The Elliptic Algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and the Deformation of W_N Algebra

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Abstract

After reviewing the recent results on the Drinfeld realization of the face type elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ by the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$, we investigate a fusion of the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$. The basic generating functions $\Lambda_j(z)$ $(j = 1, 2, .., N-1)$ of the deformed W_N algebra are derived explicitly.

1 Introduction

In a recent paper [1, 2, 3], we showed that the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ provides the Drinfeld realization of the face type elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ tensored by a Heisenberg algebra. Based on this fact, we defined the $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ counterparts of the intertwining operators of the $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ modules and obtained their free field realization in the level one representation. The resultant vertex operators, called the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$, are identified with the vertex operators of the RSOS model associated with $\widehat{\mathfrak{sl}}_N$ in the algebraic analysis formulation[5]. In general, we expect that the elliptic algebra $U_{q,p}(\mathfrak{g})$ with $\mathfrak{g}$ being an affine Lie algebra provides the Drinfeld realization for the elliptic quantum group $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ and enables us to perform an algebraic analysis of the $\mathfrak{g}$ type RSOS model.

On the other hand, the $\widehat{\mathfrak{sl}}_N$ RSOS model is known as an off-critical deformation of the W_N minimal model[6]. In this relation, it is remarkable that the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ in the $c = 1$ representation coincides with the algebra of the screening currents of the deformed W_N algebra [7, 8, 9]. In general, we expect that the elliptic algebra $U_{q,p}(\mathfrak{g})$ provides an algebra of screening currents of the deformation of the coset CFT associated with $(\mathfrak{g})_c \oplus (\mathfrak{g})_{r-c-2}/(\mathfrak{g})_{r-2}$[1, 2].

The purpose of this paper is to continue to find an explicit relation among the elliptic algebra $U_{q,p}(\mathfrak{g})$, the $\mathfrak{g}$ type RSOS model and the deformation of $W(\bar{\mathfrak{g}})$ algebra in the case $\mathfrak{g} = \widehat{\mathfrak{sl}}_N$. We here investigate a fusion of the type II vertex operator of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and its dual, and show that the generating functions of the deformed W_N algebra can be extracted from it. The idea of fusion of the vertex operators was used in [10, 11] to derive the generating function of the deformed Virasoro algebra (corresponding to the $\phi_{1,3}$ perturbation) from the ABF model in regime III, in [12]for the deformed W_N algebra with the central charge $c_N = (N-1)\left(1 - \frac{N(N+1)}{r(r-1)}\right)$ at special point $r = N + 2$ (the $\mathbb{Z}_N$ parafermion point) from the ABF model in regime II, and in [13] for the deformed Virasoro algebra (corresponding to the $\phi_{1,2}$ perturbation) from the dilute A_L model.

This article is organised as follows. In the next section, we briefly review the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ as the Drinfeld realization of the elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$. We give a summary of the results on the free field realization of the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ obtained in [3]. In the section 3, we discuss a fusion of the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and a derivation of the generators of the deformed W_N algebra.

Through this paper, we use the following symbols. $p = q^{2r}$, $p^* = pq^{-2c} = q^{2r^*}$ $(r^* = r - c;\ r, r^* \in \mathbb{R}_{>0})$,

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$

$$\Theta_p(z) = (z,p)_\infty (pz^{-1};p)_\infty (p;p)_\infty,$$
$$\{z\} = (z;p,q^{2N})_\infty, \qquad \{z\}^* = \{z\}|_{p\to p^*},$$
$$(z;t_1,\cdots,t_k)_\infty = \prod_{n_1,\cdots,n_k \geqq 0} (1 - z t_1^{n_1}\cdots t_k^{n_k}).$$

We also use the Jacobi theta functions

$$[v] = q^{\frac{v^2}{r}-v}\frac{\Theta_p(q^{2v})}{(p;p)_\infty^3}, \quad [v]^* = q^{\frac{v^2}{r^*}-v}\frac{\Theta_{p^*}(q^{2v})}{(p^*;p^*)_\infty^3},$$

which satisfy $[-v] = -[v]$ and the quasi-periodicity property

$$[v+r] = -[v], \quad [v+r\tau] = -e^{-\pi i\tau - \frac{2\pi i v}{r}}[v].$$

We take the normalization of the theta function to be

$$\oint_{C_0} \frac{dz}{2\pi i z}\frac{1}{[-v]} = 1,$$

where C_0 is a simple closed curve in the v-plane encircling $v=0$ anticlockwise. The same holds for $[v]^*$, with r replaced by r^*, except for the normalization

$$\oint_{C_0} \frac{dz}{2\pi i z}\frac{1}{[-v]^*} = \frac{[v]}{[v]^*}\Big|_{v\to 0}.$$

2 The Elliptic Algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$

2.1 Definition

Definition 2.1 (Elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$) *We define the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ to be the associative algebra of the currents $E_j(v)$, $F_j(v)$ $(1 \square j \square N-1)$ and $K_j(v)$ $(1 \square j \square N)$ satisfying the following relations.*

$$E_i(v_1)E_j(v_2) = \frac{[v_1 - v_2 + \frac{A_{ij}}{2}]^*}{[v_1 - v_2 - \frac{A_{ij}}{2}]^*}E_j(v_2)E_i(v_1), \tag{2.1}$$

$$F_i(v_1)F_j(v_2) = \frac{[v_1 - v_2 - \frac{A_{ij}}{2}]}{[v_1 - v_2 + \frac{A_{ij}}{2}]}F_j(v_2)F_i(v_1), \tag{2.2}$$

$$[E_i(v_1), F_j(v_2)] = \frac{\delta_{i,j}}{q-q^{-1}}\left(\delta(q^{-c}z_1/z_2)H_j^+\left(v_2+\frac{c}{4}\right) - \delta(q^c z_1/z_2)H_j^-\left(v_2-\frac{c}{4}\right)\right), \tag{2.3}$$

$$H_j^\pm\left(v \mp \frac{1}{2}\left(r - \frac{c}{2}\right)\right) = \square K_j\left(v + \frac{N-j}{2}\right)K_{j+1}\left(v+\frac{N-j}{2}\right)^{-1}, \tag{2.4}$$

$$K_j(v_1)K_j(v_2) = \rho(v_1 - v_2)K_j(v_2)K_j(v_1), \tag{2.5}$$

$$K_{j_1}(v_1)K_{j_2}(v_2) = \rho(v_1 - v_2)\frac{[v_1 - v_2 - 1]^*[v_1 - v_2]}{[v_1 - v_2]^*[v_1 - v_2 - 1]}K_{j_2}(v_2)K_{j_1}(v_1)$$
$$(1 \leqq j_1 < j_2 \leqq N), \tag{2.6}$$

$$K_j(v_1)E_j(v_2) = \frac{[v_1 - v_2 + \frac{j+r^*-N}{2}]^*}{[v_1 - v_2 + \frac{j+r^*-N}{2} - 1]^*}E_j(v_2)K_j(v_1), \tag{2.7}$$

$$K_{j+1}(v_1)E_j(v_2) = \frac{[v_1 - v_2 + \frac{j+r^*-N}{2}]^*}{[v_1 - v_2 + \frac{j+r^*-N}{2} + 1]^*}E_j(v_2)K_{j+1}(v_1), \tag{2.8}$$

$$K_{j_1}(v_1)E_{j_2}(v_2) = E_{j_2}(v_2)K_{j_1}(v_1) \qquad (j_1 \neq j_2, j_2 + 1), \tag{2.9}$$

$$K_j(v_1)F_j(v_2) = \frac{[v_1 - v_2 + \frac{j+r-N}{2} - 1]}{[v_1 - v_2 + \frac{j+r-N}{2}]}F_j(v_2)K_j(v_1), \tag{2.10}$$

$$K_{j+1}(v_1)F_j(v_2) = \frac{[v_1 - v_2 + \frac{j+r-N}{2} + 1]}{[v_1 - v_2 + \frac{j+r-N}{2}]}F_j(v_2)K_{j+1}(v_1), \tag{2.11}$$

$$K_{j_1}(v_1)F_{j_2}(v_2) = F_{j_2}(v_2)K_{j_1}(v_1) \qquad (j_1 \neq j_2, j_2 + 1), \tag{2.12}$$

$$\begin{aligned}
&z_1^{-\frac{1}{r^*}}\frac{(p^*q^2z_2/z_1;p^*)_\infty}{(p^*q^{-2}z_2/z_1;p^*)_\infty}\left\{(z_2/z)^{\frac{1}{r^*}}\frac{(p^*q^{-1}z/z_1;p^*)_\infty(p^*q^{-1}z/z_2;p^*)_\infty}{(p^*qz/z_1;p^*)_\infty(p^*qz/z_2;p^*)_\infty}E_i(v_1)E_i(v_2)E_j(v)\right.\\
&-[2]_q\frac{(p^*q^{-1}z/z_1;p^*)_\infty(p^*q^{-1}z_2/z;p^*)_\infty}{(p^*qz/z_1;p^*)_\infty(p^*qz_2/z;p^*)_\infty}E_i(v_1)E_j(v)E_i(v_2)\\
&\left.+(z/z_1)^{\frac{1}{r^*}}\frac{(p^*q^{-1}z_1/z;p^*)_\infty(p^*q^{-1}z_2/z;p^*)_\infty}{(p^*qz_1/z;p^*)_\infty(p^*qz_2/z;p^*)_\infty}E_j(v)E_i(v_1)E_i(v_2)\right\}+(z_1\leftrightarrow z_2)=0,
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
&z_1^{\frac{1}{r}}\frac{(pq^{-2}z_2/z_1;p)_\infty}{(pq^2z_2/z_1;p)_\infty}\left\{(z/z_2)^{\frac{1}{r}}\frac{(pqz/z_1;p)_\infty(pqz/z_2;p)_\infty}{(pq^{-1}z/z_1;p)_\infty(pq^{-1}z/z_2;p)_\infty}F_i(v_1)F_i(v_2)F_j(v)\right.\\
&-[2]_q\frac{(pqz/z_1;p)_\infty(pqz_2/z;p)_\infty}{(pq^{-1}z/z_1;p)_\infty(pq^{-1}z_2/z;p)_\infty}F_i(v_1)F_j(v)F_i(v_2)\\
&\left.+(z_1/z)^{\frac{1}{r}}\frac{(pqz_1/z;p)_\infty(pqz_2/z;p)_\infty}{(pq^{-1}z_1/z;p)_\infty(pq^{-1}z_2/z;p)_\infty}F_j(v)F_i(v_1)F_i(v_2)\right\}+(z_1\leftrightarrow z_2)=0 \quad (|i-j|=1).
\end{aligned} \tag{2.14}$$

Here $A = (A_{jk})$ is the Cartan matrix of $\mathfrak{sl}_N$. The constant $\square$ and the functions $\rho(v)$ are given by

$$\square = \frac{(p;p)_\infty(p^*q^2;p^*)_\infty}{(p^*;p^*)_\infty(pq^2;p)_\infty}, \tag{2.15}$$

$$\rho(v) = \frac{\rho^{+*}(v)}{\rho^+(v)}, \tag{2.16}$$

$$\rho^+(v) = q^{\frac{N-1}{N}}z^{\frac{N-1}{rN}}\frac{\{pq^2z\}\{pq^{2N-2}z\}\{1/z\}\{q^{2N}/z\}}{\{pz\}\{pq^{2N}z\}\{q^2/z\}\{q^{2N-2}/z\}}, \quad \rho^{+*}(v) = \rho^+(v)|_{r\to r^*}. \tag{2.17}$$

2.2 Realization of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$

The elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ can be realized by using the Drinfeld generators of $U_q(\widehat{\mathfrak{sl}}_N)$ and a Heisenberg algebra. Let $h_i,\ a^i_m,\ x^{\pm}_{i,n}$ $(i=1,\cdots,N-1 : m\in\mathbb{Z}_{\neq 0},\ n\in\mathbb{Z}),\ c,\ d$ be the standard Drinfeld generators of $U_q(\widehat{\mathfrak{sl}}_N)$[14]. Their generating functions $x^{\pm}_i(z)$, $\psi_i(z)$, $\varphi_i(z)$ are called the Drinfeld currents.

$$x^{\pm}_i(z)=\sum_{n\in\mathbb{Z}} x^{\pm}_{i,n}z^{-n}, \tag{2.18}$$

$$\psi_i(q^{\frac{c}{2}}z)=q^{h_i}\exp\left((q-q^{-1})\sum_{m>0}a_{i,m}z^{-m}\right), \tag{2.19}$$

$$\varphi_i(q^{-\frac{c}{2}}z)=q^{-h_i}\exp\left(-(q-q^{-1})\sum_{m>0}a_{i,-m}z^{m}\right) \qquad (i=1,\cdots,N-1). \tag{2.20}$$

Let us define the auxiliary currents $u^{\pm}_i(z,p)$ $(i=1,2,..,N-1)$ by

$$u^{+}_i(z,p)=\exp\left(\sum_{m>0}\frac{1}{[r^*m]_q}a_{i,-m}(q^r z)^m\right), \tag{2.21}$$

$$u^{-}_i(z,p)=\exp\left(-\sum_{m>0}\frac{1}{[rm]_q}a_{i,m}(q^{-r} z)^{-m}\right). \tag{2.22}$$

Definition 2.1 *We define "dressed" currents $e_i(z,p), f_i(z,p),\ \psi^{\pm}_i(z,p),\ (i=1,\cdots,N-1)$ by*

$$e_i(z,p)=u^{+}_i(z,p)x^{+}_i(z), \tag{2.23}$$

$$f_i(z,p)=x^{-}_i(z)u^{-}_i(z,p), \tag{2.24}$$

$$\psi^{+}_i(z,p)=u^{+}_i(q^{\frac{c}{2}}z,p)\psi_i(z)u^{-}_i(q^{-\frac{c}{2}}z,p), \tag{2.25}$$

$$\psi^{-}_i(z,p)=u^{+}_i(q^{-\frac{c}{2}}z,p)\varphi_i(z)u^{-}_i(q^{\frac{c}{2}}z,p). \tag{2.26}$$

Setting $b_{j,m}=\dfrac{[r^*m]_q}{[rm]_q}a_{j,m}$ (for $m>0$), $q^{c|m|}a_{j,m}$ (for $m<0$), we introduce new generators, B^j_m $(j=1,\cdots,N; m\in\mathbb{Z})$, according to the formula

$$-B^j_m+B^{j+1}_m=\frac{m}{[m]_q}b_{j,m}q^{(N-j)m}, \qquad \sum_{j=1}^{N}q^{2jm}B^j_m=0. \tag{2.27}$$

From this and the commutation relation of the Drinfeld generators $a_{j,m}$, we derive the following commutation relations.

$$[B^j_m,B^k_{m'}]=m\delta_{m+m',0}\frac{[r^*m]_q[cm]_q}{[rm]_q[m]_q[Nm]_q}\times\begin{cases}[(N-1)m]_q & (j=k)\\ -q^{-mN\mathrm{sgn(j-k)}}[m]_q & (j\neq k)\end{cases}, \tag{2.28}$$

for $m, m' \in \mathbb{Z}_{\neq 0},\ j, k = 1, \cdots, N$. Then we define new currents $k_j(z,p)$ $(1 \square j \square N)$ by

$$k_j(z,p) =: \exp\left(\sum_{m\neq 0} \frac{[m]_q}{m[r^* m]_q} B_m^j z^{-m}\right) : . \tag{2.29}$$

This yields the following decomposition.

$$\psi_j^{\pm}(q^{\pm(r-\frac{c}{2})}z, p) = \square\, q^{\pm h_j} k_j(q^{N-j}z,p) k_{j+1}(q^{N-j}z,p)^{-1}. \tag{2.30}$$

On the other hand, let ϵ_j $(1 \square j \square N)$ be the orthonormal basis in $\mathbb{R}^N$ with the inner product $\langle \epsilon_j, \epsilon_k \rangle = \delta_{j,k}$. Setting $\bar{\epsilon}_j = \epsilon_j - \epsilon,\ \ \epsilon = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \epsilon_j$, we have the weight lattice P of type $A_{N-1}^{(1)}$; $P = \oplus_{j=1}^{N} \mathbb{Z}\ \bar{\epsilon}_j$. Then, for example, the simple roots α_j $(1 \square j \square N-1)$ of $\mathfrak{sl}_N$ are given by $\alpha_j = -\bar{\epsilon}_j + \bar{\epsilon}_{j+1}$. Let us introduce operators h_α, β $(\alpha, \beta \in P)$ by

$$[h_{\bar{\epsilon}_j}, \bar{\epsilon}_k] = \langle \bar{\epsilon}_j, \bar{\epsilon}_k \rangle, \qquad [h_{\bar{\epsilon}_j}, h_{\bar{\epsilon}_k}] = 0 = [\bar{\epsilon}_j, \bar{\epsilon}_k], \tag{2.31}$$

$h_\alpha = \sum_j n_j h_{\bar{\epsilon}_j}$ for $\alpha = \sum_j n_j \bar{\epsilon}_j$ and $h_0 = 0$. Note that $\langle \bar{\epsilon}_j, \bar{\epsilon}_k \rangle = \delta_{j,k} - \frac{1}{N}$ and $[h_{\alpha_j}, \alpha_k] = 2\delta_{j,k} - \delta_{j,k+1} - \delta_{j,k-1} = A_{jk}$. Hence, we identify $h_{\alpha_j} = -h_{\bar{\epsilon}_j} + h_{\bar{\epsilon}_{j+1}}$ with h_j in the Drinfeld generators of $U_q(\widehat{\mathfrak{sl}}_N)$

Definition 2.2 *We define the (centrally extended) Heisenberg algebra $\mathbb{C}\{\hat{\mathcal{H}}\}$ as an associative algebra generated by $P_{\bar{\epsilon}_j},\ Q_{\bar{\epsilon}_j}$ $(1 \square j \square N)$ and η_j $(1 \square j \square N-1)$ with the relations*

$$[P_{\bar{\epsilon}_j}, Q_{\bar{\epsilon}_k}] = \langle \bar{\epsilon}_j, \bar{\epsilon}_k \rangle, \qquad [P_{\bar{\epsilon}_j}, P_{\bar{\epsilon}_k}] = 0, \tag{2.32}$$

$$[Q_{\bar{\epsilon}_j}, Q_{\bar{\epsilon}_k}] = \left(\frac{1}{r} - \frac{1}{r^*}\right) \mathrm{sgn}(j-k) \log\ q, \tag{2.33}$$

$$[Q_{\bar{\epsilon}_j}, \eta_k] = \frac{1}{r} \mathrm{sgn}(j-k) \log\ q, \tag{2.34}$$

$$[\eta_j, \eta_k] = \frac{1}{r} \mathrm{sgn}(j-k) \log\ q, \tag{2.35}$$

$$[P_{\bar{\epsilon}_j}, \eta_k] = 0, \qquad \sum_{j=1}^{N} \eta_j = 0, \tag{2.36}$$

$$[\eta_j, \alpha] = [P_{\bar{\epsilon}_j}, U_q(\widehat{\mathfrak{sl}}_N)] = [Q_{\bar{\epsilon}_j}, U_q(\widehat{\mathfrak{sl}}_N)] = [\eta_j, U_q(\widehat{\mathfrak{sl}}_N)] = 0. \tag{2.37}$$

Now we define the currents $E_j(v), F_j(v), H_j^{\pm}(v)$ $(1 \square j \square N-1)$ and $K_j(v)$ $(1 \square j \square N)$ by

$$E_j(v) = e_j(z,p) e^{\bar{\alpha}_j} e^{-Q_{\alpha_j}} (q^{-j+N}z)^{-\frac{P_{\alpha_j}-1}{r^*}}, \tag{2.38}$$

$$F_j(v) = f_j(z,p) e^{-\bar{\alpha}_j} (q^{-j+N}z)^{\frac{P_{\alpha_j}-1}{r}} (q^{-j+N}z)^{\frac{h_j}{r}}, \tag{2.39}$$

$$H_j^{\pm}(v) = \psi_j^{\pm}(z,p) q^{\mp h_j} e^{-Q_{\alpha_j}} (q^{-j+N\pm(r-\frac{c}{2})}z)^{(-\frac{1}{r^*}+\frac{1}{r})(P_{\alpha_j}-1)+\frac{1}{r}h_j}, \tag{2.40}$$

$$K_j(v) = k_j(z,p) e^{Q_{\bar{\epsilon}_j}} z^{(\frac{1}{r^*}-\frac{1}{r})P_{\bar{\epsilon}_j}} z^{-\frac{1}{r}h_{\bar{\epsilon}_j}+(\frac{1}{r^*}-\frac{1}{r})\frac{N-1}{2N}}, \tag{2.41}$$

where $\bar{\alpha}_j = -\eta_j + \eta_{j+1}$. Then it is easy to show that $E_j(v), F_j(v), H_j^{\pm}(v)$ and $K_j(v)$ satisfy the defining relations of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$.

2.3 RLL relation

We next discuss a relation between two elliptic algebras $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$. We construct a L-operator by using the half currents and show that it satisfies the dynamical RLL-relation which characterizes $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$. In order to construct a L-operator, we introduce the half currents $E^+_{l,j}(v), F^+_{j,l}(v)$ and $K^+_j(v)$. We use the following abbreviations

$$P_{j,l} = -P_{\bar{\epsilon}_j} + P_{\bar{\epsilon}_l} = P_{\alpha_j} + P_{\alpha_{j+1}} + \cdots + P_{\alpha_{l-1}} \tag{2.42}$$

$$h_{j,l} = -h_{\bar{\epsilon}_j} + h_{\bar{\epsilon}_l} = h_j + h_{j+1} + \cdots + h_{l-1} \tag{2.43}$$

for $j < l$. From the definition of $\mathbb{C}\{\hat{\mathcal{H}}\}$ and (2.38)-(2.41), we have

$$[K_j(v), P_{k,l}] = (\delta_{j,k} - \delta_{j,l})K_j(v) = [K_j(v), P_{k,l} + h_{k,l}], \tag{2.44}$$

$$[E_j(v), P_{k,l}] = (\delta_{j,k} + \delta_{j+1,l} - \delta_{j,l} - \delta_{j+1,k})E_j(v), \tag{2.45}$$

$$[F_j(v), P_{j,l} + h_{j,l}] = (\delta_{j,k} + \delta_{j+1,l} - \delta_{j,l} - \delta_{j+1,k})F_j(v), \tag{2.46}$$

$$[F_j(v), P_{k,l}] = 0 = [E_j(v), P_{k,l} + h_{k,l}]. \tag{2.47}$$

Now we define the half currents of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ as follows.

Definition 2.2 (Half currents) *We define the half currents $F^+_{j,l}(v), E^+_{l,j}(v), (1 \leq j < l \leq N)$ and $K^+_j(v)$ $(j = 1, \cdots, N)$ by*

$$K^+_j(v) = K_j\left(v + \frac{r+1}{2}\right) \quad (1 \leq j \leq N), \tag{2.48}$$

$$\begin{aligned} F^+_{j,l}(v) = a_{j,l} &\oint_{C(j,l)} \prod_{m=j}^{l-1} \frac{dz_m}{2\pi i z_m} F_{l-1}(v_{l-1})F_{l-2}(v_{l-2})\cdots F_j(v_j) \\ &\times \frac{[v - v_{l-1} + P_{j,l} + h_{j,l} + \frac{l-N}{2} - 1][1]}{[v - v_{l-1} + \frac{l-N}{2}][P_{j,l} + h_{j,l} - 1]} \\ &\times \prod_{m=j}^{l-2} \frac{[v_{m+1} - v_m + P_{j,m+1} + h_{j,m+1} - \frac{1}{2}][1]}{[v_{m+1} - v_m + \frac{1}{2}][P_{j,m+1} + h_{j,m+1}]}, \end{aligned} \tag{2.49}$$

$$\begin{aligned} E^+_{l,j}(v) = a^*_{j,l} &\oint_{C^*(j,l)} \prod_{m=j}^{l-1} \frac{dz_m}{2\pi i z_m} E_j(v_j)E_{j+1}(v_{j+1})\cdots E_{l-1}(v_{l-1}) \\ &\times \frac{[v - v_{l-1} - P_{j,l} + \frac{l-N}{2} + \frac{c}{2} + 1]^*[1]^*}{[v - v_{l-1} + \frac{l-N}{2} + \frac{c}{2}]^*[P_{j,l} - 1]^*} \\ &\times \prod_{m=j}^{l-2} \frac{[v_{m+1} - v_m - P_{j,m+1} + \frac{1}{2}]^*[1]^*}{[v_{m+1} - v_m + \frac{1}{2}]^*[P_{j,m+1} - 1]^*}. \end{aligned} \tag{2.50}$$

Here the integration contour $C(j,l)$ and $C^*(j,l)$ are given by

$$C(j,l) \quad : \quad |pq^{l-N}z| < |z_{l-1}| < |q^{l-N}z|,$$

$$
\begin{aligned}
& |pqz_{k+1}| < |z_k| < |qz_{k+1}|, && (2.51)\\
C^*(j,l) \quad : \quad & |p^* q^{l-N+c} z| < |z_{l-1}| < |q^{l-N+c} z|, \\
& |p^* q z_{k+1}| < |z_k| < |q z_{k+1}|, && (2.52)
\end{aligned}
$$

where $k = j, j+1, .., l-2$. The constants $a_{j,l}$ and $a^*_{j,l}$ are chosen to satisfy

$$
\frac{\square\ a_{j,l} a^*_{j,l}[1]}{q - q^{-1}} = 1. \tag{2.53}
$$

2.4 L-operator

Definition 2.3 (L-operator) *By using the half currents, we define the L-operator $\widehat{L}^+(v) \in \mathrm{End}(\mathbb{C}^N) \otimes U_{q,p}(\widehat{\mathfrak{sl}}_N)$ as follows.*

$$
\widehat{L}^+(u) = \begin{pmatrix} 1 & F^+_{1,2}(u) & F^+_{1,3}(u) & \cdots & F^+_{1,N}(u) \\ 0 & 1 & F^+_{2,3}(u) & \cdots & F^+_{2,N}(u) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & F^+_{N-1,N}(u) \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} K^+_1(u) & 0 & \cdots & 0 \\ 0 & K^+_2(u) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K^+_N(u) \end{pmatrix}
$$

$$
\times \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ E^+_{2,1}(u) & 1 & \ddots & & \vdots \\ E^+_{3,1}(u) & E^+_{3,2}(u) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ E^+_{N,1}(u) & E^+_{N,2}(u) & \cdots & E^+_{N,N-1}(u) & 1 \end{pmatrix}. \tag{2.54}
$$

Then by a direct comparison with the relations of the half currents, we conjecture the following statement.

Conjecture 2.3 *The L-operator $\widehat{L}^+(v)$ satisfies the following $RLL = LLR^*$ relation.*

$$
R^{+(12)}(v_1 - v_2, P + h) \widehat{L}^{+(1)}(v_1) \widehat{L}^{+(2)}(v_2) = \widehat{L}^{+(2)}(v_2) \widehat{L}^{+(1)}(v_1) R^{+*(12)}(v_1 - v_2, P). \tag{2.55}
$$

Here the R-matrix $R^+(v,P)$ is the image of the universal R-matrix $\mathcal{R}(r, \{s_j\})$ of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ in the evaluation representation $(\pi_{V,z} \otimes \pi_{V,1})$, $V \cong \mathbb{C}^N$, given by

$$
\begin{aligned}
R^+(v,s) &= \rho^+(v) \bar{R}(v,s), && (2.56)\\
\bar{R}(v,s) &= \sum_{j=1}^{N} E_{jj} \otimes E_{jj} + \sum_{1 \square j < l \square N} \left(b(v, s_{j,l}) E_{jj} \otimes E_{ll} + \bar{b}(v) E_{ll} \otimes E_{jj} \right) \\
&\quad + \sum_{1 \square j < l \square N} \left(c(v, s_{j,l}) E_{jl} \otimes E_{lj} + \bar{c}(v, s_{j,l}) E_{lj} \otimes E_{jl} \right), && (2.57)
\end{aligned}
$$

where $s_{j,l} = \sum_{m=j}^{l-1} s_j$ $(1 \square j < l \square N)$ *and*

$$b(u,s) = \frac{[s+1][s-1][u]}{[s]^2[u+1]}, \quad \bar{b}(u) = \frac{[u]}{[u+1]}, \tag{2.58}$$

$$c(u,s) = \frac{[1][s+u]}{[s][u+1]}, \quad \bar{c}(u,s) = \frac{[1][s-u]}{[s][u+1]}. \tag{2.59}$$

And $R^{+*}(v,s) = R^{+}(v,s)|_{r\to r^*}$. *Up to a gauge transformation,* $R^{+}(v,P)$ *coincides with the Boltzmann weight of the* $\widehat{\mathfrak{sl}}_N$ *RSOS model introduced in [6].*

The $c=1$ case, the statement was proved by using the free field realization [3].

Now let us define the modified L-operator $L^{+}(v,P)$ by

$$L+(v,P) = \widehat{L}+(v) \begin{pmatrix} e^{-Q_{\bar{\epsilon}_1}} & 0 & \cdots & 0 \\ 0 & e^{-Q_{\bar{\epsilon}_2}} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & e^{-Q_{\bar{\epsilon}_N}} \end{pmatrix} = \widehat{L}^{+}(v) \ \exp\left\{ \sum_{m=1}^{N} h^{(1)}_{\epsilon_m} Q_{\bar{\epsilon}_m} \right\}. \tag{2.60}$$

Here $h^{(1)}_{\epsilon_j} = h_{\epsilon_j} \otimes 1$, $h_{\epsilon_m} \equiv -E_{mm}$ (a $N \times N$ matrix unit). We then show that the modified L-operator depends on neither $Q_{\bar{\epsilon}_j}$ nor η_j and satisfies the dynamical RLL relation of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$[4].

Corollary 2.4

$$R^{+(12)}(v,P+h)L^{+(1)}(v_1,P)L^{+(2)}(v_2,P+h^{(1)}) = L^{+(2)}(v_2,P)L^{+(1)}(v_1,P+h^{(2)})R^{+*(12)}(v,P), \tag{2.61}$$

where $v = v_1 - v_2$.

Hence, we regard the elliptic currents $E_j(v)$, $F_j(v)$ $(1 \square j \square N-1)$ and $K_j(v)$ $(1 \square j \square N)$ in $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ as the Drinfeld realization of the elliptic algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ tensored by the Heisenberg algebra.

$$U_{q,p}(\widehat{\mathfrak{sl}}_N) = \mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N) \otimes_{\mathbb{C}\{P_{\bar{\epsilon}_1}, P_{\bar{\epsilon}_2}, .., P_{\bar{\epsilon}_{N-1}}\}} \mathbb{C}\{\hat{\mathcal{H}}\}. \tag{2.62}$$

3 Vertex Operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$

We here summarize a construction of the type II vertex operator of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and its dual vertex operator.

3.1 Definition

Let us first define an extension of the $U_q(\widehat{\mathfrak{sl}}_N)(\cong \mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N))$ modules by

$$\widehat{\mathcal{F}} = \bigoplus_{\mu_1,\cdots,\mu_N\in\mathbb{Z}} \mathcal{F}\otimes e^{\mu_1 Q_{\bar{\epsilon}_1}\cdots+\mu_N Q_{\bar{\epsilon}_N}}.$$

Let $\Psi^*_W(z,P)$ be the type II intertwining operator of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ [4]. We define the type II vertex operator $\widehat{\Psi}^*_W(z)$ of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ as the following extension.

$$\widehat{\Psi}^*_W(z) = \Psi^*_W(z,P)\exp\left\{\sum_{j=1}^N h_{\epsilon_j}Q_{\bar{\epsilon}_j}\right\} \qquad : W_z\otimes\widehat{\mathcal{F}}\longrightarrow\widehat{\mathcal{F}}'. \tag{3.1}$$

From the intertwining relation of the $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ intertwining operators, we derive the following relation for the new operator $\widehat{\Psi}^*_W(z)$.

$$\widehat{L}^{+(1)}_V(z_1)\widehat{\Psi}^{*(2)}_W(z_2) \;=\; \widehat{\Psi}^{*(2)}_W(z_2)\widehat{L}^{+(1)}_V(z_1)R^{+*(12)}_{VW}(v_1-v_2,P-h^{(1)}-h^{(2)}). \tag{3.2}$$

Let us consider the vector representation V of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$. We denote a basis of V by $\{\mathbf{v}_m\}_{m=1}^N$. In this representation, the R-matrix $R^+_{VV}(v,P)$ is given by $R^+(v,P)$ in (2.56) and the L-operator $\widehat{L}^+_V(z)$ by $\widehat{L}^+(z)$ in (2.54). We define the components of the vertex operators by

$$\widehat{\Psi}^*_V\left(q^{-c-1}z\right)(\mathbf{v}_m\otimes\cdot) = \Psi^*_m(z), \tag{3.3}$$

and the matrix elements of the L-operator $\widehat{L}^+(z)$ by

$$\widehat{L}^+(z)\mathbf{v}_j = \sum_{1\square m\square N}\mathbf{v}_m L^+(z)_{mj}. \tag{3.4}$$

3.2 Free field realizations

We here construct a free field realization of the vertex operators fixing $c=1$. Let α_j be the simple root operator. We make the standard central extension $[\alpha_j,\alpha_k]=\pi i A_{jk}$ and set $\hat{\alpha}_j=\alpha_j+\bar{\alpha}_j$, where $\bar{\alpha}_j$ is an element of the Heisenberg algebra $\mathbb{C}\{\hat{H}\}$. Then we have

Proposition 3.1 *The currents $E_j(v)$ and $F_j(v)$ given by*

$$E_j(v) \;=\; :\exp\left(-\sum_{m\neq 0}\frac{[rm]_q}{m[r^*m]_q}(-B^j_m+B^{j+1}_m)(q^{N-j}z)^{-m}\right): e^{\hat{\alpha}_j}z^{h_j}e^{-Q_{\alpha_j}}(q^{-j+N}z)^{-\frac{P_{\alpha_j}-1}{r^*}}, \tag{3.5}$$

$$F_j(v) \;=\; :\exp\left(\sum_{m\neq 0}\frac{1}{m}(-B^j_m+B^{j+1}_m)(q^{N-j}z)^{-m}\right): e^{-\hat{\alpha}_j}z^{-h_j}(q^{-j+N}z)^{\frac{P_{\alpha_j}-1}{r}+\frac{h_j}{r}}, \tag{3.6}$$

together with $H^\pm_j(v), K_j(v)$ given in (2.40)-(2.41) satisfy the commutation relations in Definition 2.1 for level $c=1$.

Now using this free field realization in (2.48)- (2.50), we obtain a realization of the half currents $E_j^+(v),\ F_j^+(v),\ K_j^+(v)$ as well as the L-operator $\widehat{L}^+(v)$ for $c=1$. Using the resultant L-operator in the "intertwining relation" (3.2), one can solve it for the II vertex operator.

Theorem 3.2 *The highest components of the type II vertex operator $\Psi_N^*(z)$ is realized in terms of a free field by*

$$\Psi_N^*(z) \;=\; :\exp\left(\sum_{m\neq 0}\frac{[rm]}{m[r^*m]}B_m^N z^{-m}\right): e^{-\bar{\Lambda}_{N-1}} z^{-h_{\bar{\epsilon}_N}} e^{Q_{\bar{\epsilon}_N}} z^{\frac{1}{r^*}P_{\bar{\epsilon}_N}} z^{(-1+\frac{1}{r^*})\frac{N-1}{2N}} q^{-\frac{N-1}{2N}}, \tag{3.7}$$

where

$$\bar{\Lambda}_{N-1} \;=\; \frac{1}{N}(\hat{\alpha}_1+2\hat{\alpha}_2+\cdots+(N-1)\hat{\alpha}_{N-1}). \tag{3.8}$$

The other components of the type II vertex $\Psi_j^*(z)$ $(j=1,\cdots,N)$ are given by

$$\begin{aligned}\Psi_j^*(z) \;&=\; a_{j,N}^*\oint_{C^*}\prod_{m=j}^{N-1}\frac{dz_m}{2\pi i z_m}E_j(v_j)\cdots E_{N-1}(v_{N-1})\Psi_N^*(v_N)\\ &\quad\times\prod_{m=j}^{N-1}\frac{[v_{m+1}-v_m-P_{j,m+1}+\frac{1}{2}]^*[1]^*}{[v_{m+1}-v_m+\frac{1}{2}]^*[P_{j,m+1}-1]^*}\\ &=\; a_{j,N}^*\oint_{C^*}\prod_{m=j}^{N-1}\frac{dz_m}{2\pi i z_m}\Psi_N^*(v_N)E_{N-1}(v_{N-1})\cdots E_j(v_j)\\ &\quad\times\prod_{m=j}^{N-1}\frac{[v_{m+1}-v_m-P_{j,m+1}+\frac{1}{2}]^*[1]^*}{[v_{m+1}-v_m-\frac{1}{2}]^*[P_{j,m+1}-1]^*}.\end{aligned} \tag{3.9}$$

The integration contour C^* is specified as follows.

$$|p^*q^{-1}z_{m+1}|,|q^{-1}z_{m+1}|<|z_m|<|qz_{m+1}|,|p^{*-1}qz_{m+1}| \qquad (j\ \square\ m\ \square\ N-1). \tag{3.10}$$

Here the integration variable z_m $(j\ \square\ m\ \square\ N-1)$ should encircle the poles $p^*q^{-1}z_{m+1},q^{-1}z_{m+1}$ but not the poles $p^{*-1}qz_{m+1},qz_{m+1}$.

We also have

Theorem 3.3 *The free field realizations of the type-II vertex operator $\Psi_\mu^*(z)$ satisfies the following commutation relation.*

$$\Psi_{j_1}^*(z_1)\Psi_{j_2}^*(z_2)=\sum_{j_1',j_2'=1}^{N}\Psi_{j_2'}^*(z_2)\Psi_{j_1'}^*(z_1)\ R_{j_1'j_2'}^{*j_1j_2}(v_1-v_2,P) \tag{3.11}$$

Here we set $R^*(v,P)=\mu^*(v)\bar{R}^*(v,P)$ with

$$\mu^*(v)=z^{(\frac{1}{r^*}-1)\frac{N-1}{N}}\frac{\{pq^{2N-2}z\}^*\{q^2z\}^*\{p/z\}^*\{q^{2N}/z\}^*}{\{pz\}^*\{q^{2N}z\}^*\{pq^{2N-2}/z\}^*\{q^2/z\}^*}. \tag{3.12}$$

The dual of the type II vertex operator is an operator satisfying

$$\Psi(z) \ : \mathcal{F} \to V_z \otimes \mathcal{F}'. \tag{3.13}$$

We define its components by

$$\Psi(z) = \sum_{j=1}^{N} \mathbf{v}_j \otimes \Psi_j(z). \tag{3.14}$$

The following inversion relations hold.

$$\sum_{j=1}^{N} \Psi_j(z)\Psi_j^*(z') = \frac{g'_N}{1-q^N\frac{z'}{z}}, \qquad \sum_{j=1}^{N} \Psi_j^*(z)\Psi_j(z') = \frac{g'_N}{1-q^N\frac{z'}{z}}, \tag{3.15}$$

where

$$g'_N = (-1)^{N-1}\frac{q^{\frac{r(1-N)}{2(r-1)}}}{(p^*;p^*)_\infty^{2N-3}(q^{-2};p^*)_\infty^N}\frac{(p;q^{2N},p^*)_\infty(q^{-2};q^{2N},p^*)_\infty}{(p^*;q^{2N},p^*)_\infty^2(q^{2N};q^{2N})_\infty}, \tag{3.16}$$

as $z' \to zq^{-N}$, as well as

$$\Psi_j(z)\Psi_k^*(z') = \delta_{j,k}\frac{g_N}{1-q^{-N}\frac{z'}{z}}, \tag{3.17}$$

$$g_N = (-1)^{N-1}q^{-\frac{r(1-N)}{2(r-1)}}\left(\frac{(p;p^*)_\infty}{(p^*;p^*)_\infty}\right)^N\frac{(pq^{2N};q^{2N},p^*)_\infty(q^{2N}q^{-2};q^{2N},p^*)_\infty}{(q^{2N}p^{-1}q^{-2};q^{2N},p^*)_\infty^2(q^{2N};q^{2N},p^*)_\infty}, \tag{3.18}$$

as $z' \to zq^N$. The free field realization is given as follows.

$$\begin{aligned}
\Psi_j(z) &= \oint_C \prod_{m=1}^{j-1}\frac{dz_m}{2\pi i z_m}\Psi_1(z)E_1(v_1)\cdots E_{j-1}(v_{j-1}) \\
&\qquad \times \prod_{m=1}^{j-1}\frac{[v_{m-1}-v_m-P_{m-1,j}+\frac{1}{2}]^*[1]^*}{[v_{m-1}-v_m-\frac{1}{2}]^*[P_{m-1,j}-1]^*} \\
&= \oint_C \prod_{m=1}^{j-1}\frac{dz_m}{2\pi i z_m}E_{j-1}(v_{j-1})\cdots E_1(v_1)\Psi_1(v) \\
&\qquad \times \prod_{m=1}^{j-1}\frac{[v_{m-1}-v_m-P_{-1,j}+\frac{1}{2}]^*[1]^*}{[v_{m-1}-v_m+\frac{1}{2}]^*[P_{m-1,j}-1]^*},
\end{aligned} \tag{3.19}$$

$$\Psi_1(z) = :\exp\left(-\sum_{m\neq 0}\frac{[rm]}{m[r^*m]}B_m^1(q^Nz)^{-m}\right): \ e^{\Lambda_1 z^{h_{\bar{\epsilon}_1}}}e^{-Q_{\bar{\epsilon}_1}}(q^Nz)^{-\frac{1}{r^*}P_{\bar{\epsilon}_1}+\frac{N-1}{2Nr^*}}z^{\frac{N-1}{2N}} \tag{3.20}$$

where $v = v_0$ and the integration contour C is specified by the condition : the poles $z_m = q^{-1}z_{m-1}p^{*n}$ $(n = 0,1,2,...)$ are inside and $z_m = qz_{m-1}p^{*-n}$ $(n = 0,1,2,...)$ are outside for $1 \square m \square j-1$.

Remark The free field realizations of the vertex operators in Theorem 3.2 and of the dual vertex operators are essentially the same as those of the $\widehat{\mathfrak{sl}}_N$ RSOS model obtained in [15].

4 Fusion of the Vertex Operators

We now consider the fusion of the type II vertex operator $\Psi_1^*(z_2)$ and its dual $\Psi_1(z_1)$. Namely, we consider a product $\Psi_1(z_1)\Psi_1^*(z_2)$ and investigate the limits to the fusion points $z_1 = q^{-N}p^{*n}z_2$ ($n = 0, 1, 2, .., N$), where the contour in (3.9) for w_1 gets pinches.

For example, let us consider the case $n = 1$. If we take residues for the poles $w_{N-1} = q^{-1}z_2,\ w_{j-1} = q^{-1}w_j$ ($j = N-1, N-2, .., 3$), the limit $z_1 \to q^{-N}p^*z_2$ causes pinches in the contour for w_1 at two points $w_1 = q^{-(N-1)}z_2,\ q^{-(N-1)}p^*z_2$. Similarly, for $1 \square l \square N-2$, if we take residues at the poles $w_{N-1} = q^{-1}z_2,\ w_{j-1} = q^{-1}w_j$ ($j = N-1, N-2, .., N-l+1$), $w_{N-l} = q^{-1}p^*w_{N-l+1},\ w_{j-1} = q^{-1}w_j$ ($j = N-l-1, N-l-2, .., 3$), the same limit $z_1 \to q^{-N}p^*z_2$ causes a pinch in the contour for w_1 at a point $w_1 = q^{-(N-1)}p^*z_2$. Hence in the limit $z_1 \to q^{-N}p^*z_2$, we obtain totally N terms of contributions from the residues at the N pinching points. Similar consideration leads us to the following results. As $z_1 \to q^{-N}p^{*n}z_2$,

$$
\begin{aligned}
&\Psi_1(z_1)\Psi_1^*(z_2) \\
&= \frac{1}{1-q^{-N}p^{*n}\frac{z_2}{z_1}}\Big\{C_n\ \tilde{T}_n(q^{(n-1)r^*}z_2) \\
&\quad + \sum\nolimits'_{1\square j_1\square j_2\square ..\square j_n\square N} C_{j_1,j_2,..,j_n} : \Lambda_{j_1}(z_2q^{(2n-1)r^*})\Lambda_{j_1}(z_2q^{(2n-3)r^*})\cdots\Lambda_{j_n}(z_2q^{r^*}) :\Big\}.
\end{aligned}
$$

Here

$$
\tilde{T}_n(z) = \sum\nolimits_{1\square j_1<j_2<..<j_n\square N} : \Lambda_{j_1}(z_2q^{(n-1)r^*})\Lambda_{j_2}(z_2q^{(n-3)r^*})\cdots\Lambda_{j_n}(z_2q^{-(n-1)r^*}) :, \tag{4.1}
$$

$$
\Lambda_j(z) =: \exp\left(\sum_{m\neq 0}\frac{q^{rm}-q^{-rm}}{m}B_m^j z^{-m}\right) : q^{-2P_{\bar{\epsilon}_j}}p^{*h_{\bar{\epsilon}_j}}q^{\frac{2(1-N)}{N}}p^{*-\frac{1}{N}-j}, \tag{4.2}
$$

$$
\begin{aligned}
C_n &= z_1^{1-N}\sqrt{-1}^N q^{\frac{N+1}{2r^*}+\frac{N^2-1}{2}}\left(\frac{(p^*q^2;p^*)_\infty}{(p^*;p^*)_\infty}\right)^N\left(\frac{1-pq^{-N}}{1-q^{-N}}\right)^n \\
&\quad\times\frac{(pq^{2N}p^{*-n};q^{2N},p^*)_\infty(q^{2N-2}p^{*-n};q^{2N},p^*)_\infty}{(q^{2N}p^{*-n};q^{2N},p^*)_\infty(q^{2N}p^{*1-n};q^{2N},p^*)_\infty}.
\end{aligned} \tag{4.3}
$$

Here $\sum'$ denotes the sum over the complementary set to $1 \square j_1 < j_2 < .. < j_n \square N$. $C_{j_1,j_2,..,j_n}$ are constants not important here.

The basic operators $\Lambda_j(z)$ ($j = 1, 2, .., N-1$) coincides with those in the deformed W_N algebra[7, 8]. The expressions for $\tilde{T}_n$ ($1 \square n \square N$) are almost same as those of the generating "currents" of the deformed W_N algebra, but the unit of the q-shift in the arguments in $\Lambda_j(z)$ is different. In an identification of the parameters $p_W = q^{-2},\ q_W = p = q^{2r}$, where p_W and q_W are p and q in [7, 8], respectively, the unit of the q-shift in [7, 8] is given by p_W, whereas it is

$p^* = q^{2(r-1)}$ in our $\tilde{T}_n(z)$. As a consequence, we have

$$\tilde{T}_N(z) =: \Lambda_1(z_2 q^{(N-1)r^*})\Lambda_2(z_2 q^{(N-3)r^*}) \cdots \Lambda_N(z_2 q^{-(N-1)r^*}) :\neq 1. \tag{4.4}$$

Therefore, our deformed W algebra generated by $\tilde{T}_n$ ($1 \square n \square N$) is $\mathfrak{gl}_N$ type instead of $\mathfrak{sl}_N$ type.

On the other hand, since the type II vertex operator $\Psi^*(z)$ and its dual $\Psi(z)$ are the creation operators of the physical excited particle and anti-particle, it is natural to identify the operators $\tilde{T}_n(z)$ ($1 \square n \square N$) with the creation operator of their bound states. The S-matrix of the bound state particles are calculated as follows.

$$\tilde{T}_n(z)\tilde{T}_m(w) = S_{n,m}(w/z)\ \tilde{T}_m(w)\tilde{T}_n(z), \tag{4.5}$$

$$S_{n,m}(z) = \prod_{k=1}^{n}\prod_{l=1}^{m} \varphi_N\left(zq^{r^*(n-m+2(l-k))}\right), \tag{4.6}$$

$$\varphi_N(z) = \frac{\Theta_{q^{2N}}(q^2 z)\Theta_{q^{2N}}(p^* z)\Theta_{q^{2N}}(p^{*-1}q^{-2}z)}{\Theta_{q^{2N}}(q^{-2}z)\Theta_{q^{2N}}(p^{*-1}z)\Theta_{q^{2N}}(p^* q^2 z)}. \tag{4.7}$$

Again, this S-matrix is different from the one obtained by Feigin and Frenkel (sec7.2 in [7]) only by the choice of the unit of the q-shift.

The scaling limit of the $\widehat{\mathfrak{sl}}_N$ RSOS model is expected to be the RSOS restriction of the affine Toda field theory with imaginary coupling constant. It is interesting to compare the scaling limit of our S-matrices $R^*(v, P)$ for the excited particle and $S_{n,m}(z)$ for the bound states with the bootstrap results [16, 17].

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