + 1 - \frac{s}{N}\right]_r}{\left[u^{(t)}_{\sigma_t(i)} - u^{(t)}_{\sigma_t(j)}\right]_r \left[u^{(t)}_{\sigma_t(j)} - u^{(t)}_{\sigma_t(i)} - 1\right]_r}. \qquad (4.196)
\end{aligned}$$

This is a special case $\nu_{\alpha,t} = \nu_{\beta,t} = 0$, $(1 \leqq t \leqq N)$ of the theta identity in Proposition 15. Now we have shown Theorem. Q.E.D.

4.3. Proof of $[\mathcal{I}_m, \mathcal{G}_n] = 0$. In this section we give proof of the commutation relation $[\mathcal{I}_m, \mathcal{G}_n] = 0$. The fundamental operators $\Lambda_j(z)$ and $F_j(z)$ commute almost everywhere.

$$\begin{aligned}
[\Lambda_j(z_1), F_j(z_2)] &= (-x^{r^*} + x^{-r^*}) \delta\left(x^{\frac{2s}{N}j - r} \frac{z_2}{z_1} \right) \mathcal{A}_j(x^{-r+\frac{2s}{N}j} z_1), \\
[\Lambda_{j+1}(z_1), F_j(z_2)] &= (x^{r^*} - x^{-r^*}) \delta\left(x^{\frac{2s}{N}j + r} \frac{z_2}{z_1} \right) \mathcal{A}_j(x^{r+\frac{2s}{N}j} z_1).
\end{aligned}$$

Hence, in order to show the commutation relations, remaining task for us is to check whether delta-function factors cancell out or not. The Dynkin-automorphism invariance $\eta(\mathcal{I}_m) = \mathcal{I}_m$, $\eta(\mathcal{G}_m) = \mathcal{G}_m$, which we will show later, plays an important role in proof of this commutation relation $[\mathcal{I}_m, \mathcal{G}_n] = 0$.

Proof of Theorem 6. For a while we consider upon the regime: $0 < \mathrm{Re}(s) < N$, $0 < \mathrm{Re}(r) < 1$. At first we show the simple case, $[\mathcal{I}_1, \mathcal{G}_n] = 0$, for reader's convenience. Using Leibnitz-rule of adjoint action $[A, BC] = [A, B]C + B[A, C]$

and the invariance $\eta(\mathcal{I}_1)=\mathcal{I}_1$, we have

$$
\begin{aligned}
[\mathcal{I}_1,\mathcal{G}_n]=&(x^{-r^*}-x^{r^*})\sum_{t=1}^{N-1}\sum_{j=1}^{n}\prod_{u=1}^{N}\prod_{k=1}^{n}\int_{\widetilde{C}(t,j)}\frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}}F_1(z_1^{(1)})\cdots\\
&\times\cdots F_t(z_1^{(t)})\cdots F_t(z_{j-1}^{(t)})\mathcal{A}_t(x^{-r+\frac{2s}{N}t}z_j^{(t)})F_t(z_{j+1}^{(t)})\cdots F_t(z_n^{(t)})\cdots F_N(z_n^{(N)})\\
&\times\frac{\displaystyle\prod_{t=1}^{N}\prod_{1\leqq i<j\leqq n}\left[u_i^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_i^{(t)}-1\right]_r}{\displaystyle\prod_{t=1}^{N-1}\prod_{i,j=1}^{n}\left[u_i^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_r\prod_{i,j=1}^{n}\left[u_i^{(1)}-u_j^{(N)}+\frac{s}{N}\right]_r}\\
&\times\vartheta\left(\sum_{j=1}^{n}u_j^{(1)}\middle|\sum_{j=1}^{n}u_j^{(2)}\middle|\cdots\middle|\sum_{j=1}^{n}u_j^{(N)}\right)\\
&+(x^{r^*}-x^{-r^*})\sum_{t=1}^{N-1}\sum_{j=1}^{n}\prod_{u=1}^{N}\prod_{k=1}^{n}\int_{\widetilde{C}'(t,j)}\frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}}F_1(z_1^{(1)})\cdots\\
&\times\cdots F_t(z_1^{(t)})\cdots F_t(z_{j-1}^{(t)})\mathcal{A}_t(x^{r+\frac{2s}{N}t}z_j^{(t)})F_t(z_{j+1}^{(t)})\cdots F_t(z_n^{(t)})\cdots F_N(z_n^{(N)})\\
&\times\frac{\displaystyle\prod_{t=1}^{N}\prod_{1\leqq i<j\leqq n}\left[u_i^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_i^{(t)}-1\right]_r}{\displaystyle\prod_{t=1}^{N-1}\prod_{i,j=1}^{n}\left[u_i^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_r\prod_{i,j=1}^{n}\left[u_i^{(1)}-u_j^{(N)}+\frac{s}{N}\right]_r}\\
&\times\vartheta\left(\sum_{j=1}^{n}u_j^{(1)}\middle|\sum_{j=1}^{n}u_j^{(2)}\middle|\cdots\middle|\sum_{j=1}^{n}u_j^{(N)}\right)\\
&+\eta\Bigg(\sum_{j=1}^{n}\prod_{u=1}^{N}\prod_{k=1}^{n}\int_{C}\frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}}\;F_2(z_1^{(1)})\cdots F_2(z_n^{(1)})\cdots F_N(z_1^{(N-1)})\cdots F_N(z_n^{(N-1)})\\
&\times F_1(z_1^{(N)})\cdots F_1(z_{j-1}^{(N)})[\mathcal{I}_1,F_1(z_j^{(N)})]F_1(z_{j+1}^{(N)})\cdots F_1(z_n^{(N)})\\
&\times\frac{\displaystyle\prod_{t=1}^{N}\prod_{1\leqq i<j\leqq n}\left[u_i^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_i^{(t)}-1\right]_r}{\displaystyle\prod_{t=1}^{N-1}\prod_{i,j=1}^{n}\left[u_i^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_r\prod_{i,j=1}^{n}\left[u_i^{(1)}-u_j^{(N)}+\frac{s}{N}\right]_r}\\
&\times\vartheta\left(\sum_{j=1}^{n}u_j^{(N)}\middle|\sum_{j=1}^{n}u_j^{(1)}\middle|\cdots\middle|\sum_{j=1}^{n}u_j^{(N-1)}\right)\Bigg).
\end{aligned}
\tag{4.197}
$$

Here we have set

$$\begin{aligned}\widetilde{C}(t,j):\ & |x^{4r-2+\frac{2s}{N}t}z_{j+1}^{(t)}|,\cdots,|x^{4r-2+\frac{2s}{N}t}z_m^{(t)}|<|z_j^{(t)}|<|x^{-2r+2-\frac{2s}{N}t}z_1^{(t)}|,\cdots,|x^{-2r+2-\frac{2s}{N}t}z_{j-1}^{(t)}|,\\ & |x^{2r+2-\frac{2s}{N}}z_k^{(t-1)}|<|z_j^{(t)}|<|x^{2r-\frac{2s}{N}}z_k^{(t-1)}|,\ (1\leqq k\leqq m),\\ & |x^{\frac{2s}{N}}z_k^{(t+1)}|<|z_j^{(t)}|<|x^{-2+\frac{2s}{N}}z_k^{(t+1)}|,\ (1\leqq k\leqq m),\end{aligned}\tag{4.198}$$

$$\begin{aligned}\widetilde{C}'(t,j):\ & |x^{2r-2+\frac{2s}{N}t}z_{j+1}^{(t)}|,\cdots,|x^{2r-2+\frac{2s}{N}t}z_m^{(t)}|<|z_j^{(t)}|<|x^{-4r+2-\frac{2s}{N}t}z_1^{(t)}|,\cdots,|x^{-4r+2-\frac{2s}{N}t}z_{j-1}^{(t)}|,\\ & |x^{2-\frac{2s}{N}}z_k^{(t-1)}|<|z_j^{(t)}|<|x^{-\frac{2s}{N}}z_k^{(t-1)}|,\ (1\leqq k\leqq m),\\ & |x^{-2r+\frac{2s}{N}}z_k^{(t+1)}|<|z_j^{(t)}|<|x^{-2r-2+\frac{2s}{N}}z_k^{(t+1)}|,\ (1\leqq k\leqq m).\end{aligned}\tag{4.199}$$

Let us change the variable $z_j^{(t)}\to x^{2r}z_j^{(t)}$ in the first term of (4.197). Using the periodicity of the integrand, we deform the first term to the second term of (4.197).

$$\begin{aligned}&\prod_{u=1}^{N}\prod_{k=1}^{n}\int_{\widetilde{I}(t,j)}\frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}}F_1(z_1^{(1)})\cdots F_t(z_1^{(t)})\cdots F_t(z_{j-1}^{(t)})\\ &\times\mathcal{A}_t(x^{-r}z_j^{(t)})F_t(z_{j+1}^{(t)})\cdots F_t(z_n^{(t)})\cdots F_N(z_n^{(N)})\\ &\times\frac{\displaystyle\prod_{t=1}^{N}\prod_{1\leqq i<j\leqq n}\left[u_i^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_i^{(t)}-1\right]_r}{\displaystyle\prod_{t=1}^{N-1}\prod_{i,j=1}^{n}\left[u_i^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_r\prod_{i,j=1}^{n}\left[u_i^{(1)}-u_j^{(N)}+\frac{s}{N}\right]_r}\vartheta\left(\sum_{j=1}^{n}u_j^{(1)}\middle|\sum_{j=1}^{n}u_j^{(2)}\middle|\cdots\middle|\sum_{j=1}^{n}u_j^{(N)}\right)\\ &=\prod_{u=1}^{N}\prod_{k=1}^{n}\int_{\widetilde{I}'(t,j)}\frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}}F_1(z_1^{(1)})\cdots F_t(z_1^{(t)})\cdots F_t(z_{j-1}^{(t)})\\ &\times\mathcal{A}_t(x^{r}z_j^{(t)})F_t(z_{j+1}^{(t)})\cdots F_t(z_n^{(t)})\cdots F_N(z_n^{(N)})\\ &\times\frac{\displaystyle\prod_{t=1}^{N}\prod_{1\leqq i<j\leqq n}\left[u_i^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_i^{(t)}-1\right]_r}{\displaystyle\prod_{t=1}^{N-1}\prod_{i,j=1}^{n}\left[u_i^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_r\prod_{i,j=1}^{n}\left[u_i^{(1)}-u_j^{(N)}+\frac{s}{N}\right]_r}\vartheta\left(\sum_{j=1}^{n}u_j^{(1)}\middle|\sum_{j=1}^{n}u_j^{(2)}\middle|\cdots\middle|\sum_{j=1}^{n}u_j^{(N)}\right).\end{aligned}\tag{4.200}$$

Hence we have

$$
\begin{aligned}
&\eta([\mathcal{I}_1,\mathcal{G}_n])\\
&=(x^{-r^*}-x^{r^*})\sum_{j=1}^{n}\prod_{u=1}^{N}\prod_{k=1}^{n}\int_{C(N,j)}\frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}}F_2(z_1^{(1)})\cdots F_2(z_n^{(1)})\cdots F_N(z_1^{(N-1)})\cdots F_N(z_n^{(N-1)})\\
&\times F_1(z_1^{(N)})\cdots F_1(z_{j-1}^{(N)})\mathcal{A}_1(x^{-r+\frac{2s}{N}}z_j^{(N)})]F_1(z_{j+1}^{(N)})\cdots F_1(z_n^{(N)})\\
&\times\frac{\prod_{t=1}^{N}\prod_{1\leqq i<j\leqq n}\left[u_i^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_i^{(t)}-1\right]_r}{\prod_{t=1}^{N-1}\prod_{i,j=1}^{n}\left[u_i^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_r\prod_{i,j=1}^{n}\left[u_i^{(1)}-u_j^{(N)}+\frac{s}{N}\right]_r}\\
&\times\vartheta\left(\sum_{j=1}^{n}u_j^{(1)}\middle|\sum_{j=1}^{n}u_j^{(2)}\middle|\cdots\middle|\sum_{j=1}^{n}u_j^{(N)}\right)\\
&-(x^{-r^*}-x^{r^*})\sum_{j=1}^{n}\prod_{u=1}^{N}\prod_{k=1}^{n}\int_{C'(N,j)}\frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}}F_2(z_1^{(1)})\cdots F_2(z_n^{(1)})\cdots F_N(z_1^{(N-1)})\cdots F_N(z_n^{(N-1)})\\
&\times F_1(z_1^{(N)})\cdots F_1(z_{j-1}^{(N)})\mathcal{A}_1(x^{r+\frac{2s}{N}}z_j^{(N)})]F_1(z_{j+1}^{(N)})\cdots F_1(z_n^{(N)})\\
&\times\frac{\prod_{t=1}^{N}\prod_{1\leqq i<j\leqq n}\left[u_i^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_i^{(t)}-1\right]_r}{\prod_{t=1}^{N-1}\prod_{i,j=1}^{n}\left[u_i^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_r\prod_{i,j=1}^{n}\left[u_i^{(1)}-u_j^{(N)}+\frac{s}{N}\right]_r}\\
&\times\vartheta\left(\sum_{j=1}^{n}u_j^{(N)}\middle|\sum_{j=1}^{n}u_j^{(1)}\middle|\cdots\middle|\sum_{j=1}^{n}u_j^{(N-1)}\right).
\end{aligned}
\tag{4.201}
$$

Here we have set

$$
\begin{aligned}
\widetilde{C}(N,j):\;&|x^{4r-2+\frac{2s}{N}}z_{j+1}^{(N)}|,\cdots,|x^{4r-2+\frac{2s}{N}}z_m^{(N)}|<|z_j^{(N)}|<|x^{-2r+2-\frac{2s}{N}}z_1^{(N)}|,\cdots,|x^{-2r+2-\frac{2s}{N}}z_{j-1}^{(N)}|,\\
&|x^{2r+2-\frac{2s}{N}}z_k^{(N-1)}|<|z_j^{(N)}|<|x^{2r-\frac{2s}{N}}z_k^{(N-1)}|,\ (1\leqq k\leqq m),\\
&|x^{\frac{2s}{N}}z_k^{(1)}|<|z_j^{(N)}|<|x^{-2+\frac{2s}{N}}z_k^{(1)}|,\ (1\leqq k\leqq m),
\end{aligned}
\tag{4.202}
$$

$$
\begin{aligned}
\widetilde{C}'(N,j):\;&|x^{2r-2+\frac{2s}{N}}z_{j+1}^{(N-1)}|,\cdots,|x^{2r-2+\frac{2s}{N}}z_m^{(N-1)}|\\
&<|z_j^{(N)}|<|x^{-4r+2-\frac{2s}{N}}z_1^{(N-1)}|,\cdots,|x^{-4r+2-\frac{2s}{N}}z_{j-1}^{(N-1)}|,\\
&|x^{2-\frac{2s}{N}}z_k^{(N-1)}|<|z_j^{(N)}|<|x^{-\frac{2s}{N}}z_k^{(N-1)}|,\ (1\leqq k\leqq m),\\
&|x^{-2r+\frac{2s}{N}}z_k^{(1)}|<|z_j^{(N)}|<|x^{-2r-2+\frac{2s}{N}}z_k^{(1)}|,\ (1\leqq k\leqq m).
\end{aligned}
\tag{4.203}
$$

Using the priodicity of the integtrand, we have $[\mathcal{I}_1,\mathcal{G}_n]=0$. For the second, we consider the commutation relation $[\mathcal{I}_m,\mathcal{G}_n]=0$. By using the invariance of the local integrals of motion, $\eta(\mathcal{I}_m)=\mathcal{I}_m$, we have

$$[\mathcal{I}_m,\mathcal{G}_n]$$

$$=(x^{-r^*}-x^{r^*})\sum_{t=1}^{N-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\prod_{u=1}^{N}\prod_{k=1}^{n}\int_{\widetilde{C}(t,j)}\frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}}\vartheta\left(\sum_{j=1}^{n}u_j^{(1)}\middle|\sum_{j=1}^{n}u_j^{(2)}\middle|\cdots\middle|\sum_{j=1}^{n}u_j^{(N)}\right)$$

$$\times\frac{\displaystyle\prod_{t=1}^{N}\prod_{1\leqq i<j\leqq n}\left[u_i^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_i^{(t)}-1\right]_r}{\displaystyle\prod_{t=1}^{N-1}\prod_{i,j=1}^{n}\left[u_i^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_r\prod_{i,j=1}^{n}\left[u_i^{(1)}-u_j^{(N)}+\frac{s}{N}\right]_r}$$

$$\times\left(\prod_{k=1}^{m}\int_{I\cap\left\{\left|\frac{x^{-r+\frac{2s}{N}t}z_j^{(t)}}{w_i}\right|<1\right\}}\frac{dw_k}{2\pi\sqrt{-1}w_k}\frac{1}{\left(1-\frac{x^{-r+\frac{2s}{N}t}z_j^{(t)}}{w_i}\right)}\right.$$

$$\left.+\prod_{k=1}^{m}\int_{I\cap\left\{\left|\frac{x^{-r+\frac{2s}{N}t}z_j^{(t)}}{w_i}\right|>1\right\}}\frac{dw_k}{2\pi\sqrt{-1}w_k}\frac{\frac{x^{r-\frac{2s}{N}t}w_i}{z_j^{(t)}}}{\left(1-\frac{x^{r-\frac{2s}{N}t}w_i}{z_j^{(t)}}\right)}\right)$$

$$\times\prod_{1\leqq k<l\leqq m}h(v_k-v_l)T_1(w_1)\cdots T_1(w_{i-1})F_1(z_1^{(t)})\cdots F_t(z_{j-1}^{(t)})$$

$$\times\mathcal{A}_t(x^{-r+\frac{2s}{N}t}z_j^{(t)})F_t(z_{j+1}^{(t)})\cdots F_N(z_n^{(t)})T_1(w_{i+1})\cdots T_1(w_m)$$

$$-(x^{-r^*}-x^{r^*})\sum_{t=1}^{N-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\prod_{u=1}^{N}\prod_{k=1}^{n}\int_{\widetilde{C}'(t,j)}\frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}}\vartheta\left(\sum_{j=1}^{n}u_j^{(1)}\middle|\sum_{j=1}^{n}u_j^{(2)}\middle|\cdots\middle|\sum_{j=1}^{n}u_j^{(N)}\right)$$

$$\times\frac{\displaystyle\prod_{t=1}^{N}\prod_{1\leqq i<j\leqq n}\left[u_i^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_i^{(t)}-1\right]_r}{\displaystyle\prod_{t=1}^{N-1}\prod_{i,j=1}^{n}\left[u_i^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_r\prod_{i,j=1}^{n}\left[u_i^{(1)}-u_j^{(N)}+\frac{s}{N}\right]_r}$$

$$\times\left(\prod_{k=1}^{m}\int_{I\cap\left\{\left|\frac{x^{r+\frac{2s}{N}t}z_j^{(t)}}{w_i}\right|<1\right\}}\frac{dw_k}{2\pi\sqrt{-1}w_k}\frac{1}{\left(1-\frac{x^{r+\frac{2s}{N}t}z_j^{(t)}}{w_i}\right)}\right.$$

$$\left.+\prod_{k=1}^{m}\int_{I\cap\left\{\left|\frac{x^{r+\frac{2s}{N}t}z_j^{(t)}}{w_i}\right|>1\right\}}\frac{dw_k}{2\pi\sqrt{-1}w_k}\frac{\frac{x^{-r-\frac{2s}{N}t}w_i}{z_j^{(t)}}}{\left(1-\frac{x^{-r-\frac{2s}{N}t}w_i}{z_j^{(t)}}\right)}\right)$$

$$\times\prod_{1\leqq k<l\leqq m}h(v_k-v_l)T_1(w_1)\cdots T_1(w_{i-1})F_1(z_1^{(t)})\cdots F_t(z_{j-1}^{(t)})$$

$$\times\mathcal{A}_t(x^{r+\frac{2s}{N}}z_j^{(t)})F_t(z_{j+1}^{(t)})\cdots F_N(z_n^{(t)})T_1(w_{i+1})\cdots T_1(w_m)$$

$$+\eta\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\prod_{u=1}^{N}\prod_{k=1}^{n}\int_{C}\frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}}\vartheta\left(\sum_{j=1}^{n}u_j^{(N)}\middle|\sum_{j=1}^{n}u_j^{(1)}\middle|\cdots\middle|\sum_{j=1}^{n}u_j^{(N-1)}\right)\right.$$

$$\times\frac{\displaystyle\prod_{t=1}^{N}\prod_{1\leqq i<j\leqq m}\left[u_i^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_i^{(t)}-1\right]_r}{\displaystyle\prod_{t=1}^{N-1}\prod_{i,j=1}^{n}\left[u_i^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_r\prod_{i,j=1}^{n}\left[u_i^{(1)}-u_j^{(N)}+\frac{s}{N}\right]_r}$$

$$\times \prod_{k=1}^{m} \int_I \frac{dw_k}{2\pi\sqrt{-1}w_k} \prod_{1\leqq k<l\leqq m} h(v_k - v_l) T_1(w_1)\cdots T_1(w_{i-1})$$
$$\times F_2(z_1^{(1)})\cdots F_2(z_n^{(1)})\cdots F_N(z_1^{(N-1)})\cdots F_N(z_n^{(N-1)}) F_1(z_1^{(N)})\cdots F_1(z_{j-1}^{(N)})$$
$$\times [T_1(w_i), F_1(z_j^{(N)})] F_1(z_{j+1}^{(N)})\cdots F_1(z_n^{(N)}) T_1(w_{i+1})\cdots T_1(w_m)). \qquad (4.204)$$

Here $\widetilde{C}(t,j)$, $\widetilde{C}'(t,j)$ are given as the same manner in proof of $[\mathcal{I}_1, \mathcal{G}_n] = 0$. Because the integral contour I is not annulus, we have used

$$\prod_{k=1}^{m} \int_{I\cap\{|z/w_i|<1\}} \frac{dw_k}{w_k} \frac{1}{1-z/w_i} + \prod_{k=1}^{m} \int_{I\cap\{|z/w_i|>1\}} \frac{dw_k}{w_k} \frac{w_i/z}{1-w_i/z}$$

instead of $\delta(z/w_i) = \sum_{m\in\mathbb{Z}} (z/w_i)^m$. Let us change the variable $z_j^{(t)} \to x^{2r} z_j^{(t)}$ upon the conditions $0 < \mathrm{Re}(s) < N$, $0 < \mathrm{Re}(r) < 1$. Using periodicity of the integrands, we have $[\mathcal{I}_m, \mathcal{G}_n] = 0$ as the same manner as $[\mathcal{I}_1, \mathcal{G}_n] = 0$. Other commutation relations : $[\mathcal{I}_m, \mathcal{G}_n^*] = [\mathcal{I}_m^*, \mathcal{G}_n] = [\mathcal{I}_m^*, \mathcal{G}_n^*] = 0$ are shown by similar way. Q.E.D.

4.4. Proof of $[\mathcal{G}_m, \mathcal{G}_n^*] = 0$. In this section we give proof of the commutation relation $[\mathcal{I}_m, \mathcal{G}_n] = 0$. The fundamental operators $E_j(z)$ and $F_j(z)$ commute almost everywhere.

$$[E_j(z_1), F_j(z_2)] = \frac{1}{x - x^{-1}} (\delta(xz_2/z_1) H_j(x^r z_2) - \delta(xz_1/z_2) H_j(x^{-r} z_2)).$$

Hence, in order to show the commutation relations, remaining task for us is to check whether delta-function factors cancell out or not.

Proof of Theorem 5 We consider the regime $\mathrm{Re}(r) > 0$ and $\mathrm{Re}(r^*) < 0$. Using Leibnitz-rule of adjoint action and the commutation relations of screening currents $E_j(z), F_j(z)$, we have

$$[\mathcal{G}_m^*, \mathcal{G}_n] = \sum_{t=1}^{N} \sum_{i=1}^{n} \sum_{j=1}^{m} \prod_{q=1}^{N} \prod_{k=1}^{n} \prod_{p=1}^{N} \prod_{\substack{l=1 \\ l\neq t}}^{m} \oint_{C_{i,j}^{(t)}} \frac{dz^{(q)_k}}{2\pi\sqrt{-1} z_k^{(q)}} \frac{dw^{(p)_l}}{2\pi\sqrt{-1} w_l^{(q)}}$$
$$\times B_{i,j}^{(t)} \left(x^{-r} z_i^{(t)}; \{z_j^{(q)}\}_{\substack{1\leqq q\leqq N \\ 1\leqq k\leqq n}}, \{w^{(q)}\}_{\substack{1\leqq q\leqq N \\ 1\leqq l\neq j\leqq m}} \right)$$
$$- \sum_{t=1}^{N} \sum_{i=1}^{n} \sum_{j=1}^{m} \prod_{q=1}^{N} \prod_{k=1}^{n} \prod_{p=1}^{N} \prod_{\substack{l=1 \\ l\neq t}}^{m} \oint_{C_{i,j}^{(t)}} \frac{dz^{(q)_k}}{2\pi\sqrt{-1} z_k^{(q)}} \frac{dw^{(p)_l}}{2\pi\sqrt{-1} w_l^{(q)}}$$
$$\times B_{i,j}^{(t)} \left(x^{r} z_i^{(t)}; \{z_j^{(q)}\}_{\substack{1\leqq q\leqq N \\ 1\leqq k\leqq n}}, \{w^{(q)}\}_{\substack{1\leqq q\leqq N \\ 1\leqq l\neq j\leqq m}} \right). \qquad (4.205)$$

Here we have set

$$
\begin{aligned}
&B_{i,j}^{(t)}\left(z;\{z_j^{(q)}\}_{\substack{1\leqq q\leqq N\\1\leqq k\leqq n}},\{w^{(q)}\}_{\substack{1\leqq q\leqq N\\1\leqq l\neq j\leqq m}}\right)\\
&=\frac{1}{x-x^{-1}}F_1(z_1^{(1)})\cdots F_1(z_n^{(1)})E_1(w_1^{(1)})\cdots E_1(w_m^{(1)})\cdots\\
&\times\cdots F_t(z_1^{(t)})\cdots F_t(z_{i-1}^{(t)})E_t(w_1^{(t)})\cdots E_t(w_{j-1}^{(t)})\\
&\times H_t(z)E_t(w_{j+1}^{(t)})\cdots E_t(w_m^{(t)})F_t(z_{i+1}^{(t)})\cdots F_t(z_n^{(t)})\cdots\\
&\times\cdots E_1(w_1^{(N)})\cdots E_1(w_m^{(N)})F_1(z_1^{(N)})\cdots F_1(z_n^{(N)})\\
&\times\frac{\displaystyle\prod_{q=1}^{N}\prod_{1\leqq k<l\leqq n}\left[u_k^{(q)}-u_l^{(q)}\right]_r\left[u_l^{(q)}-u_k^{(q)}-1\right]_r}{\displaystyle\prod_{q=1}^{N-1}\prod_{k,l=1}^{n}\left[u_k^{(q)}-u_l^{(q+1)}+1-\frac{s}{N}\right]_r\prod_{k,l=1}^{n}\left[u_k^{(1)}-u_l^{(N)}+\frac{s}{N}\right]_r}\\
&\times\frac{\displaystyle\prod_{q=1}^{N}\prod_{1\leqq k<l\leqq m}\left[v_k^{(q)}-v_l^{(q)}\right]_{-r^*}\left[v_l^{(q)}-v_k^{(q)}-1\right]_{-r^*}}{\displaystyle\prod_{q=1}^{N-1}\prod_{k,l=1}^{n}\left[v_k^{(q)}-v_l^{(q+1)}+1-\frac{s}{N}\right]_{-r^*}\prod_{k,l=1}^{n}\left[v_k^{(1)}-v_l^{(N)}+\frac{s}{N}\right]_{-r^*}}\\
&\times\vartheta\left(\sum_{k=1}^{n}u_k^{(1)}\middle|\sum_{k=1}^{n}u_k^{(2)}\middle|\cdots\middle|\sum_{k=1}^{n}u_k^{(N)}\right)\vartheta\left(\sum_{k=1}^{m}v_k^{(1)}\middle|\sum_{k=1}^{m}v_k^{(2)}\middle|\cdots\middle|\sum_{k=1}^{m}v_k^{(N)}\right)\Bigg|_{v_j=u-\frac{r^*}{2}}.
\end{aligned}
\tag{4.206}
$$

Here the contours $C_{i,j}^{(t)}$ and $C_{i,j}^{(t)}$ are characterized by

$$
\begin{aligned}
C_{i,j}^{(t)}:\ &|x^{2-\frac{2s}{N}}z_k^{(t-1)}|,|x^{\frac{2s}{N}}z_k^{(t+1)}|<|z_i^{(t)}|<|x^{-2r-\frac{2s}{N}}z_k^{(t-1)}|,|x^{-2-2r+\frac{2s}{N}}z_k^{(t+1)}|,(1\leqq k\leqq n)\\
&|x^{-2r+3-\frac{2s}{N}}w_l^{(t-1)}|,|x^{-2r+1+\frac{2s}{N}}w_l^{(t+1)}|<|z^{(t)}|<|x^{-1-\frac{2s}{N}}w_i^{(t-1)}|,|x^{-3+\frac{2s}{N}}w_i^{(t+1)}|,(1\leqq l\leqq m),\\
&|x^{\frac{2s}{N}}z_l^{(q+1)}|<|z_k^{(q)}|<|x^{-2+\frac{2s}{N}}z_l^{(q+1)}|,(1\leqq q\leqq N;(q,k)\neq(t,i),(q,l)\neq(t-1,i)),\\
&|x^{\frac{2s}{N}}w_l^{(q+1)}|<|w_k^{(q)}|<|x^{-2+\frac{2s}{N}}w_l^{(q+1)}|,(1\leqq q\leqq N;(q,k)\neq(t,j),(q,l)\neq(t-1,j)),
\end{aligned}
\tag{4.207}
$$

$$
\begin{aligned}
\widetilde{C}^{(t)}:\ &|x^{2r+2-\frac{2s}{N}}z_k^{(t-1)}|,|x^{2r+\frac{2s}{N}}z_k^{(t+1)}|<|z_i^{(t)}|<|x^{-\frac{2s}{N}}z_k^{(t-1)}|,|x^{-2+\frac{2s}{N}}z_k^{(t+1)}|,(1\leqq k\leqq n),\\
&|x^{3-\frac{2s}{N}}w_l^{(t-1)}|,|x^{1+\frac{2s}{N}}w_l^{(t+1)}|<|z_i^{(t)}|<|x^{2r-1-\frac{2s}{N}}w_l^{(t-1)}|,|x^{2r-3+\frac{2s}{N}}w_l^{(t+1)}|,(1\leqq l\leqq m),\\
&|x^{\frac{2s}{N}}z_l^{(q+1)}|<|z_k^{(q)}|<|x^{-2+\frac{2s}{N}}z_l^{(q+1)}|,(1\leqq q\leqq N;(q,k)\neq(t,i),(q,l)\neq(t-1,i)),\\
&|x^{\frac{2s}{N}}w_l^{(q+1)}|<|w_k^{(q)}|<|x^{-2+\frac{2s}{N}}w_l^{(q+1)}|,(1\leqq q\leqq N;(q,k)\neq(t,j),(q,l)\neq(t-1,j)).
\end{aligned}
\tag{4.208}
$$

Let us change the variable $z_i^{(t)}\to x^{2r}z_i^{(t)}$ of the integrand $B^{(t)}(x^{-r}z_i^{(t)};\{z_k^{(q)}\}_{\substack{1\leqq q\leqq N\\1\leqq k\leqq n}},\{w_k^{(q)}\}_{\substack{1\leqq q\leqq N\\1\leqq k\neq j\leqq m}})$ and the contour $C_{i,j}^{(t)}$. Using periodic condition of theta function $[u+r]_r=-[u]_r$, we have $B^{(t)}(x^r z_i^{(t)}; \{z_k^{(q)}\}_{\substack{1\leqq q\leqq N\\ 1\leqq k\leqq n}}, \{w_k^{(q)}\}_{\substack{1\leqq q\leqq N\\ 1\leqq k\neq j\leqq m}})$ and the contour $\widetilde{C}_{i,j}^{(t)}$. Hence we have

$$\prod_{q=1}^{N}\prod_{k=1}^{n}\prod_{p=1}^{N}\prod_{\substack{l=1\\ l\neq t}}^{m}\oint_{C_{i,j}^{(t)}}\frac{dz^{(q)_k}}{2\pi\sqrt{-1}z_k^{(q)}}\frac{dw^{(p)_l}}{2\pi\sqrt{-1}w_l^{(q)}}B_{i,j}^{(t)}\left(x^{-r}z_i^{(t)}; \{z_j^{(q)}\}_{\substack{1\leqq q\leqq N\\ 1\leqq k\leqq n}}, \{w^{(q)}\}_{\substack{1\leqq q\leqq N\\ 1\leqq l\neq j\leqq m}}\right)$$
$$=\prod_{q=1}^{N}\prod_{k=1}^{n}\prod_{p=1}^{N}\prod_{\substack{l=1\\ l\neq t}}^{m}\oint_{C_{i,j}^{(t)}}\frac{dz^{(q)_k}}{2\pi\sqrt{-1}z_k^{(q)}}\frac{dw^{(p)_l}}{2\pi\sqrt{-1}w_l^{(q)}}B_{i,j}^{(t)}\left(x^{r}z_i^{(t)}; \{z_j^{(q)}\}_{\substack{1\leqq q\leqq N\\ 1\leqq k\leqq n}}, \{w^{(q)}\}_{\substack{1\leqq q\leqq N\\ 1\leqq l\neq j\leqq m}}\right). \tag{4.209}$$

Therefore we have shown the commutation relation $[\mathcal{G}_m^*, \mathcal{G}_n] = 0$. Generalization to generic parameter $0 < \mathrm{Re}(r) < 1$ and $s \in \mathbb{C}$ should be understood as analytic continuation. Q.E.D.

5. Dynkin-Automorphism Invariance

In this section we consider the Dynkin-automorphism invariance of the integrals of motion.

5.1. Dynkin-Automorphism Invariance. The integrals of motion are invariant under the action of the Dynkin-automorphism.

Theorem 7. *The local integrals of motion $\mathcal{I}_n$, $\mathcal{I}_n^*$ are invariant under the action of Dynkin-automorphism η*

$$\eta(\mathcal{I}_n) = \mathcal{I}_n, \quad \eta(\mathcal{I}_n^*) = \mathcal{I}_n^*, \quad (n \in \mathbb{N}). \tag{5.210}$$

Theorem 8. *The nonlocal integrals of motion $\mathcal{G}_n$, $\mathcal{G}_n^*$ are invariant ynder the action of Dynkin-automorphism η*

$$\eta(\mathcal{G}_n) = \mathcal{G}_n, \quad \eta(\mathcal{G}_n^*) = \mathcal{G}_n^*, \quad (n \in \mathbb{N}). \tag{5.211}$$

This theorem plays an important role in proof of the commutation relation $[\mathcal{I}_m, \mathcal{G}_n] = 0$.

5.2. Proof of Dynkin-Automorphism Invariance $\eta(\mathcal{I}_n) = \mathcal{I}_n$. In this section we show Dynkin-Automorphism Invariance $\eta(\mathcal{I}_n) = \mathcal{I}_n$, by using Laurent series formulae $\mathcal{I}_n = [\prod_{j<k} s(z_k/z_j)\mathcal{O}_n(z_1, \cdots, z_n)]$. We have $\eta^N = id$. Let us set the functions $h_{J,K}^{\eta^p,\eta^q}(z)$ for $0 \leqq J \leqq K \leqq N$, $0 \leqq p \leqq J-1$, $0 \leqq q \leqq K-1$,

$$\begin{aligned} h_{J,K}^{\eta^p,\eta^q}(z) = & \prod_{j=1}^{J-p}\prod_{k=1}^{K-q} h_{11}(x^{-K+J+2(k-j)+\frac{2s}{N}(q-p)}z) \\ & \times \prod_{j=J-p+1}^{J}\prod_{k=K-q+1}^{K} h_{11}(x^{-K+J+2(k-j)+\frac{2s}{N}(q-p)}z) \\ & \times \prod_{j=1}^{J-p}\prod_{k=K-q+1}^{K} h_{11}(x^{-K+J+2(k-j)+\frac{2s}{N}(q-p)-2s}z) \\ & \times \prod_{j=J-p+1}^{J}\prod_{k=1}^{K-q} h_{11}(x^{-K+J+2(k-j)+\frac{2s}{N}(q-p)+2s}z). \end{aligned} \tag{5.212}$$

Here we have set $h_{1,1}(z) = h(u)$ for $z = x^{2u}$. We use notation $h_{J,K}^{\eta^0,\eta^0}(z) = h_{J,K}^{id,id}(z) = h_{J,K}(z)$.

Proof of Theorem 7 Let us study from the invariance of $\mathcal{I}_2 = [s(z_2/z_1)\mathcal{O}_2(z_1, z_2)]_{1,z_1z_2}$. The action of η is given by

$$\begin{aligned} & \eta\left([h_{1,1}(z_2/z_1)T_1(z_1)T_1(z_2)]_{1,z_1,z_2}\right) \\ = & [h_{1,1}(z_2/z_1)T_1(z_1)T_1(z_2)]_{1,z_1,z_2} + [h_{1,1}(z_2/z_1)(\Lambda_1(x^{-s}z_1) - \Lambda_1(x^s z_1))\sum_{j=2}^{N}\Lambda_j(x^s z_2)]_{1,z_1,z_2} \\ & + [h_{1,1}(z_2/z_1)\sum_{j=2}^{N}\Lambda_j(x^s z_1)(\Lambda_1(x^{-s}z_2) - \Lambda_1(x^s z_2))]_{1,z_1,z_2}. \end{aligned} \tag{5.213}$$

By using the relation

$$h_{1,1}(z) - h_{1,1}(x^{2s}z) = c_{11}(\delta(x^2 z) - \delta(x^{2r-2+2s}z)), \tag{5.214}$$
$$c_{11} = -\frac{(x^2; x^{2s})_\infty (x^{2r-2}; x^{2s})_\infty (x^{2s-2}; x^{2s})_\infty (x^{2s-2r+2}; x^{2s})_\infty}{(x^{2r-4}; x^{2s})_\infty (x^{2s}; x^{2s})_\infty (x^{2s}; x^{2s})_\infty (x^{2s-2r+4}; x^{2s})_\infty},$$

we have

$$\begin{aligned} & [h_{1,1}(z_2/z_1)\Lambda_1(x^{-s}z_1)\sum_{j=2}^{N}\Lambda_j(x^s z_2)]_{1,z_1z_2} = [h_{1,1}(z_2/z_1)\Lambda_1(z_1)\sum_{j=2}^{N}\Lambda_j(z_2)]_{1,z_1z_2} \\ & +cs_{11}(x^{-2})[\delta(x^2 z_2/z_1) : \Lambda_1(x^{-2s}z_1)\sum_{j=2}^{N}\Lambda_j(x^{-2}z_1) :]_{1,z_1z_2}, \end{aligned} \tag{5.215}$$

$$\begin{aligned} & [h_{1,1}(z_2/z_1)\sum_{j=2}^{N}\Lambda_j(x^s z_1)\Lambda_1(x^{-s}z_2)]_{1,z_1z_2} = [h_{1,1}(z_2/z_1)\sum_{j=2}^{N}\Lambda_j(z_1)\Lambda_1(z_2)]_{1,z_1z_2} \\ & -cs_{11}(x^{-2})[\delta(x^{2-2s}z_2/z_1) : \Lambda_1(x^{-2}z_1)\sum_{j=2}^{N}\Lambda_j(z_1) :]_{1,z_1z_2}, \end{aligned} \tag{5.216}$$

Here we have used

$$\delta(x^{2r-2+2s}z_2/z_1)\Lambda_1(x^{-s}z_1)\Lambda_j(x^sz_2)=\delta(x^{2r-2}z_2/z_1)\Lambda_j(x^sz_1)\Lambda_1(x^{-s}z_2)=0. \tag{5.217}$$

Summing up every terms, we have

$$\begin{aligned}&\eta\left([h_{1,1}(z_2/z_1)T_1(z_1)T_1(z_2)]_{1,z_1,z_2}\right)=[h_{1,1}(z_2/z_1)T_1(z_1)T_1(z_2)]_{1,z_1,z_2}\\&+cs(x^{-2})[\delta(x^2z_2/z_1)\eta(T_2(x^{-1}z_1))]_{1,z_1z_2}-cs(x^{-2})[\delta(x^2z_2/z_1)T_2(x^{-1}z_1)]_{1,z_1z_2}.\end{aligned} \tag{5.218}$$

Summing up every termes of $\eta([s(z_2/z_1)\mathcal{O}_2(z_1,z_2)]_{1,z_1z_2})$, we conclude $\eta(\mathcal{I}_2)=\mathcal{I}_2$. Next we study $\eta(\mathcal{I}_3)=\mathcal{I}_3$. We use weakly sense equation for the basic operator $\Lambda_j(z)$

$$\begin{aligned}&g_{1,1}\left(\frac{z_2}{z_1}\right)\Lambda_j(z_1)\Lambda_i(z_2)-g_{1,1}\left(\frac{z_1}{z_2}\right)\Lambda_i(z_2)\Lambda_j(z_1)\\&\sim c\delta\left(\frac{x^2z_2}{z_1}\right):\Lambda_i(z_2)\Lambda_j(z_1):,(i<j).\end{aligned} \tag{5.219}$$

We have

$$\begin{aligned}&\eta\left(\left[\prod_{1\leqq j<k\leqq 3}h_{1,1}(z_k/z_j)T_1(z_1)T_1(z_2)T_1(z_3)\right]_{1,z_1z_2z_3}\right)\\&=\Bigg[\prod_{1\leqq j<k\leqq 3}h_{1,1}(z_k/z_j)T_1(z_1)T_1(z_2)T_1(z_3)\\&-cs(x^{-2})h_{1,2}(x^{-1}z_2/z_1)T_1(z_1)T_2(x^{-1}z_2)\delta(x^2z_3/z_2)\\&-cs(x^{-2})h_{1,2}(x^{-1}z_1/z_2)T_1(z_2)T_2(x^{-1}z_1)\delta(x^2z_3/z_1)\\&-cs(x^{-2})h_{1,2}(x^{-1}z_1/z_3)T_1(z_3)T_2(x^{-1}z_1)\delta(x^2z_2/z_1)\\&+cs(x^{-2})h_{1,2}^{id,\eta}(x^{-1}z_2/z_1)T_1(z_1)\eta(T_2(x^{-1}z_2))\delta(x^2z_3/z_2)\\&+cs(x^{-2})h_{1,2}^{id,\eta}(x^{-1}z_1/z_2)T_1(z_2)\eta(T_2(x^{-1}z_1))\delta(x^2z_3/z_1)\\&+cs(x^{-2})h_{1,2}^{id,\eta}(x^{-1}z_1/z_3)T_1(z_3)\eta(T_2(x^{-1}z_1))\delta(x^2z_2/z_1)\\&+c^2s(x^{-2})^2s(x^{-4})\Delta(x^3)\delta(x^2z_1/z_2)\delta(x^2z_3/z_1)\eta^2(T_3(z_1))\\&+c^2s(x^{-2})^2s(x^{-4})\Delta(x^3)\delta(x^2z_1/z_3)\delta(x^2z_2/z_1)\eta^2(T_3(z_1))\\&+c^2s(x^{-2})^2s(x^{-4})\Delta(x^3)\delta(x^2z_2/z_1)\delta(x^2z_3/z_2)\eta^2(T_3(z_2))\\&+c^2s(x^{-2})^2s(x^{-4})\Delta(x^3)(\delta(x^2z_1/z_2)\delta(x^2z_3/z_1)+\delta(x^2z_1/z_3)\delta(x^2z_2/z_1))T_3(z_1)\\&-c^2s(x^{-2})^2s(x^{-4})\Delta(x^3)(\delta(x^2z_1/z_2)\delta(x^2z_3/z_1)+\delta(x^2z_1/z_3)\delta(x^2z_2/z_1))\eta(T_3(z_1))\Bigg]_{1,z_1z_2z_3}.\end{aligned} \tag{5.220}$$

$$\begin{aligned}&\eta([cs(x^{-2})h_{1,2}(x^{-1}z_2/z_1)T_1(z_1)T_2(x^{-1}z_2)\delta(x^2z_3/z_2)]_{1,z_1z_2z_3})\\&=[cs(x^{-2})h_{1,2}^{id,\eta}(x^{-1}z_2/z_1)T_1(z_1)\eta(T_2(x^{-1}z_2))\delta(x^2z_3/z_2)\\&+c^2s(x^{-2})^2s(x^{-4})\Delta(x^3)\delta(x^2z_3/z_2)\delta(x^2z_2/z_1)\eta^2(T_3(z_2))]_{1,z_1z_2z_3},\end{aligned} \tag{5.221}$$

$$\eta([\delta(x^2z_1/z_2)\delta(x^2z_3/z_1)T_3(z_1)]_{1,z_1z_2z_3})=[\delta(x^2z_1/z_2)\delta(x^2z_3/z_1)\eta(T_3(z_1))]_{1,z_1z_2z_3}. \tag{5.222}$$

Summing up every term of $\eta([s(z_2/z_1)s(z_3/z_1)s(z_3/z_2)\mathcal{O}_3(z_1,z_2,z_3)]_{1,z_1z_2z_3})$, we have $\eta(\mathcal{I}_3)=\mathcal{I}_3$. We consider the case of general $\mathcal{I}_n$. The action of η is given by

$$
\begin{aligned}
&\eta\left(\left[\prod_{1\leqq j<k\leqq n} h(z_k/z_j)T_1(z_1)T_1(z_2)T_1(z_3)\cdots T_1(z_n)\right]_{1,z_1\cdots z_n}\right)=\sum_{\substack{\alpha_1,\alpha_2,\cdots,\alpha_N\geqq 0\\ \alpha_1+2\alpha_2+\cdots+N\alpha_N=n}}\\
&\times \sum_{\substack{\alpha_1^{(1)}=\alpha_1\\ \alpha_2^{(1)},\alpha_2^{(2)}\geqq 0,\ \alpha_2^{(1)}+\alpha_2^{(2)}=\alpha_2\\ \alpha_3^{(1)},\alpha_3^{(2)},\alpha_3^{(3)}\geqq 0,\ \alpha_3^{(1)}+\alpha_3^{(2)}+\alpha_3^{(3)}=\alpha_3\\ \cdots\\ \alpha_N^{(1)},\alpha_N^{(2)},\cdots,\alpha_N^{(N)}\geqq 0,\ \alpha_N^{(1)}+\alpha_N^{(2)}+\cdots+\alpha_N^{(N)}=\alpha_N}} \quad \sum_{\substack{\{A_j^{(t,q)}\}_{1\leqq q\leqq t\leqq N,\ 1\leqq j\leqq \alpha_t^{(q)}}\\ A_j^{(t,q)}\subset\{1,2,\cdots,n\},\ |A_j^{(t,q)}|=t,\ \cup_{t=1}^N\cup_{q=1}^t\cup_{j=1}^{\alpha_t^{(q)}}A_j^{(t,q)}\\ Min(A_1^{(t,q)})<Min(A_2^{(t,q)})<\cdots<Min(A_{\alpha_s^{(t)}}^{(t,q)})}}\\
&\times \sum_{\substack{\widetilde{A_j^{(3,3)}}\subset A_j^{(3,3)},\ |\widetilde{A_j^{(3,3)}}|=2\\ \widetilde{A_j^{(4,3)}}\subset A_j^{(4,3)},\ \widetilde{A_j^{(4,4)}}\subset A_j^{(4,4)},\ |\widetilde{A_j^{(4,3)}}|=3,\ |\widetilde{A_j^{(4,4)}}|=2\\ \cdots\\ \widetilde{A_j^{(N,3)}}\subset A_j^{(N,3)},\ \widetilde{A_j^{(N,4)}}\subset A_j^{(N,4)},\cdots,\widetilde{A_j^{(N,N)}}\subset A_j^{(N,N)},\ |\widetilde{A_j^{(N,3)}}|=N-2,\ |\widetilde{A_j^{(N,4)}}|=N-3,\ \cdots,|\widetilde{A_j^{(N,N)}}|=2}}\\
&\times (-1)^{\sum_{t=1}^{[\frac{N}{2}]}\sum_{u=1}^t\alpha_{2t}^{(2u-1)}+\sum_{t=1}^{[\frac{N-1}{2}]}\sum_{u=1}^t\alpha_{2t+1}^{(2u)}}\prod_{t=2}^N\left(c^{t-1}\prod_{u=1}^{t-2}\varDelta(x^{2u+1})^{t-u-1}\prod_{u=1}^{t-1}s(x^{-2u})^{t-u}\right)^{\alpha_t}\\
&\times\left[\prod_{t=1}^N\prod_{q=1}^t\prod_{\substack{j<k\\ j,k\in A_{Min}^{(t,q)}}} h_{t,t}^{\eta^{q-1},\eta^{q-1}}(z_k/z_j)\right.\\
&\times\prod_{1\leqq t<u\leqq N}\prod_{q=1}^t\prod_{p=1}^u\prod_{j\in A_{Min}^{(t,q)}}\prod_{k\in A_{Min}^{(u,p)}} h_{t,u}^{\eta^{q-1},\eta^{p-1}}(x^{u-t-2[\frac{u}{2}]+2[\frac{t}{2}]}z_k/z_j)\\
&\times\left\{\prod_{\overrightarrow{j\in A_{Min}^{(1,1)}}}T_1(z_j)\right\}\left\{\prod_{\overrightarrow{j\in A_{Min}^{(2,1)}}}T_2(x^{-1}z_j)\prod_{\overrightarrow{j\in A_{Min}^{(2,2)}}}\eta(T_2(x^{-1}z_j))\right\}\cdots\\
&\times\left\{\prod_{\overrightarrow{j\in A_{Min}^{(N,1)}}}T_N(x^{-1+N-2[\frac{N}{2}]}z_j)\cdots\prod_{\overrightarrow{j\in A_{Min}^{(N,N)}}}\eta^{N-1}(T_N(x^{-1+N-2[\frac{N}{2}]}z_j))\right\}\\
&\times\prod_{q=1}^2\prod_{j=1}^{\alpha_2^{(q)}}\delta\left(\frac{x^2z_{j_2}}{z_{j_1}}\right)\prod_{t=3}^N\prod_{q=1}^2\prod_{\substack{j=1\\ j_1=A_{j,1}^{(t,q)}\\ \cdots\\ j_t=A_{j,t}^{(t,q)}}}^{\alpha_t^{(q)}}\sum_{\substack{\sigma\in S_t\\ \sigma(1)=1}}\prod_{\substack{u=1\\ u\neq[\frac{t}{2}]+1}}^t\delta\left(\frac{x^2z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}}\right)\\
&\times\prod_{t=3}^N\prod_{q=3}^t\prod_{\substack{j=1\\ j_1=\widetilde{A_{j,1}^{(t,q)}}\\ \cdots\\ j_{t-q+2}=\widetilde{A_{j,t-q+2}^{(t,q)}}}}^{\alpha_t^{(q)}}\prod_{\substack{k_1,k_2,\cdots,k_{q-2}\in A_j^{(t,q)}-\widetilde{A_j^{(t,q)}}\\ k_1<k_2<\cdots<k_{q-2}}}\sum_{\substack{\sigma\in S_{t-q+2}\\ \sigma(1)=1}}\prod_{\substack{u=1\\ u\neq[\frac{t-q}{2}]+2}}^{t-q+2}\delta\left(\frac{x^2z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}}\right)\sum_{\tau\in S_{q-2}}\\
&\times\prod_{t-q\in 2\mathbb{Z}}\delta\left(\frac{x^2z_{k_{\tau([\frac{q+1}{2}])}}}{z_{j_{\sigma([\frac{t-q}{2}]+2)}}}\right)\delta\left(\frac{x^2z_{j_{\sigma([\frac{t-q}{2}]+3)}}}{z_{k_{\tau(1)}}}\right)\prod_{u=[\frac{q+3}{2}]}^{q-2}\delta\left(\frac{x^2z_{k_{\tau(u)}}}{z_{k_{\tau(u-1)}}}\right)\prod_{u=2}^{[\frac{q-1}{2}]}\delta\left(\frac{x^2z_{k_{\tau(u-1)}}}{z_{k_{\tau(u)}}}\right)\\
&\times\left.\prod_{t-q+1\in 2\mathbb{Z}}\delta\left(\frac{x^2z_{k_{\tau(1)}}}{z_{j_{\sigma([\frac{t-q}{2}]+2)}}}\right)\delta\left(\frac{x^2z_{j_{\sigma([\frac{t-q}{2}]+3)}}}{z_{k_{\tau([\frac{q+1}{2}])}}}\right)\prod_{u=2}^{[\frac{q-1}{2}]}\delta\left(\frac{x^2z_{k_{\tau(u)}}}{z_{k_{\tau(u-1)}}}\right)\prod_{u=[\frac{q+3}{2}]}^{q-2}\delta\left(\frac{x^2z_{k_{\tau(u-1)}}}{z_{k_{\tau(u)}}}\right)\right]_{1,z_1\cdots z_n}.
\end{aligned}
\tag{5.223}
$$

Here we have set $A_{Min}^{(t,q)}=\{Min(A_1^{(t,q)}),Min(A_2^{(t,q)}),\cdots,Min(A_{\alpha_t^{(q,t)}}^{(t,q)})\}$ for $q=0,1$, and have set $A_{Min}^{(t,q)}=\{Min(\widetilde{A_1^{(t,q)}}),Min(\widetilde{A_2^{(t,q)}}),\cdots,Min(\widetilde{A_{\alpha_t^{(q,t)}}^{(t,q)}})\}$ for $q=2,\cdots,t$. Here we have set $A_{j,1}^{(u,t)},A_{j,2}^{(u,t)},\cdots,A_{j,u}^{(u,t)}$ such that $A_j^{(u,t)}=\{A_{j,1}^{(u,t)}<A_{j,2}^{(u,t)}<\cdots<A_{j,u}^{(u,t)}\}$, and have set $\widetilde{A_{j,1}^{(u,t)}},\cdots,\widetilde{A_{j,N+2-t}^{(u,t)}}$ such that $A_j^{(u,t)}=\{\widetilde{A_{j,1}^{(u,t)}}<\widetilde{A_{j,2}^{(u,t)}}<\cdots<\widetilde{A_{j,N+2-t}^{(u,t)}}\}$.

We give the action of η for more general case. We prepare notations. Let us set $\beta_1,\beta_2,\cdots,\beta_N\geqq 0$ such that $\beta_1+2\beta_2+3\beta_3+\cdots+N\beta_N=n$. Let us set subset $B_j^{(t)}\subset\{1,2,\cdots,n\}$, $(1\leqq t\leqq N, 1\leqq j\leqq\beta_t)$ such that $|B_j^{(t)}|=t$, $\cup_{t=1}^N\cup_{j=1}^{\alpha_t}B_j^{(t)}=\{1,2,\cdots,n\}$ and $Min(B_1^{(t)})<Min(B_2^{(t)})<\cdots<Min(B_{\alpha_t}^{(t)})$. Let us set the index $B_{j,k}^{(t)}=j_k$ for $B_j^{(t)}=\{j_1,j_2,\cdots,j_t|j_1<j_2<\cdots<j_t\}$, $(1\leqq t\leqq N, 1\leqq j\leqq\alpha_t)$, and $B_{Min}^{(t)}=\{B_{1,1}^{(t)},B_{2,1}^{(t)},\cdots,B_{\alpha_t,1}^{(t)}\}$. The action of η is given by followings.

$$\eta\left(\left[\prod_{\overrightarrow{j\in B_{Min}^{(1)}}}T_1(z_j)\prod_{\overrightarrow{j\in B_{Min}^{(2)}}}T_2(x^{-1}z_j)\cdots\prod_{\overrightarrow{j\in B_{Min}^{(N)}}}T_N(x^{-1+N-2[\frac{N}{2}]}z_j)\right.\right.$$

$$\times\prod_{t=1}^N\left((-c)^{t-1}\prod_{u=1}^{t-1}\Delta(x^{2u+1})^{t-u-1}\right)^{\beta_t}\prod_{t=2}^N\prod_{\substack{j=1\\ j_1=B_{j,1}^{(t)}\\ \cdots\\ j_t=B_{j,t}^{(t)}}}^{\beta_t}\sum_{\substack{\sigma\in S_t\\ \sigma(1)=1}}\prod_{\substack{u=1\\ u\neq[\frac{t}{2}]+1}}^{t}\delta\left(\frac{x^2z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}}\right)$$

$$\left.\left.\times\prod_{t=1}^N\prod_{\substack{j<k\\ j,k\in B_{Min}^{(t)}}}g_{t,t}\left(\frac{z_k}{z_j}\right)\prod_{1\leqq t<u\leqq N}\prod_{\substack{j\in B_{Min}^{(t)}\\ k\in B_{Min}^{(u)}}}g_{t,u}\left(x^{u-t-2[\frac{u}{2}]+2[\frac{t}{2}]}\frac{z_k}{z_j}\right)\right]_{1,z_1\cdots z_n}\right)$$

$$=\sum_{\substack{\alpha_1,\alpha_2,\cdots,\alpha_N\geqq 0\\ \alpha_1+2\alpha_2+\cdots+N\alpha_N=n}}\sum_{\substack{\alpha_1^{(1)}=\alpha_1\\ \alpha_2^{(1)},\alpha_2^{(2)}\geqq 0,\ \alpha_2^{(1)}+\alpha_2^{(2)}=\alpha_2\\ \alpha_3^{(1)},\alpha_3^{(2)},\alpha_3^{(3)}\geqq 0,\ \alpha_3^{(1)}+\alpha_3^{(2)}+\alpha_3^{(3)}=\alpha_3\\ \cdots\\ \alpha_N^{(1)},\alpha_N^{(2)},\cdots,\alpha_N^{(N)}\geqq 0,\ \alpha_N^{(1)}+\alpha_N^{(2)}+\cdots+\alpha_N^{(N)}=\alpha_N}}\sum_{\substack{\{A_j^{(t,q)}\}_{1\leqq q\leqq t\leqq N,\ 1\leqq j\leqq\alpha_t^{(q)}}\\ A_j^{(t,q)}\subset\{1,2,\cdots,n\},\ |A_j^{(t,q)}|=t,\ \cup_{t=1}^N\cup_{q=1}^t\cup_{j=1}^{\alpha_t^{(q)}}A_j^{(t,q)}\\ Min(A_1^{(t,q)})<Min(A_2^{(t,q)})<\cdots<Min(A_{\alpha_s^{(t)}}^{(t,q)})}}$$

$$\times\sum_{\substack{\widetilde{A_j^{(3,3)}}\subset A_j^{(3,3)},\ |\widetilde{A_j^{(3,3)}}|=2\\ \widetilde{A_j^{(4,3)}}\subset A_j^{(4,3)},\ \widetilde{A_j^{(4,4)}}\subset A_j^{(4,4)},\ |\widetilde{A_j^{(4,3)}}|=3,\ |\widetilde{A_j^{(4,4)}}|=2\\ \cdots\\ \widetilde{A_j^{(N,3)}}\subset A_j^{(N,3)},\ \widetilde{A_j^{(N,4)}}\subset A_j^{(N,4)},\cdots,\widetilde{A_j^{(N,N)}}\subset A_j^{(N,N)},\ |\widetilde{A_j^{(N,3)}}|=N-2,\ |\widetilde{A_j^{(N,4)}}|=N-3,\ \cdots,|\widetilde{A_j^{(N,N)}}|=2}}$$

$$
\times \sum_{\substack{\{B_j^{(N-1)}\}_{1\leqq j\leqq \beta_{N-1}} \subset \{A_j^{(N,2)}\}_{1\leqq j \leqq \alpha_N^{(2)}} \\ \{B_j^{(N)}\}_{1\leqq j\leqq \beta_N} \subset \{A_j^{(N,1)}\}_{1\leqq j\leqq \alpha_N^{(1)}}}} \sum_{\substack{\{B_j^{(2)}\}_{1\leqq j\leqq \beta_2} \subset \{\widetilde{A_j^{(t,t)}}\}_{3\leqq t\leqq N, 1\leqq j\leqq \alpha_t^{(t)}} \\ \{B_j^{(3)}\}_{1\leqq j\leqq \beta_3} \subset \{\widetilde{A_j^{(t,t-1)}}\}_{4\leqq t\leqq N, 1\leqq j\leqq \alpha_t^{(t-1)}} \\ \cdots \\ \{B_j^{(N-2)}\}_{1\leqq j\leqq \beta_{N-2}} \subset \{\widetilde{A_j^{(N,3)}}\}_{1\leqq j\leqq \alpha_N^{(3)}}}}
$$

$$
\times (-1)^{\sum_{t=1}^{[\frac{N}{2}]}\sum_{u=1}^{t}\alpha_{2t}^{(2u-1)}+\sum_{t=1}^{[\frac{N-1}{2}]}\sum_{u=1}^{t}\alpha_{2t+1}^{(2u)}+\sum_{j=2}^{N}\beta_j} \prod_{t=2}^{N}\left(c^{t-1}\prod_{u=1}^{t-2}\varDelta(x^{2u+1})^{t-u-1}\prod_{u=1}^{t-1}s(x^{-2u})^{t-u}\right)^{\alpha_t}
$$

$$
\times \left[\prod_{t=1}^{N}\prod_{q=1}^{t}\prod_{\substack{j<k \\ j,k\in A_{Min}^{(t,q)}}} h_{t,t}^{\eta^{q-1},\eta^{q-1}}(z_k/z_j) \prod_{1\leqq t<u\leqq N}\prod_{q=1}^{t}\prod_{p=1}^{u}\prod_{j\in A_{Min}^{(t,q)}}\prod_{k\in A_{Min}^{(u,p)}} h_{t,u}^{\eta^{q-1},\eta^{p-1}}(x^{u-t-2[\frac{u}{2}]+2[\frac{t}{2}]}z_k/z_j)\right.
$$

$$
\times \left\{\prod_{\overrightarrow{j\in A_{Min}^{(1,1)}}} T_1(z_j)\right\}\left\{\prod_{\overrightarrow{j\in A_{Min}^{(2,1)}}} T_2(x^{-1}z_j)\prod_{\overrightarrow{j\in A_{Min}^{(2,2)}}}\eta(T_2(x^{-1}z_j))\right\}\cdots
$$

$$
\times \left\{\prod_{\overrightarrow{j\in A_{Min}^{(N,1)}}} T_N(x^{-1+N-2[\frac{N}{2}]}z_j)\cdots\prod_{\overrightarrow{j\in A_{Min}^{(N,N)}}}\eta^{N-1}(T_N(x^{-1+N-2[\frac{N}{2}]}z_j))\right\}
$$

$$
\times \prod_{q=1}^{2}\prod_{j=1}^{\alpha_2^{(q)}}\delta\left(\frac{x^2z_{j_2}}{z_{j_1}}\right)\prod_{t=3}^{N}\prod_{q=1}^{2}\prod_{\substack{j=1 \\ j_1=A_{j,1}^{(t,q)} \\ \cdots \\ j_t=A_{j,t}^{(t,q)}}}^{\alpha_t^{(q)}}\sum_{\substack{\sigma\in S_t \\ \sigma(1)=1}}\prod_{\substack{u=1 \\ u\neq[\frac{t}{2}]+1}}^{t}\delta\left(\frac{x^2z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}}\right)
$$

$$
\times \prod_{t=3}^{N}\prod_{q=3}^{t}\prod_{\substack{j=1 \\ j_1=\widetilde{A_{j,1}^{(t,q)}} \\ \cdots \\ j_{t-q+2}=\widetilde{A_{j,t-q+2}^{(t,q)}}}}^{\alpha_t^{(q)}}\prod_{\substack{k_1,k_2,\cdots,k_{q-2}\in A_j^{(t,q)}-\widetilde{A_j^{(t,q)}} \\ k_1<k_2<\cdots<k_{q-2}}}\sum_{\substack{\sigma\in S_{t-q+2} \\ \sigma(1)=1}}\prod_{\substack{u=1 \\ u\neq[\frac{t-q}{2}]+2}}^{t-q+2}\delta\left(\frac{x^2z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}}\right)\sum_{\tau\in S_{q-2}}
$$

$$
\times \prod_{t-q\in 2\mathbb{Z}}\delta\left(\frac{x^2z_{k_{\tau([\frac{q+1}{2}])}}}{z_{j_{\sigma([\frac{t-q}{2}]+2)}}}\right)\delta\left(\frac{x^2z_{j_{\sigma([\frac{t-q}{2}]+3)}}}{z_{k_{\tau(1)}}}\right)\prod_{u=[\frac{q+3}{2}]}^{q-2}\delta\left(\frac{x^2z_{k_{\tau(u)}}}{z_{k_{\tau(u-1)}}}\right)\prod_{u=2}^{[\frac{q-1}{2}]}\delta\left(\frac{x^2z_{k_{\tau(u-1)}}}{z_{k_{\tau(u)}}}\right)
$$

$$
\times \prod_{t-q+1\in 2\mathbb{Z}}\delta\left(\frac{x^2z_{k_{\tau(1)}}}{z_{j_{\sigma([\frac{t-q}{2}]+2)}}}\right)\delta\left(\frac{x^2z_{j_{\sigma([\frac{t-q}{2}]+3)}}}{z_{k_{\tau([\frac{q+1}{2}])}}}\right)\prod_{u=2}^{[\frac{q-1}{2}]}\delta\left(\frac{x^2z_{k_{\tau(u)}}}{z_{k_{\tau(u-1)}}}\right)\prod_{u=[\frac{q+3}{2}]}^{q-2}\delta\left(\frac{x^2z_{k_{\tau(u-1)}}}{z_{k_{\tau(u)}}}\right)\Bigg]_{1,z_1\cdots z_n} \qquad (5.224)
$$

We note that differnces between the equations (5.223) and (5.224) is only the signature and the restriction condition

$$
\sum_{\substack{\{B_j^{(N-1)}\}_{1\leqq j\leqq \beta_{N-1}} \subset \{A_j^{(N,2)}\}_{1\leqq j \leqq \alpha_N^{(2)}} \\ \{B_j^{(N)}\}_{1\leqq j\leqq \beta_N} \subset \{A_j^{(N,1)}\}_{1\leqq j\leqq \alpha_N^{(1)}}}} \sum_{\substack{\{B_j^{(2)}\}_{1\leqq j\leqq \beta_2} \subset \{\widetilde{A_j^{(t,t)}}\}_{3\leqq t\leqq N, 1\leqq j\leqq \alpha_t^{(t)}} \\ \cdots \\ \{B_j^{(N-2)}\}_{1\leqq j\leqq \beta_{N-2}} \subset \{\widetilde{A_j^{(N,3)}}\}_{1\leqq j\leqq \alpha_N^{(3)}}}} .
$$

Hence, summing up every terms of $\eta([\prod_{1\leqq j<k\leqq n} s(z_k/z_j)\mathcal{O}_n(z_1,z_2,\cdots,z_n)]_{1,z_1\cdots z_n})$, we have shown $\eta(\mathcal{I}_n)=\mathcal{I}_n$. Q.E.D.

5.3. Proof of Dynkin-Automorphism Invariance $\eta(\mathcal{G}_n)=\mathcal{G}_n$. In this section we show Dynkin-automorphism invariance $\eta(\mathcal{G}_n)=\mathcal{G}_n$.

Proof of Theorem 8 For reader's convenience, we explain $\eta(\mathcal{G}_1)=\mathcal{G}_1$ at first. We have the action of η as following.

$$\eta\left(\prod_{t=1}^{N}\oint_C \frac{dz^{(t)}}{2\pi\sqrt{-1}z^{(t)}} \frac{F_1(z^{(1)})F_2(z^{(2)})\cdots F_N(z^{(N)})\vartheta(u^{(1)}|u^{(2)}|\cdots|u^{(N)})}{\prod_{t=1}^{N-1}\left[u^{(t)}-u^{(t+1)}+1-\frac{s}{N}\right]_r\left[u^{(1)}-u^{(N)}+\frac{s}{N}\right]_r}\right)$$
$$=\prod_{t=1}^{N}\oint_C \frac{dz^{(t)}}{2\pi\sqrt{-1}z^{(t)}} \frac{\eta(\vartheta(u^{(1)}|u^{(2)}|\cdots|u^{(N)}))}{\prod_{t=2}^{N-1}\left[u^{(t)}-u^{(t+1)}+1-\frac{s}{N}\right]_r\left[u^{(N)}-u^{(1)}+1-\frac{s}{N}\right]_r\left[u^{(2)}-u^{(1)}+\frac{s}{N}\right]_r}$$
$$\times F_2(z^{(1)})F_3(z^{(2)})\cdots F_N(z^{(N-1)})F_1(z^{(N)}). \tag{5.225}$$

Here we have used $\eta(F_1(z_1)F_2(z_2)\cdots F_N(z_N))=F_2(z_1)\cdots F_N(z_{N-1})F_1(z_N)$. Let us change variables $u^{(1)}\to u^{(N)}, u^{(2)}\to u^{(1)}, u^{(2)}\to u^{(1)},\cdots,u^{(N)}\to u^{(N-1)}$, and move $F_1(z^{(1)})$ from the right to the left. We have

$$\prod_{t=1}^{N}\oint_C \frac{dz^{(t)}}{2\pi\sqrt{-1}z^{(t)}} \frac{F_1(z^{(1)})F_2(z^{(2)})\cdots F_N(z^{(N)})\eta(\vartheta(u^{(2)}|u^{(3)}|\cdots|u^{(N)}|u^{(1)}))}{\prod_{t=1}^{N-1}\left[u^{(t)}-u^{(t+1)}+1-\frac{s}{N}\right]_r\left[u^{(1)}-u^{(N)}+\frac{s}{N}\right]_r}. \tag{5.226}$$

We conclude $\eta(\mathcal{G}_1)=\mathcal{G}_1$ from theta property $\eta(\vartheta(u^{(2)}|u^{(3)}|\cdots|u^{(N)}|u^{(1)}))=\vartheta(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$. Let us show $\eta(\mathcal{G}_m)=\mathcal{G}_m$. After changing the variables $u_j^{(1)}\to u_j^{(N)}$, $u_j^{(2)}\to u_j^{(1)}$, $u_j^{(3)}\to u_j^{(2)},\cdots$ $u_j^{(N)}\to u_j^{(N-1)}$, we have $\eta(\mathcal{G}_m)$ as following.

$$\prod_{t=1}^{N}\prod_{j=1}^{m}\oint_C \frac{dz_j^{(t)}}{2\pi\sqrt{-1}z_j^{(t)}}F_2(z_1^{(1)})\cdots F_2(z_m^{(1)})F_3(z_1^{(2)})\cdots F_3(z_m^{(2)})\cdots F_N(z_1^{(N-1)})\cdots F_N(z_m^{(N-1)})$$
$$\times F_1(z_1^{(N)})\cdots F_1(z_m^{(N)})\eta\left(\vartheta\left(\sum_{j=1}^{m}u_j^{(1)}\middle|\sum_{j=1}^{m}u_j^{(2)}\middle|\cdots\middle|\sum_{j=1}^{m}u_j^{(N)}\right)\right) \tag{5.227}$$
$$\times\frac{\prod_{t=1}^{N}\prod_{1\leqq i<j\leqq m}\left[u_i^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_i^{(t)}-1\right]_r}{\prod_{t=2}^{N-1}\prod_{i,j=1}^{m}\left[u_i^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_r\prod_{i,j=1}^{m}\left[u_i^{(N)}-u_j^{(1)}+1-\frac{s}{N}\right]_r\prod_{i,j=1}^{m}\left[u_i^{(1)}-u_j^{(2)}+\frac{s}{N}\right]_r}.$$

Here we have used

$$\begin{aligned}&\eta(F_1(z_1^{(1)})\cdots F_1(z_m^{(1)})\cdots F_{N-1}(z_1^{(N-1)})\cdots F_{N-1}(z_m^{(N-1)})F_N(z_1^{(N)})\cdots F_N(z_m^{(N)}))\\ &=F_2(z_1^{(1)})\cdots F_2(z_m^{(1)})F_3(z_1^{(2)})\cdots F_3(z_m^{(2)})\cdots F_N(z_1^{(N-1)})\cdots F_N(z_m^{(N-1)})F_1(z_1^{(N)})\cdots F_1(z_m^{(N)}).\end{aligned} \tag{5.228}$$

Let us change the variables $u_j^{(1)}\to u_j^{(2)}$, $u_j^{(2)}\to u_j^{(3)},\cdots\ u_j^{(N-1)}\to u_j^{(N)}$, $u_j^{(N)}\to u_j^{(1)}$, and move $F_1(z_1^{(N)})\cdots F_1(z_m^{(N)})$ from the right to the left. We have

$$\begin{aligned}&\prod_{t=1}^{N}\prod_{j=1}^{m}\oint_C\frac{dz_j^{(t)}}{2\pi\sqrt{-1}z_j^{(t)}}F_1(z_1^{(1)})\cdots F_1(z_m^{(1)})F_2(z_1^{(2)})\cdots F_2(z_m^{(2)})\cdots F_N(z_1^{(N)})\cdots F_N(z_m^{(N)})\\ &\times\frac{\displaystyle\prod_{t=1}^{N}\prod_{1\leqq i<j\leqq m}\left[u_i^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_i^{(t)}-1\right]_r}{\displaystyle\prod_{t=1}^{N-1}\prod_{i,j=1}^{m}\left[u_i^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_r\prod_{i,j=1}^{m}\left[u_i^{(1)}-u_j^{(N)}+\frac{s}{N}\right]_r}\\ &\times\eta\left(\vartheta\left(\sum_{j=1}^{m}u_j^{(2)}\left|\sum_{j=1}^{m}u_j^{(3)}\right|\cdots\left|\sum_{j=1}^{m}u_j^{(N)}\right|\sum_{j=1}^{m}u_j^{(1)}\right)\right).\end{aligned} \tag{5.229}$$

We conclude $\eta(\mathcal{G}_m)=\mathcal{G}_m$ from $\eta(\vartheta(u^{(1)}|\cdots|u^{(N)}))=\vartheta(u^{(N)}|u^{(1)})|\cdots|u^{(N-1)})$. Proof of $\eta(\mathcal{G}_m^*)=\mathcal{G}_m^*$ is given as the same manner as above. Q.E.D.

A. Normal Ordering

We summarize the normal orderings of the basic operators. For generic $\mathrm{Re}(s)>0$, $r\in\mathbb{C}$, we have

$$\Lambda_i(z_1)\Lambda_i(z_2)=::(1-z_2/z_1)\frac{(x^2z_2/z_1;x^{2s})_\infty(x^{2r+2s-2}z_2/z_1;x^{2s})_\infty(x^{2s-2r}z_2/z_1;x^{2s})_\infty}{(x^{2s-2}z_2/z_1;x^{2s})_\infty(x^{2r}z_2/z_1;x^{2s})_\infty(x^{2-2r}z_2/z_1;x^{2s})_\infty}, \tag{A.230}$$

$$\Lambda_i(z_1)\Lambda_j(z_2)=::\frac{(x^2z_2/z_1;x^{2s})_\infty(x^{-2r}z_2/z_1;x^{2s})_\infty(x^{2r-2}z_2/z_1;x^{2s})_\infty}{(x^{-2}z_2/z_1;x^{2s})_\infty(x^{2r}z_2/z_1;x^{2s})_\infty(x^{2-2r}z_2/z_1;x^{2s})_\infty}, \tag{A.231}$$

$$\Lambda_j(z_1)\Lambda_i(z_2)=::\frac{(x^{2+2s}z_2/z_1;x^{2s})_\infty(x^{2s-2r}z_2/z_1;x^{2s})_\infty(x^{2s+2r-2}z_2/z_1;x^{2s})_\infty}{(x^{2s-2}z_2/z_1;x^{2s})_\infty(x^{2s+2r}z_2/z_1;x^{2s})_\infty(x^{2s+2-2r}z_2/z_1;x^{2s})_\infty}. \tag{A.232}$$

$$\Lambda_j(z_1)F_j(z_2) = :: x^{-2r^*}\frac{(1-x^{r-2+\frac{2s}{N}j}z_2/z_1)}{(1-x^{-r+\frac{2s}{N}j}z_2/z_1)}, \tag{A.233}$$

$$F_j(z_1)\Lambda_j(z_2) = :: \frac{(1-x^{2-r-\frac{2s}{N}j}z_2/z_1)}{(1-x^{r-\frac{2s}{N}j}z_2/z_1)}, \tag{A.234}$$

$$\Lambda_{j+1}(z_1)F_j(z_2) = :: x^{2r^*}\frac{(1-x^{2-r+\frac{2s}{N}j}z_2/z_1)}{(1-x^{r+\frac{2s}{N}j}z_2/z_1)}, \tag{A.235}$$

$$F_j(z_1)\Lambda_{j+1}(z_2) = :: \frac{(1-x^{r-2-\frac{2s}{N}j}z_2/z_1)}{(1-x^{-r-\frac{2s}{N}j}z_2/z_1)}, \tag{A.236}$$

$$\Lambda_1(z_1)F_N(z_2) = :: x^{2r^*}\frac{(1-x^{2-r}z_2/z_1)}{(1-x^{r}z_2/z_1)}, \tag{A.237}$$

$$F_N(z_1)\Lambda_1(z_2) = :: \frac{(1-x^{r-2}z_2/z_1)}{(1-x^{-r}z_2/z_1)}, \tag{A.238}$$

$$\Lambda_N(z_1)F_N(z_2) = :: x^{-2r^*}\frac{(1-x^{r-2+2s}z_2/z_1)}{(1-x^{-r+2s}z_2/z_1)}, \tag{A.239}$$

$$F_N(z_1)\Lambda_N(z_2) = :: \frac{(1-x^{2-r-2s}z_2/z_1)}{(1-x^{r-2s}z_2/z_1)}, \tag{A.240}$$

$$\Lambda_j(z_1)E_j(z_2) = :: x^{2r}\frac{(1-x^{-r-1+\frac{2s}{N}j}z_2/z_1)}{(1-x^{r-1+\frac{2s}{N}j}z_2/z_1)}, \tag{A.241}$$

$$E_j(z_1)\Lambda_j(z_2) = :: \frac{(1-x^{r+1-\frac{2s}{N}j}z_2/z_1)}{(1-x^{-r+1-\frac{2s}{N}j}z_2/z_1)}, \tag{A.242}$$

$$\Lambda_{j+1}(z_1)E_j(z_2) = :: x^{-2r}\frac{(1-x^{r+1+\frac{2s}{N}j}z_2/z_1)}{(1-x^{-r+1+\frac{2s}{N}j}z_2/z_1)}, \tag{A.243}$$

$$E_j(z_1)\Lambda_{j+1}(z_2) = :: \frac{(1-x^{-r-1-\frac{2s}{N}j}z_2/z_1)}{(1-x^{r-1-\frac{2s}{N}j}z_2/z_1)}, \tag{A.244}$$

$$\Lambda_1(z_1)E_N(z_2) = :: x^{-2r}\frac{(1-x^{r+1}z_2/z_1)}{(1-x^{-r+1}z_2/z_1)}, \tag{A.245}$$

$$E_N(z_1)\Lambda_1(z_2) = :: \frac{(1-x^{-r-1}z_2/z_1)}{(1-x^{r-1}z_2/z_1)}, \tag{A.246}$$

$$\Lambda_N(z_1)E_N(z_2) = :: x^{2r}\frac{(1-x^{-r^*-2+2s}z_2/z_1)}{(1-x^{r^*+2s}z_2/z_1)}, \tag{A.247}$$

$$E_N(z_1)\Lambda_N(z_2) = :: \frac{(1-x^{r^*+2-2s}z_2/z_1)}{(1-x^{-r^*-2s}z_2/z_1)}, \tag{A.248}$$

$$E_j(z_1)F_j(z_2) = :: \frac{x^{(1-\frac{2s}{N})2j}}{z_1^2(1-xz_2/z_1)(1-x^{-1}z_2/z_1)}, \tag{A.249}$$

$$F_j(z_1)E_j(z_2) = :: \frac{x^{(1-\frac{2s}{N})2j}}{z_1^2(1-xz_2/z_1)(1-x^{-1}z_2/z_1)}, \tag{A.250}$$

$$E_j(z_1)F_{j+1}(z_2) = :: x^{(\frac{2s}{N}-1)j}z_1(1-x^{-1+\frac{2s}{N}}z_2/z_1), \tag{A.251}$$

$$F_{j+1}(z_1)E_j(z_2) = :: x^{(\frac{2s}{N}-1)(j+1)}z_1(1-x^{1-\frac{2s}{N}}z_2/z_1), \tag{A.252}$$

$$E_{j+1}(z_1)F_j(z_2) = :: x^{(\frac{2s}{N}-1)(j+1)}z_1(1-x^{-1+\frac{2s}{N}}z_2/z_1), \tag{A.253}$$

$$E_j(z_1)F_{j+1}(z_2) = :: x^{(\frac{2s}{N}-1)j}z_1(1-x^{1-\frac{2s}{N}}z_2/z_1). \tag{A.254}$$

For $\mathrm{Re}(r^*) > 0$ we have

$$E_j(z_1)E_j(z_2) = :: z_1^{\frac{2r}{r^*}}(1 - z_2/z_1)\frac{(x^{-2}z_2/z_1; x^{2r^*})_\infty}{(x^{2r^*+2}z_2/z_1; x^{2r^*})_\infty}, \tag{A.255}$$

$$E_j(z_1)E_{j+1}(z_2) = :: (x^{\frac{2s}{N}-j}z_1)^{\frac{r}{r^*}}\frac{(x^{2r-2+\frac{2s}{N}}\frac{z_2}{z_1}; x^{2r^*})_\infty}{(x^{\frac{2s}{N}-2}z_2/z_1; x^{2r^*})_\infty}, \tag{A.256}$$

$$E_{j+1}(z_1)E_j(z_2) = :: (x^{\frac{2s}{N}-j-1}z_1)^{\frac{r}{r^*}}\frac{(x^{2r-\frac{2s}{N}}\frac{z_2}{z_1}; x^{2r^*})_\infty}{(x^{-\frac{2s}{N}}z_2/z_1; x^{2r^*})_\infty}, \tag{A.257}$$

$$\mathcal{B}_j(z_1)E_j(z_2) = :: (x^{-j}z_1)^{\frac{2r}{r^*}}\frac{(x^{-r^*-2+\frac{2s}{N}j}z_2/z_1; x^{2r^*})_\infty}{(x^{3r^*+2+\frac{2s}{N}j}z_2/z_1; x^{2r^*})_\infty}, \tag{A.258}$$

$$E_j(z_1)\mathcal{B}_j(z_2) = :: (x^{(\frac{2s}{N}-1)j})^{\frac{2r}{r^*}}\frac{(x^{-r^*-2-\frac{2s}{N}j}z_2/z_1; x^{2r^*})_\infty}{(x^{3r^*+2-\frac{2s}{N}j}z_2/z_1; x^{2r^*})_\infty}, \tag{A.259}$$

$$\mathcal{B}_j(z_1)E_{j+1}(z_2) = :: x^{r(1-\frac{1}{N})}(x^{-j}z_1)^{-\frac{r}{r^*}}\frac{(x^{3r^*+\frac{2s}{N}(j+1)}z_2/z_1; x^{2r^*})_\infty}{(x^{r^*-2+\frac{2s}{N}(j+1)}z_2/z_1; x^{2r^*})_\infty}, \tag{A.260}$$

$$E_{j+1}(z_1)\mathcal{B}_j(z_2) = :: (x^{(\frac{2s}{N}-1)(j+1)}z_1)^{-\frac{r}{r^*}}\frac{(x^{r^*-\frac{2s}{N}(j+1)}z_2/z_1; x^{2r^*})_\infty}{(x^{-r^*+2-\frac{2s}{N}(j+1)}z_2/z_1; x^{2r^*})_\infty}, (1 \leqq j \leqq N-2), \tag{A.261}$$

$$E_N(z_1)\mathcal{B}_{N-1}(z_2) = :: (x^{2s-N}z_1)^{-\frac{r}{r^*}(1-\frac{1}{N})}\frac{(x^{r^*-2s}z_2/z_1; x^{2r^*})_\infty}{(x^{-r^*+2-2s}z_2/z_1; x^{2r^*})_\infty}, \tag{A.262}$$

$$\mathcal{B}_j(z_1)E_{j-1}(z_2) = :: x^{-r(1-\frac{1}{N})}(x^{-j}z_1)^{-\frac{r}{r^*}}\frac{(x^{r^*+2+\frac{2s}{N}(j-1)}z_2/z_1; x^{2r^*})_\infty}{(x^{-r^*+\frac{2s}{N}(j-1)}z_2/z_1; x^{2r^*})_\infty}, \tag{A.263}$$

$$E_{j-1}(z_1)\mathcal{B}_j(z_2) = :: (x^{(\frac{2s}{N}-1)(j-1)})^{-\frac{r}{r^*}}\frac{(x^{3r^*-\frac{2s}{N}(j-1)}z_2/z_1; x^{2r^*})_\infty}{(x^{r^*-2-\frac{2s}{N}(j-1)}z_2/z_1; x^{2r^*})_\infty}, (2 \leqq j \leqq N-1), \tag{A.264}$$

$$E_N(z_1)\mathcal{B}_1(z_2) = :: z_1^{-\frac{r}{r^*}(1-\frac{1}{N})}\frac{(x^{3r^*}z_2/z_1; x^{2r^*})_\infty}{(x^{r^*-2}z_2/z_1; x^{2r^*})_\infty}. \tag{A.265}$$

For $\mathrm{Re}(r^*) < 0$, we have

$$E_j(z_1)E_j(z_2) = :: z_1^{\frac{2r}{r^*}}(1 - z_2/z_1)\frac{(x^{2}z_2/z_1; x^{-2r^*})_\infty}{(x^{-2r^*-2}z_2/z_1; x^{-2r^*})_\infty}, \tag{A.266}$$

$$E_j(z_1)E_{j+1}(z_2) = :: (x^{\frac{2s}{N}-j}z_1)^{\frac{r}{r^*}}\frac{(x^{-2r^*-2+\frac{2s}{N}}\frac{z_2}{z_1}; x^{-2r^*})_\infty}{(x^{\frac{2s}{N}}z_2/z_1; x^{-2r^*})_\infty}, \tag{A.267}$$

$$E_{j+1}(z_1)E_j(z_2) = :: (x^{\frac{2s}{N}-j-1}z_1)^{\frac{r}{r^*}}\frac{(x^{-2r^*-\frac{2s}{N}}\frac{z_2}{z_1}; x^{-2r^*})_\infty}{(x^{2-\frac{2s}{N}}z_2/z_1; x^{-2r^*})_\infty}, \tag{A.268}$$

$$E_{j-1}(z_1)H_j(z_2) = :: (x^{(\frac{2s}{N}-1)(j-1)}z_1)^{-\frac{1}{r^*}}\frac{(x^{-r^*-2+\frac{2s}{N}}z_2/z_1; x^{-2r^*})_\infty}{(x^{-r^*+\frac{2s}{N}}z_2/z_1; x^{-2r^*})_\infty}, \tag{A.269}$$

$$E_j(z_1)H_j(z_2) = :: (x^{(\frac{2s}{N}-1)j}z_1)^{\frac{2}{r^*}}\frac{(x^{-r^*+2}z_2/z_1; x^{-2r^*})_\infty}{(x^{-r^*-2}z_2/z_1; x^{-2r^*})_\infty}, \tag{A.270}$$

$$H_j(z_1)E_j(z_2) = :: (x^{(\frac{2s}{N}-1)j}z_1)^{\frac{2}{r^*}} \frac{(x^{-r^*+2}z_2/z_1; x^{-2r^*})_\infty}{(x^{-r^*-2}z_2/z_1; x^{-2r^*})_\infty}, \tag{A.271}$$

$$H_j(z_1)E_{j+1}(z_2) = :: (x^{(\frac{2s}{N}-1)j}z_1)^{-\frac{1}{r^*}} \frac{(x^{-r^*-2+\frac{2s}{N}}z_2/z_1; x^{-2r^*})_\infty}{(x^{-r^*+\frac{2s}{N}}z_2/z_1; x^{-2r^*})_\infty}. \tag{A.272}$$

For Re$(r) > 0$ we have

$$F_j(z_1)F_j(z_2) = :: x^{\frac{2r^*}{r}}(1-z_2/z_1) \frac{(x^2 z_2/z_1; x^{2r})_\infty}{(x^{2r-2}z_2/z_1; x^{2r})_\infty}, \tag{A.273}$$

$$F_j(z_1)F_{j+1}(z_2) = :: (x^{\frac{2s}{N}-j}z_1)^{-\frac{r^*}{r}} \frac{(x^{2r-2+\frac{2s}{N}}z_2/z_1; x^{2r})_\infty}{(x^{\frac{2s}{N}}z_2/z_1; x^{2r})_\infty}, \tag{A.274}$$

$$F_{j+1}(z_1)F_j(z_2) = :: (x^{\frac{2s}{N}-j-1}z_1)^{-\frac{r^*}{r}} \frac{(x^{2r-\frac{2s}{N}}z_2/z_1; x^{2r})_\infty}{(x^{2-\frac{2s}{N}}z_2/z_1; x^{2r})_\infty}, \tag{A.275}$$

$$\mathcal{A}_j(z_1)F_j(z_2) = :: (x^{-j}z_1)^{\frac{2r^*}{r}} \frac{(x^{-r+2+\frac{2s}{N}j}z_2/z_1; x^{2r})_\infty}{(x^{3r-2+\frac{2s}{N}j}z_2/z_1; x^{2r})_\infty}, \tag{A.276}$$

$$F_j(z_1)\mathcal{A}_j(z_2) = :: (x^{(\frac{2s}{N}-1)j}z_1)^{\frac{2r^*}{r}} \frac{(x^{-r+2-\frac{2s}{N}j}z_2/z_1; x^{2r})_\infty}{(x^{3r-2-\frac{2s}{N}j}z_2/z_1; x^{2r})_\infty}, \tag{A.277}$$

$$\mathcal{A}_j(z_1)F_{j+1}(z_2) = :: x^{-r^*(1-\frac{1}{N})}(x^{-j}z_1)^{-\frac{r^*}{r}} \frac{(x^{r-2+\frac{2s}{N}(j+1)}z_2/z_1; x^{2r})_\infty}{(x^{-r+\frac{2s}{N}(j+1)}z_2/z_1; x^{2r})_\infty}, \tag{A.278}$$

$$F_{j+1}(z_1)\mathcal{A}_j(z_2) = :: (x^{(\frac{2s}{N}-1)(j+1)}z_1)^{-\frac{r^*}{r}} \frac{(x^{3r-\frac{2s}{N}(j+1)}z_2/z_1; x^{2r})_\infty}{(x^{r+2-\frac{2s}{N}(j+1)}z_2/z_1; x^{2r})_\infty}, (1 \leqq j \leqq N-2),$$

$$F_N(z_1)\mathcal{A}_{N-1}(z_2) = :: (x^{2s-N}z_1)^{-\frac{r^*}{r}(1-\frac{1}{N})} \frac{(x^{3r-2s}z_2/z_1; x^{2r})_\infty}{(x^{r+2-2s}z_2/z_1; x^{2r})_\infty}, \tag{A.279}$$

$$\mathcal{A}_j(z_1)F_{j-1}(z_2) = :: x^{r^*(1-\frac{1}{N})}(x^{-j}z_1)^{-\frac{r^*}{r}} \frac{(x^{3r+\frac{2s}{N}(j-1)}z_2/z_1; x^{2r})_\infty}{(x^{r+2+\frac{2s}{N}(j-1)}z_2/z_1; x^{2r})_\infty}, \tag{A.280}$$

$$F_{j-1}(z_1)\mathcal{A}_j(z_2) = :: (x^{(\frac{2s}{N}-1)(j-1)}z_1)^{-\frac{r^*}{r}} \frac{(x^{r-2-\frac{2s}{N}(j-1)}z_2/z_1; x^{2r})_\infty}{(x^{-r-\frac{2s}{N}(j-1)}z_2/z_1; x^{2r})_\infty}, \tag{A.281}$$

$$F_N(z_1)\mathcal{A}_1(z_2) = :: z_1^{-\frac{r^*}{r}(1-\frac{1}{N})} \frac{(x^{r-2}z_2/z_1; x^{2r})_\infty}{(x^{-r}z_2/z_1; x^{2r})_\infty}, \tag{A.282}$$

$$F_{j-1}(z_1)H_j(z_2) = :: (x^{(\frac{2s}{N}-1)(j-1)}z_1)^{\frac{1}{r}} \frac{(x^{r-2+\frac{2s}{N}}z_2/z_1; x^{2r})_\infty}{(x^{r+\frac{2s}{N}}z_2/z_1; x^{2r})_\infty}, \tag{A.283}$$

$$F_j(z_1)H_j(z_2) = :: (x^{(\frac{2s}{N}-1)j}z_1)^{-\frac{2}{r}} \frac{(x^{r+2}z_2/z_1; x^{2r})_\infty}{(x^{r-2}z_2/z_1; x^{2r})_\infty}, \tag{A.284}$$

$$H_j(z_1)F_j(z_2) = :: (x^{(\frac{2s}{N}-1)j}z_1)^{-\frac{2}{r}} \frac{(x^{r+2}z_2/z_1; x^{2r})_\infty}{(x^{r-2}z_2/z_1; x^{2r})_\infty}, \tag{A.285}$$

$$H_j(z_1)F_{j+1}(z_2) = :: (x^{(\frac{2s}{N}-1)j}z_1)^{\frac{1}{r}} \frac{(x^{r-2+\frac{2s}{N}}z_2/z_1; x^{2r})_\infty}{(x^{r+\frac{2s}{N}}z_2/z_1; x^{2r})_\infty}, \tag{A.286}$$

For Re$(r) < 0$ we have

$$F_j(z_1)F_j(z_2) = :: x^{\frac{2r^*}{r}}(1 - z_2/z_1)\frac{(x^{-2}z_2/z_1; x^{-2r})_\infty}{(x^{2-2r}z_2/z_1; x^{-2r})_\infty}, \tag{A.287}$$

$$F_j(z_1)F_{j+1}(z_2) = :: (x^{\frac{2s}{N}-j}z_1)^{-\frac{r^*}{r}}\frac{(x^{-2r+\frac{2s}{N}}z_2/z_1; x^{-2r})_\infty}{(x^{-2+\frac{2s}{N}}z_2/z_1; x^{-2r})_\infty}, \tag{A.288}$$

$$F_{j+1}(z_1)F_j(z_2) = :: (x^{\frac{2s}{N}-j-1}z_1)^{-\frac{r^*}{r}}\frac{(x^{-2r+2-\frac{2s}{N}}z_2/z_1; x^{-2r})_\infty}{(x^{-\frac{2s}{N}}z_2/z_1; x^{-2r})_\infty}, \tag{A.289}$$

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