The Elliptic Algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and the Deformation of $W_N$ Algebra

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Abstract

After reviewing the recent results on the Drinfeld realization of the face type elliptic quantum group $B_{h,\lambda}(\widehat{\mathfrak{sl}}_N)$ by the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$, we investigate a fusion of the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$. The basic generating functions $\Lambda_j(z)$ ($j = 1, 2, ..., N - 1$) of the deformed $W_N$ algebra are derived explicitly.
1 Introduction

In a recent paper [1, 2, 3], we showed that the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ provides the Drinfeld realization of the face type elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ tensored by a Heisenberg algebra. Based on this fact, we defined the $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ counterparts of the intertwining operators of the $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ modules and obtained their free field realization in the level one representation. The resultant vertex operators, called the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$, are identified with the vertex operators of the RSOS model associated with $\widehat{\mathfrak{sl}}_N$ in the algebraic analysis formulation[5]. In general, we expect that the elliptic algebra $U_{q,p}(\mathfrak{g})$ with $\mathfrak{g}$ being an affine Lie algebra provides the Drinfeld realization for the elliptic quantum group $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ and enables us to perform an algebraic analysis of the $\mathfrak{g}$ type RSOS model.

On the other hand, the $\widehat{\mathfrak{sl}}_N$ RSOS model is known as an off-critical deformation of the $W_N$ minimal model[6]. In this relation, it is remarkable that the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ in the $c = 1$ representation coincides with the algebra of the screening currents of the deformed $W_N$ algebra [7, 8, 9]. In general, we expect that the elliptic algebra $U_{q,p}(\mathfrak{g})$ provides an algebra of screening currents of the deformation of the coset CFT associated with $(\mathfrak{g}_c \oplus (\mathfrak{g})_{r-c-2}/(\mathfrak{g})_{r-2})[1, 2]$.

The purpose of this paper is to continue to find an explicit relation among the elliptic algebra $U_{q,p}(\mathfrak{g})$, the $\mathfrak{g}$ type RSOS model and the deformation of $W(\mathfrak{g})$ algebra in the case $\mathfrak{g} = \widehat{\mathfrak{sl}}_N$. We here investigate a fusion of the type II vertex operator of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and its dual, and show that the generating functions of the deformed $W_N$ algebra can be extracted from it. The idea of fusion of the vertex operators was used in [10, 11] to derive the generating function of the deformed Virasoro algebra (corresponding to the $\phi_{1,3}$ perturbation) from the ABF model in regime III, in [12] for the deformed $W_N$ algebra with the central charge $c_N = (N - 1) \left(1 - \frac{N(N+1)}{r(r-1)}\right)$ at special point $r = N + 2$ (the $\mathbb{Z}_N$ parafermion point) from the ABF model in regime II, and in [13] for the deformed Virasoro algebra (corresponding to the $\phi_{1,2}$ perturbation) from the dilute $A_L$ model.

This article is organised as follows. In the next section, we briefly review the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ as the Drinfeld realization of the elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$. We give a summary of the results on the free field realization of the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ obtained in [3]. In the section 3, we discuss a fusion of the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and a derivation of the generators of the deformed $W_N$ algebra.

Through this paper, we use the following symbols. $p = q^{2r}$, \( p^* = pq^{-2c} = q^{2r^*} \) \( (r^* = r - c; \ r, r^* \in \mathbb{R}_{>0}) \),

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$
\[ \Theta_p(z) = (z, p)_\infty (p z^{-1}, p)_\infty (p, p)_\infty, \]
\[ \{ z \} = (z; p, q^{2N})_\infty, \quad \{ z \}^* = \{ z \}^p \]
\[ (z; t_1, \cdots, t_k)_\infty = \prod_{n_1, \cdots, n_k \geq 0} (1 - z t_1^{n_1} \cdots t_k^{n_k}). \]

We also use the Jacobi theta functions
\[ [v] = q^{\frac{v}{2} - v} \Theta_p(q^{2v}) (p; p)_\infty^3, \quad [v]^* = q^{v - v} \Theta_{p^*}(q^{2v}) (p^*; p^*)_\infty^3, \]
which satisfy \([-v] = -[v]\) and the quasi-periodicity property
\[ [v + r] = -[v], \quad [v + r \tau] = -e^{-\pi i r - 2\pi i v} [v]. \]

We take the normalization of the theta function to be
\[ \int_{C_0} \frac{dz}{2\pi i z} \frac{1}{-[v]} = 1, \]
where \(C_0\) is a simple closed curve in the \(v\)-plane encircling \(v = 0\) anticlockwise. The same holds for \([v]^*\), with \(r\) replaced by \(r^*\), except for the normalization
\[ \int_{C_0} \frac{dz}{2\pi i z} \frac{1}{-[v]^*} = \left[ \frac{[v]}{[v]^*} \right]_{v \to 0}. \]

2 The Elliptic Algebra \(U_{q,p}(\hat{\mathfrak{sl}}_N)\)

2.1 Definition

**Definition 2.1** (Elliptic algebra \(U_{q,p}(\hat{\mathfrak{sl}}_N)\)) We define the elliptic algebra \(U_{q,p}(\hat{\mathfrak{sl}}_N)\) to be the associative algebra of the currents \(E_j(v), F_j(v)\) (\(1 \neq j \neq N - 1\)) and \(K_j(v)\) (\(1 \neq j \neq N\)) satisfying the following relations.

\[ E_i(v_1) E_j(v_2) = \frac{[v_1 - v_2 + \frac{A_{ij}}{2}]}{[v_1 - v_2 - \frac{A_{ij}}{2}]} E_j(v_2) E_i(v_1), \]
\[ F_i(v_1) F_j(v_2) = \frac{[v_1 - v_2 - \frac{A_{ij}}{2}]}{[v_1 - v_2 + \frac{A_{ij}}{2}]} F_j(v_2) F_i(v_1), \]
\[ [E_i(v_1), F_j(v_2)] = \frac{\delta_{i,j}}{q - q^{-1}} \left( \delta(q^{-c} z_1 / z_2) H_j^+ (v_2 + \frac{c}{4}) - \delta(q^c z_1 / z_2) H_j^- (v_2 - \frac{c}{4}) \right), \]
\[ H_j^\pm \left( v + \frac{1}{2} \left( r - \frac{c}{2} \right) \right) = \mp K_j \left( v + \frac{N - j}{2} \right) K_{j+1} \left( v + \frac{N - j}{2} \right)^{-1}, \]
\begin{align}
K_j(v_1)K_j(v_2) &= \rho(v_1 - v_2)K_j(v_2)K_j(v_1), \\
K_{j_1}(v_1)K_{j_2}(v_2) &= \rho(v_1 - v_2)\left[\frac{v_1 - v_2 - 1}{v_1 - v_2}\right]^*\frac{v_1 - v_2}{v_1 - v_2 - 1}K_{j_2}(v_2)K_{j_1}(v_1) \\
& \quad (1 \leq j_1 < j_2 \leq N), \\
K_j(v_1)E_{j}(v_2) &= \frac{[v_1 - v_2 + j + r - N]}{[v_1 - v_2 + j + r - N - 1]}E_{j}(v_2)K_j(v_1), \\
K_{j+1}(v_1)E_j(v_2) &= \frac{[v_1 - v_2 + j + r - N]}{[v_1 - v_2 + j + r - N + 1]}E_j(v_2)K_{j+1}(v_1), \\
K_{j_1}(v_1)E_{j_2}(v_2) &= E_{j_2}(v_2)K_{j_1}(v_1) \quad (j_1 \neq j_2, j_2 + 1),
\end{align}

\begin{align}
K_j(v_1)F_j(v_2) &= \frac{[v_1 - v_2 + j + r - N - 1]}{[v_1 - v_2 + j + r - N]}F_j(v_2)K_j(v_1), \\
K_{j+1}(v_1)F_j(v_2) &= \frac{[v_1 - v_2 + j + r - N + 1]}{[v_1 - v_2 + j + r - N]}F_j(v_2)K_{j+1}(v_1), \\
K_{j_1}(v_1)F_{j_2}(v_2) &= F_{j_2}(v_2)K_{j_1}(v_1) \quad (j_1 \neq j_2, j_2 + 1),
\end{align}

\begin{align}
& z_1^{\frac{1}{2}} \frac{(p^* q^2 z_2/z_1; p^*)_\infty}{(p^* q^{-2} z_2/z_1; p^*)_\infty} \left\{ (z_2/z)^{\frac{1}{2}} \frac{(p^* q^{-1} z/z_1; p^*)_\infty(p^* q^{-1} z_2/z_1; p^*)_\infty}{(p^* q z/z_1; p^*)_\infty(p^* q z_2/z_1; p^*)_\infty} E_i(v_1)E_i(v_2)E_j(v) \\
& \quad \quad - 2q^{\frac{1}{2}} \frac{(p^* q^{-1} z/z_1; p^*)_\infty(p^* q^{-1} z_2/z; p^*)_\infty}{(p^* q z/z_1; p^*)_\infty(p^* q z_2/z; p^*)_\infty} E_i(v_1)E_j(v)E_i(v_2) \\
& \quad \quad + (z/z_1)^{\frac{1}{2}} \frac{(p^* q^{-1} z_1/z; p^*)_\infty(p^* q^{-1} z_2/z; p^*)_\infty}{(p^* q z_1/z; p^*)_\infty(p^* q z_2/z; p^*)_\infty} E_j(v)E_i(v_1)E_i(v_2) \right\} + (z_1 \leftrightarrow z_2) = 0, \\
& z_1^{\frac{1}{2}} \frac{(p q^{-2} z_2/z_1; p)_\infty}{(p q^2 z_2/z_1; p)_\infty} \left\{ \frac{(z/z_2)^{\frac{1}{2}}}{(p q^{-1} z/z_1; p)_\infty(p q^{-1} z_2/z_1; p)_\infty} F_i(v_1)F_i(v_2)F_j(v) \\
& \quad \quad - 2q^{\frac{1}{2}} \frac{(p q^{-1} z/z_1; p)_\infty(p q z_2/z_1; p)_\infty}{(p q^{-1} z_1/z; p)_\infty(p q^{-1} z_2/z; p)_\infty} F_i(v_1)F_j(v)F_i(v_2) \\
& \quad \quad + (z_1/z)^{\frac{1}{2}} \frac{(p q z_1/z; p)_\infty(p q z_2/z; p)_\infty}{(p q^{-1} z_1/z; p)_\infty(p q^{-1} z_2/z; p)_\infty} F_j(v)F_i(v_1)F_i(v_2) \right\} + (z_1 \leftrightarrow z_2) = 0 \quad (|i - j| = 1).
\end{align}

Here $A = (A_{jk})$ is the Cartan matrix of $\mathfrak{sl}_N$. The constant $\Box$ and the functions $\rho(v)$ are given by

\begin{align}
\Box &= \frac{(p; p)_\infty(p^* q^2 z; p^*_%)_\infty}{(p^*; p^*_%)_\infty(p q^2 z; p)_\infty}, \\
\rho(v) &= \frac{\rho^{++}(v)}{\rho^+(v)}, \\
\rho^+(v) &= q^{N-1} z^{N-1} \frac{\{p q^2 z\} \{p q^{2N-2} z\} \{1/z\} \{q^{2N} z^2\}}{\{p z\} \{p q^{2N} z\} \{q^{2} z^2\} \{q^{2N-2} z^2\}} \rho^{+-}(v) = \rho^+(v)|_{r \rightarrow r^*}.
\end{align}
2.2 Realization of $U_{q,p}(\hat{sl}_N)$

The elliptic algebra $U_{q,p}(\hat{sl}_N)$ can be realized by using the Drinfeld generators of $U_q(\hat{sl}_N)$ and a Heisenberg algebra. Let $h_i$, $a_i^\pm$, $x_i^\pm$ (i = 1, \cdots, N - 1 : m \in \mathbb{Z} \neq 0$,  $n \in \mathbb{Z}$), $c$, $d$ be the standard Drinfeld generators of $U_q(\hat{sl}_N)$[14]. Their generating functions $x_i^\pm(z)$, $\psi_i(z)$, $\varphi_i(z)$ are called the Drinfeld currents.

$$x_i^\pm(z) = \sum_{n \in \mathbb{Z}} x_{i,n}^\pm z^{-n}, \quad (2.18)$$

$$\psi_i(q^\pm z) = q^{h_i} \exp \left( (q - q^{-1}) \sum_{m>0} a_{i,m} z^{-m} \right), \quad (2.19)$$

$$\varphi_i(q^{-\pm} z) = q^{-h_i} \exp \left( -(q - q^{-1}) \sum_{m>0} a_{i,m} z^m \right) \quad (i = 1, \cdots, N - 1). \quad (2.20)$$

Let us define the auxiliary currents $u_i^\pm(z,p)$ (i = 1, 2, \cdots, N - 1) by

$$u_i^+(z,p) = \exp \left( \sum_{m>0} \frac{1}{[r^m]_q} a_{i,-m} (q^r z)^m \right), \quad (2.21)$$

$$u_i^-(z,p) = \exp \left( -\sum_{m>0} \frac{1}{[r^m]_q} a_{i,m} (q^{-r} z)^{-m} \right). \quad (2.22)$$

**Definition 2.1** We define “dressed” currents $e_i(z,p)$, $f_i(z,p)$, $\psi_i^\pm(z,p)$, (i = 1, \cdots, N - 1) by

$$e_i(z,p) = u_i^+(z,p)x_i^+(z), \quad (2.23)$$

$$f_i(z,p) = x_i^+(z)u_i^-(z,p), \quad (2.24)$$

$$\psi_i^+(z,p) = u_i^+(q^\pm z,p)\psi_i(z)u_i^-(q^{-\pm} z,p), \quad (2.25)$$

$$\psi_i^-(z,p) = u_i^+(q^{-\pm} z,p)\varphi_i(z)u_i^-(q^\pm z,p). \quad (2.26)$$

Setting $b_{j,m} = \frac{[r^m]_q}{[r^m]_q} a_{j,m}$ (for $m > 0$), $q^{c|m|} a_{j,m}$ (for $m < 0$), we introduce new generators, $B^j_m$ (j = 1, \cdots, N; m \in \mathbb{Z})$, according to the formula

$$-B^j_m + B^{j+1}_m = \frac{m}{[m]_q} b_{j,m} q^{(N-j)m}, \quad \sum_{j=1}^{N} q^{2jm} B^j_m = 0. \quad (2.27)$$

From this and the commutation relation of the Drinfeld generators $a_{j,m}$, we derive the following commutation relations.

$$[B^j_m, B^{k}_{m'}] = m \delta_{m+m',0} \left[ \frac{[r^m]_q[m]_q}{[r^m]_q[m]_q[N]_q} \right] \times \left\{ \begin{array}{ll} \frac{[(N-1)m]_q}{[m]_q} & (j = k) \\ -q^{-mNsgn(j-k)[m]_q} & (j \neq k) \end{array} \right\} \quad (2.28)$$

5
for $m, m' \in \mathbb{Z}_{\neq 0}, \; j, k = 1, \ldots, N$. Then we define new currents $k_j(z, p)$ (1 $\square$ $j$ $\square$ $N$) by

$$k_j(z, p) = \exp \left( \sum_{m \neq 0} \frac{[m]_q}{m[r^m]_q} B_m^j z^{-m} \right).$$

(2.29)

This yields the following decomposition.

$$\psi_j^\pm (q^{\pm(r-\frac{1}{2})} z, p) = \Delta q^{\pm h_j} k_j(q^{N-j} z, p) k_{j+1}(q^{N-j} z, p)^{-1}.$$  

(2.30)

On the other hand, let $\epsilon_j$ (1 $\square$ $j$ $\square$ $N$) be the orthonormal basis in $\mathbb{R}^N$ with the inner product $\langle \epsilon_j, \epsilon_k \rangle = \delta_{j,k}$. Setting $\check{\epsilon}_j = \epsilon_j - \epsilon$, $\epsilon = \frac{1}{N} \sum_{j=1}^N \epsilon_j$, we have the weight lattice $P$ of type $A_{N-1}^{(1)}$: $P = \oplus_{j=1}^N \mathbb{Z} \check{\epsilon}_j$. Then, for example, the simple roots $\alpha_j$ (1 $\square$ $j$ $\square$ $N-1$) of $s\mathfrak{I}_N$ are given by $\alpha_j = -\check{\epsilon}_j + \check{\epsilon}_{j+1}$. Let us introduce operators $h_\alpha, \beta$ ($\alpha, \beta \in P$) by

$$[h_{\check{\epsilon}_j}, \check{\epsilon}_k] = \langle \check{\epsilon}_j, \check{\epsilon}_k \rangle, \quad [h_{\check{\epsilon}_j}, h_{\check{\epsilon}_k}] = 0 = [\check{\epsilon}_j, \check{\epsilon}_k],$$

(2.31)

$h_\alpha = \sum_j n_j h_{\check{\epsilon}_j}$ for $\alpha = \sum_j n_j \epsilon_j$ and $h_0 = 0$. Note that $\langle \check{\epsilon}_j, \check{\epsilon}_k \rangle = \delta_{j,k} - \frac{1}{N}$ and $[h_\alpha, \alpha] = 2\delta_{j,k} - \delta_{j,k+1} - \delta_{j,k-1} = A_{jk}$. Hence, we identify $h_\alpha_j = -h_{\check{\epsilon}_j} + h_{\check{\epsilon}_{j+1}}$ with $h_j$ in the Drinfeld generators of $U_q(\hat{\mathfrak{sI}_N})$.

**Definition 2.2** We define the (centrally extended) Heisenberg algebra $\mathbb{C}[\hat{\mathcal{H}}]$ as an associative algebra generated by $P_{\check{\epsilon}_j}$, $Q_{\check{\epsilon}_j}$ (1 $\square$ $j$ $\square$ $N$) and $\eta_j$ (1 $\square$ $j$ $\square$ $N-1$) with the relations

$$[P_{\check{\epsilon}_j}, Q_{\check{\epsilon}_k}] = \langle \check{\epsilon}_j, \check{\epsilon}_k \rangle, \quad [P_{\check{\epsilon}_j}, P_{\check{\epsilon}_k}] = 0,$$

$$[Q_{\check{\epsilon}_j}, Q_{\check{\epsilon}_k}] = \left( \frac{1}{r} - \frac{1}{r^s} \right) \text{sgn}(j-k) \log q,$$

(2.32)

(2.33)

$$[Q_{\check{\epsilon}_j}, \eta_k] = \frac{1}{r} \text{sgn}(j-k) \log q,$$

$$[\eta_j, \eta_k] = \frac{1}{r} \text{sgn}(j-k) \log q,$$

(2.34)

(2.35)

$$[P_{\check{\epsilon}_j}, \eta_k] = 0, \quad \sum_{j=1}^N \eta_j = 0,$$

$$[\eta_j, \alpha] = [P_{\check{\epsilon}_j}, U_q(\hat{\mathfrak{S}_{\mathfrak{I}_N}})] = [Q_{\check{\epsilon}_j}, U_q(\hat{\mathfrak{S}_{\mathfrak{I}_N}})] = [\eta_j, U_q(\hat{\mathfrak{S}_{\mathfrak{I}_N}})] = 0.$$

(2.36)

(2.37)

Now we define the currents $E_j(v), F_j(v), H_j^\pm(v)$ (1 $\square$ $j$ $\square$ $N-1$) and $K_j(v)$ (1 $\square$ $j$ $\square$ $N$) by

$$E_j(v) = e_j(z, p) e^{\bar{\alpha}_j} e^{-Q_{\alpha_j} (q^{-j+N} z)} \frac{r_{\alpha_j}^{-1}}{r},$$

$$F_j(v) = f_j(z, p) e^{-\bar{\alpha}_j} e^{-Q_{\alpha_j} (q^{-j+N} z)} \frac{r_{\alpha_j}^{-1}}{r},$$

$$H_j^+(v) = \psi_j^+ (z, p) q^{h_j} e^{-Q_{\alpha_j} (q^{-j+N} z) \left( \frac{1}{r} - \frac{1}{2} \right)} (P_{\alpha_j}^{-1} + \frac{1}{2} h_j),$$

$$K_j(v) = k_j(z, p) e^{Q_{\alpha_j} z \left( \frac{1}{r} - \frac{1}{2} \right)} P_{\alpha_j}^{-1} h_j - \frac{1}{r} h_j + \frac{1}{2} h_j (P_{\alpha_j}^{-1} + \frac{1}{2} h_j),$$

(2.38)

(2.39)

(2.40)

(2.41)

where $\bar{\alpha}_j = -\eta_j + \eta_{j+1}$. Then it is easy to show that $E_j(v), F_j(v), H_j^+(v)$ and $K_j(v)$ satisfy the defining relations of the elliptic algebra $U_{q,p}(\hat{\mathfrak{S}_{\mathfrak{I}_N}})$. 

6
2.3 \textit{RLL relation}

We next discuss a relation between two elliptic algebras $U_{q,p}(\hat{\mathfrak{sl}}_N)$ and $B_{q,\lambda}(\hat{\mathfrak{sl}}_N)$. We construct a $L$-operator by using the half currents and show that it satisfies the dynamical \textit{RLL}-relation which characterizes $B_{q,\lambda}(\hat{\mathfrak{sl}}_N)$. In order to construct a $L$-operator, we introduce the half currents $E_{i,j}^-(v), F_{i,j}^+(v)$ and $K_j^+(v)$. We use the following abbreviations

$$P_{j,l} = -P_{e_j} + P_{e_l} = P_{\alpha_j} + P_{\alpha_{j+1}} + \cdots + P_{\alpha_{l-1}}$$

(2.42)

$$h_{j,l} = -h_{e_j} + h_{e_l} = h_j + h_{j+1} + \cdots + h_{l-1}$$

(2.43)

for $j < l$. From the definition of $C\{\mathcal{H}\}$ and (2.38)-(2.41), we have

$$[K_j(v), P_{k,l}] = (\delta_{j,k} - \delta_{j,l})K_j(v) = [K_j(v), P_{k,l} + h_{k,l}],$$

(2.44)

$$[E_j(v), P_{k,l}] = (\delta_{j,k} + \delta_{j+1,l} - \delta_{j,l} - \delta_{j+1,k})E_j(v),$$

(2.45)

$$[F_j(v), P_{j,l} + h_{j,l}] = (\delta_{j,k} + \delta_{j+1,l} - \delta_{j,l} - \delta_{j+1,k})F_j(v),$$

(2.46)

$$[F_j(v), P_{k,l}] = 0 = [E_j(v), P_{k,l} + h_{k,l}].$$

(2.47)

Now we define the half currents of $U_{q,p}(\hat{\mathfrak{sl}}_N)$ as follows.

\textbf{Definition 2.2 (Half currents)} We define the half currents $F_{j,l}^+(v), E_{i,j}^+(v), (1 \leq j < l \leq N)$ and $K_j^+(v)$ ($j = 1, \cdots, N$) by

$$K_j^+(v) = K_j\left(v + \frac{r+1}{2}\right)\quad (1 \leq j \leq N),$$

(2.48)

$$F_{j,l}^+(v) = a_{j,l} \oint_{C(j,l)} \prod_{m=j}^{l-1} \frac{dz_m}{2\pi i z_m} F_{l-1}(v_{l-1}) F_{l-2}(v_{l-2}) \cdots F_j(v_j)$$

$$\times \frac{[v - v_{l-1} + P_{j,l} + h_{j,l} + \frac{l-N}{2} - 1][1]}{[v - v_{l-1} + \frac{l-N}{2}][P_{j,l} + h_{j,l} - 1]}$$

$$\times \prod_{m=j}^{l-2} \frac{[v_{m+1} - v_{m} + P_{j,m+1} + h_{j,m+1} - \frac{1}{2}][1]}{[v_{m+1} - v_{m} + \frac{1}{2}][P_{j,m+1} + h_{j,m+1}]},$$

(2.49)

$$E_{i,j}^+(v) = a_{i,j}^* \oint_{C^*(j,l)} \prod_{m=j}^{l-1} \frac{dz_m}{2\pi i z_m} E_j(v_j) E_{j+1}(v_{j+1}) \cdots E_{l-1}(v_{l-1})$$

$$\times \frac{[v - v_{l-1} - P_{j,l} + \frac{l-N}{2} + \frac{1}{2} + 1][1]^*}{[v - v_{l-1} + \frac{l-N}{2} + \frac{3}{2}][P_{j,l} - 1]^*}$$

$$\times \prod_{m=j}^{l-2} \frac{[v_{m+1} - v_{m} - P_{j,m+1} + \frac{1}{2}][1]^*}{[v_{m+1} - v_{m} + \frac{3}{2}][P_{j,m+1} - 1]^*}.$$\hfill (2.50)

Here the integration contour $C(j,l)$ and $C^*(j,l)$ are given by

$$C(j,l) : |pq^{l-N}z| < |z_{l-1}| < |q^{l-N}z|,$$
\[
|pqz_{k+1}| < |z_k| < |qz_{k+1}|,
\]
\[
C^*(j, l) : |p^*q^{l-N+c}z| < |z_{l-1}| < |q^{l-N+c}z|,
\]
\[
|p^*qz_{k+1}| < |z_k| < |qz_{k+1}|,
\]

where \(k = j, j + 1, \ldots, l - 2\). The constants \(a_j, l\) and \(a_j^*\) are chosen to satisfy
\[
\frac{\partial a_j, l a_j^*}{q - q^{-1}} = 1.
\]

### 2.4 L-operator

**Definition 2.3 (L-operator)** By using the half currents, we define the L-operator \(\hat{L}^+(v) \in \text{End}(C^N) \otimes U_{q,p}(\mathfrak{sl}_N)\) as follows.

\[
\hat{L}^+(u) = \left( \begin{array}{cccc}
1 & F_{1,2}^+(u) & \cdots & F_{1,N}^+(u) \\
0 & 1 & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & 1 & F_{N-1,N}^+(u) \\
0 & \cdots & \cdots & 0
\end{array} \right)
\left( \begin{array}{cccc}
K_1^+(u) & 0 & \cdots & 0 \\
0 & K_2^+(u) & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & K_N^+(u)
\end{array} \right)
\times
\left( \begin{array}{cccc}
1 & 0 & \cdots & 0 \\
E_{2,1}^+(u) & 1 & \ddots & \vdots \\
E_{3,2}^+(u) & \ddots & \ddots & \vdots \\
\vdots & \ddots & 1 & 0 \\
E_{N,1}^+(u) & E_{N,2}^+(u) & \cdots & E_{N,N-1}^+(u)
\end{array} \right).
\]

Then by a direct comparison with the relations of the half currents, we conjecture the following statement.

**Conjecture 2.3** The L-operator \(\hat{L}^+(v)\) satisfies the following RLL = LLR* relation.

\[
R^{(12)}(v_1 - v_2, P + h)\hat{L}^{(+1)}(v_1)\hat{L}^{(+2)}(v_2) = \hat{L}^{(+2)}(v_2)\hat{L}^{(+1)}(v_1)R^{(12)}(v_1 - v_2, P).
\]

Here the \(R\)-matrix \(R^+(v, P)\) is the image of the universal \(R\)-matrix \(R(r, \{s_j\})\) of \(B_{q,\lambda}(\mathfrak{sl}_N)\) in the evaluation representation \((\pi_{V,z} \otimes \pi_{V,1}), V \cong C^N\), given by

\[
R^+(v, s) = \rho^+(v)\tilde{R}(v, s),
\]

\[
\tilde{R}(v, s) = \sum_{j=1}^N E_{jj} \otimes E_{jj} + \sum_{1 \leq j < l \leq N} (b(v, s_{jj})E_{jj} \otimes E_{ll} + \tilde{b}(v)E_{ll} \otimes E_{jj})
\]

\[
+ \sum_{1 \leq j < l \leq N} (c(v, s_{jj})E_{jl} \otimes E_{lj} + \tilde{c}(v, s_{jj})E_{lj} \otimes E_{jl}),
\]

\[(254)\]
where \( s_{j,l} = \sum_{m=j}^{l-1} s_j \) (1 \( \square \) j \( \square \) l \( \square \) N) and

\[
    b(u, s) = \frac{(s+1)(s-1)}{s^2|u+1|}, \quad \bar{b}(u) = \frac{[u]}{[u+1]},
\]
\[
    c(u, s) = \frac{[1]}{s|u+1|}, \quad \bar{c}(u, s) = \frac{[1]}{s|u+1|}.
\]

(2.58)

(2.59)

And \( R^{+*}(v, s) = R^+(v, s)|_{r \rightarrow r^*} \). Up to a gauge transformation, \( R^+(v, P) \) coincides with the Boltzmann weight of the \( \mathfrak{s}_N \) RSOS model introduced in [6].

The \( c = 1 \) case, the statement was proved by using the free field realization [3].

Now let us define the modified \( L \)-operator \( L^+(v, P) \) by

\[
    L + (v, P) = \hat{L} + (v) \begin{pmatrix} e^{-Q_{\varepsilon_1}} & 0 & \ldots & 0 \\ 0 & e^{-Q_{\varepsilon_2}} & \vdots \\ \vdots & \ddots & \ldots & 0 \\ 0 & \ldots & 0 & e^{-Q_{\varepsilon_N}} \end{pmatrix} = \hat{L}^+(v) \exp \left\{ \sum_{m=1}^{N} h_{\varepsilon_m}^{(1)} Q_{\varepsilon_m} \right\}. \quad (2.60)
\]

Here \( h_{\varepsilon_j}^{(1)} = h_{\varepsilon_j} \otimes 1, \) \( h_{\varepsilon_m} \equiv -E_{mm} \) (a \( N \times N \) matrix unit). We then show that the modified \( L \)-operator depends on neither \( Q_{\varepsilon_j} \) nor \( \eta_j \) and satisfies the dynamical RLL relation of \( \mathcal{B}_{q,\lambda}(\mathfrak{s}_N)[4] \).

**Corollary 2.4**

\[
    R^{+(12)}(v, P + h)L^{+(1)}(v_1, P)L^{+(2)}(v_2, P + h^{(1)}) = L^{+(2)}(v_2, P)L^{+(1)}(v_1, P + h^{(2)})R^{+(12)}(v, P),
\]

(2.61)

where \( v = v_1 - v_2 \).

Hence, we regard the elliptic currents \( E_j(v), \) \( F_j(v) \) (1 \( \square \) j \( \square \) N - 1) and \( K_j(v) \) (1 \( \square \) j \( \square \) N) in \( U_{q,p}(\mathfrak{s}_N) \) as the Drinfeld realization of the elliptic algebra \( \mathcal{B}_{q,\lambda}(\mathfrak{s}_N) \) tensored by the Heisenberg algebra.

\[
    U_{q,p}(\mathfrak{s}_N) = \mathcal{B}_{q,\lambda}(\mathfrak{s}_N) \otimes_{\mathbb{C}} \mathbb{C}\{\hat{H}\}. \quad (2.62)
\]

**3 Vertex Operators of \( U_{q,p}(\mathfrak{s}_N) \)**

We here summarize a construction of the type II vertex operator of \( U_{q,p}(\mathfrak{s}_N) \) and its dual vertex operator.
3.1 Definition

Let us first define an extension of the $U_q(\hat{\mathfrak{sl}}_N)(\cong B_{q,\lambda}(\hat{\mathfrak{sl}}_N))$ modules by

$$\hat{\mathcal{F}} = \bigoplus_{\mu_1, \ldots, \mu_N \in \mathbb{Z}} \mathcal{F} \otimes e^{\mu_1 Q_{\mu_1} + \cdots + \mu_N Q_{\mu_N}}.$$ 

Let $\Psi^*_W(z, P)$ be the type II intertwining operator of $B_{q,\lambda}(\hat{\mathfrak{sl}}_N)$ [4]. We define the type II vertex operator $\hat{\Psi}^*_W(z)$ of $U_{q,p}(\hat{\mathfrak{sl}}_N)$ as the following extension.

$$\hat{\Psi}^*_W(z) = \Psi^*_W(z, P) \exp \left\{ N \sum_{j=1}^{N} h_{\epsilon_j} Q_{\epsilon_j} \right\} : W_z \otimes \hat{\mathcal{F}} \rightarrow \hat{\mathcal{F}}'.$$ (3.1)

From the intertwining relation of the $B_{q,\lambda}(\hat{\mathfrak{sl}}_N)$ intertwining operators, we derive the following relation for the new operator $\hat{\Psi}^*_W(z)$.

$$\hat{L}^{(1)}_{v_1}(z_1) \hat{\Psi}^*_W(z_2) = \hat{\Psi}^*_W(z_2) \hat{L}^{(1)}_{v_1}(z_1) R_{WW}^+(v_1 - v_2, P - h_1 - h_2).$$ (3.2)

Let us consider the vector representation $V$ of $B_{q,\lambda}(\hat{\mathfrak{sl}}_N)$. We denote a basis of $V$ by $\{v_m\}_{m=1}^{N}$. In this representation, the $R$-matrix $R_{WW}^+(v, P)$ is given by $R^+(v, P)$ in (2.56) and the $L$-operator $\hat{L}^+(z)$ by $\hat{L}^+(z)$ in (2.54). We define the components of the vertex operators by

$$\hat{\Psi}^*_V(q^{-c-1}z)(v_m \otimes \cdot) = \Psi^*_m(z),$$ (3.3)

and the matrix elements of the $L$-operator $\hat{L}^+(z)$ by

$$\hat{L}^+(z)v_j = \sum_{1 \leq m \leq N} v_m \hat{L}^+(z)_{mj}.$$ (3.4)

3.2 Free field realizations

We here construct a free field realization of the vertex operators fixing $c = 1$. Let $\alpha_j$ be the simple root operator. We make the standard central extension $[\alpha_j, \alpha_k] = \pi i A_{jk}$ and set $\hat{\alpha}_j = \alpha_j + \check{\alpha}_j$, where $\check{\alpha}_j$ is an element of the Heisenberg algebra $\mathbb{C}\{\hat{H}\}$. Then we have

**Proposition 3.1** The currents $E_j(v)$ and $F_j(v)$ given by

$$E_j(v) = : \exp \left( - \sum_{m \neq 0} \frac{[rm]_q}{[m^2]_q} (-B_m^j + B_{m+1}^j)(q^{N-j}z)^{-m} \right) : \equiv \hat{\alpha}_j^{\hat{z}} \hat{H}_{j+1}^v (q^{-j}z)^{-\frac{r_{j+1}}{r}},$$ (3.5)

$$F_j(v) = : \exp \left( \sum_{m \neq 0} \frac{1}{m} (-B_m^j + B_{m+1}^j)(q^{N-j}z)^{-m} \right) : \equiv \hat{\alpha}_j \hat{H}_{j}^v (q^{-j}z)^{-\frac{r_j}{r+1}} + \hat{H}_{j}^v,$$ (3.6)

together with $H_j(v), K_j(v)$ given in (2.40)-(2.41) satisfy the commutation relations in Definition 2.1 for level $c = 1$. 

10
Now using this free field realization in (2.48)–(2.50), we obtain a realization of the half currents $E_j^+(v)$, $F_j^+(v)$, $K_j^+(v)$ as well as the $L$-operator $\hat{L}^+(v)$ for $c = 1$. Using the resultant $L$-operator in the “intertwining relation” (3.2), one can solve it for the II vertex operator.

**Theorem 3.2** The highest components of the type II vertex operator $\Psi_{N}^*(z)$ is realized in terms of a free field by

$$
\Psi_{N}^*(z) = \exp \left( \sum_{m \neq 0} \frac{[rm]}{m[r^*m]} B^N_{m^2 - m} \right) e^{-\tilde{\Lambda}_{N-1} z - h_{iN} \frac{1}{2} \rho_{iN} \frac{1}{z} \frac{\pi}{\rho_{iN}} \frac{N-1}{2} q^{-\frac{N-1}{2}}},
$$

(3.7)

where

$$
\tilde{\Lambda}_{N-1} = \frac{1}{N} (\tilde{\alpha}_1 + 2\tilde{\alpha}_2 + \cdots + (N-1)\tilde{\alpha}_{N-1}).
$$

(3.8)

The other components of the type II vertex $\Psi^*_j(z)$ ($j = 1, \cdots, N$) are given by

$$
\Psi^*_j(z) = a^*_{j,N} \int_{C^*} \prod_{m=j}^{N-1} \frac{dz_m}{2\pi i z_m} \Psi^*_N(v_N) \cdots \Psi^*_1(v_1) \Psi^*_j(v_j) \times \prod_{m=j}^{N-1} \frac{[v_{m+1} - v_m - \frac{1}{2}]_1^*}{[v_{m+1} - v_m + \frac{1}{2}]_1^*} \frac{[P_{j,m+1} - 1]}{[P_{j,m+1} - 1]_1^*}
$$

$$
= a^*_{j,N} \int_{C^*} \prod_{m=j}^{N-1} \frac{dz_m}{2\pi i z_m} \Psi^*_N(v_N) \cdots \Psi^*_1(v_1) \Psi^*_j(v_j) \times \prod_{m=j}^{N-1} \frac{[v_{m+1} - v_m - \frac{1}{2}]_1^*}{[v_{m+1} - v_m + \frac{1}{2}]_1^*} \frac{[P_{j,m+1} - 1]}{[P_{j,m+1} - 1]_1^*}.
$$

(3.9)

The integration contour $C^*$ is specified as follows.

$$
|p^* q^{-1} z_{m+1}|, |q^{-1} z_{m+1}| < |z_m| < |q z_{m+1}|, |p^* q^{-1} z_{m+1}| (j \neq m \neq N-1).
$$

(3.10)

Here the integration variable $z_m (j \neq m \neq N-1)$ should encircle the poles $p^* q^{-1} z_{m+1}, q^{-1} z_{m+1}$ but not the poles $p^{-1} q^{-1} z_{m+1}, q^{-1} z_{m+1}$.

We also have

**Theorem 3.3** The free field realizations of the type-II vertex operator $\Psi^*_j(z)$ satisfies the following commutation relation.

$$
\Psi^*_{j_1}(z_1) \Psi^*_{j_2}(z_2) = \sum_{j_1', j_2'}^N \Psi^*_{j_2'}(z_2) \Psi^*_{j_1'}(z_1) R^{j_1 j_2}_{j_1' j_2'}(v_1 - v_2, P)
$$

(3.11)

Here we set $R^*(v, P) = \mu^*(v) \hat{R}^*(v, P)$ with

$$
\mu^*(v) = z^{(\frac{1}{2} + 1) \frac{N-1}{2}} \frac{p q^{2N-2} z}{p^{2}} \{q^{2N} z \}^* \{q^2 z \}^* \{p \}^* \{q^2 \}^* \{q^{2N} \}^* \{p q^{2N-2} \}^* \{q^2 \}^*.
$$

(3.12)
The dual of the type II vertex operator is an operator satisfying

$$\Psi(z) : \mathcal{F} \rightarrow V_z \otimes \mathcal{F}'$$

We define its components by

$$\Psi(z) = \sum_{j=1}^{N} v_j \otimes \Psi_j(z).$$

The following inversion relations hold.

$$\sum_{j=1}^{N} \Psi_j(z) \Psi_j'(z') = \frac{g_N'}{1 - q^{-N} z'}, \quad \sum_{j=1}^{N} \Psi_j(z) \Psi_j'(z') = \frac{g_N'}{1 - q^{-N} z'},$$

where

$$g_N' = (-1)^{N-1} \frac{r(1-N)}{q^{2(N-1)}} \left( \frac{p^*; p^* q^{2N} q^{-2}; q^{2N} q^{-2}}{q^N q^{-N} q^{2N} q^{-2}} \right)_{\infty}$$

as \( z' \rightarrow qz^{-N} \), as well as

$$\Psi_j(z) \Psi_k(z') = \delta_{j,k} \frac{g_N}{1 - q^{-N} z'},$$

$$g_N = (-1)^{N-1} q^{-r(1-N) N} \left( \frac{p^*; p^* q^{2N} q^{-2}; q^{2N} q^{-2}}{p^*; p^* q^{2N} q^{-2}; q^{2N} q^{-2}} \right)_{\infty}$$

as \( z' \rightarrow qz^{N} \). The free field realization is given as follows.

$$\Psi_j(z) = \oint_{C} \prod_{m=1}^{j-1} \frac{dz_m}{2\pi i z_m} \Psi_1(z) E_1(v_1) \cdots E_{j-1}(v_{j-1})$$

$$\times \prod_{m=1}^{j-1} \frac{[v_{m-1} - v_m - P_{m-1,j} + \frac{1}{2}]^*[1]^*}{[v_{m-1} - v_m - \frac{1}{2}]^*[P_{m-1,j} - 1]^*}$$

$$= \oint_{C} \prod_{m=1}^{j-1} \frac{dz_m}{2\pi i z_m} E_{j-1}(v_{j-1}) \cdots E_1(v_1) \Psi_1(v)$$

$$\times \prod_{m=1}^{j-1} \frac{[v_{m-1} - v_m - P_{m-1,j} + \frac{1}{2}]^*[1]^*}{[v_{m-1} - v_m + \frac{1}{2}]^*[P_{m-1,j} - 1]^*}$$

$$\Psi_1(z) = \exp \left( -\sum_{m \neq 0} \frac{r_m}{m [r^* m]} B_m(q^N z)^{-m} \right) : e^{A_1 z h_1} e^{-Q_{11} (q^N z)^{-1} P_{11} + \frac{N-1}{2N-1} \frac{N-1}{2N-1} z^{N-1}} \right)$$

where \( v = v_0 \) and the integration contour \( C \) is specified by the condition: the poles \( z_m = q^{-1} z_{m-1} p_m^* \) (\( n = 0, 1, 2, \ldots \)) are inside and \( z_m = q^2 z_{m-1} p_m \) (\( n = 0, 1, 2, \ldots \)) are outside for \( 1 \leq m \leq j - 1 \).

**Remark**: The free field realizations of the vertex operators in Theorem 3.2 and of the dual vertex operators are essentially the same as those of the \( \widehat{sl}_N \) RSOS model obtained in [15].
4 Fusion of the Vertex Operators

We now consider the fusion of the type II vertex operator $\Psi^*_1(z_2)$ and its dual $\Psi_1(z_1)$. Namely, we consider a product $\Psi_1(z_1)\Psi^*_1(z_2)$ and investigate the limits to the fusion points $z_1 = q^{-N}p_n^m z_2$ ($n = 0, 1, 2, \ldots, N$), where the contour in (3.9) for $w_1$ gets pinches.

For example, let us consider the case $n = 1$. If we take residues for the poles $w_{N-1} = q^{-1}z_2$, $w_{j-1} = q^{-1} w_j$ ($j = N - 1, N - 2, \ldots, 3$), the limit $z_1 \rightarrow q^{-N} p^* z_2$ causes pinches in the contour for $w_1$ at two points $w_1 = q^{-(N-1)z_2}$, $q^{-(N-1)p^*z_2}$. Similarly, for $1 \not\subseteq l \not\subseteq N - 2$, if we take residues at the poles $w_{N-1} = q^{-1}z_2$, $w_{j-1} = q^{-1} w_j$ ($j = N - l, N - 1, \ldots, N - l + 1$), $w_{N-l} = q^{-1}p^*w_{N-l+1}$, $w_{j-1} = q^{-1} w_j$ ($j = N - l - 1, N - l - 1, \ldots, 3$), the same limit $z_1 \rightarrow q^{-N} p^* z_2$ causes a pinch in the contour for $w_1$ at a point $w_1 = q^{-(N-1)p^*z_2}$. Hence in the limit $z_1 \rightarrow q^{-N} p^* z_2$, we obtain totally $N$ terms of contributions from the residues at the $N$ pinching points. Similar consideration leads us to the following results. As $z_1 \rightarrow q^{-N} p^* z_2$,

$$
\Psi_1(z_1)\Psi^*_1(z_2) = \frac{1}{1 - q^{-N}p_n^m z_2} \left\{ C_n \tilde{T}_n(q^{(n-1)r^*}_z) 
+ \sum'_{1 \not\subseteq j_1, j_2, \ldots, j_n \not\subseteq N} C_{j_1, j_2, \ldots, j_n} : \Lambda_{j_1}(z_2q^{(n-1)r^*}) \Lambda_{j_2}(z_2q^{(n-3)r^*}) \cdots \Lambda_{j_n}(z_2q^{r^*}) : \right\}.
$$

Here

$$
\tilde{T}_n(z) = \sum_{1 \not\subseteq j_1, j_2, \ldots, j_n \not\subseteq N} : \Lambda_{j_1}(z_2q^{(n-1)r^*}) \Lambda_{j_2}(z_2q^{(n-3)r^*}) \cdots \Lambda_{j_n}(z_2q^{-(n-1)r^*}) :, \quad (4.1)
$$

$$
\Lambda_j(z) = : \exp \left( \sum_{m \neq 0} \frac{q^{rm} - q^{-rm}}{m} B_m^j z^{-m} \right) : q^{-2P_{ij}p^{s_{ij}}q^{2(1-N)/p^*-1}j}, \quad (4.2)
$$

$$
C_n = z_1^{-N} \sqrt{\Gamma^N q^{N+1} \frac{N^2 - 1}{2}} \left( \frac{p^*; q^2; \infty}{(p^*; \infty) N \left( 1 - q^{-N} \right)^n} \right) \times \left( \frac{pq^{2N} p^{s-n}; q^{2N}, p^*; \infty}{(q^{2N} p^{s-n}; q^{2N}, p^*; \infty) (q^{2N} p^{s-n}; q^{2N}, p^*; \infty)} \right) ^{N}.
$$

Here $\sum'$ denotes the sum over the complementary set to $1 \not\subseteq j_1 < j_2 < \ldots < j_n \not\subseteq N$. $C_{j_1, j_2, \ldots, j_n}$ are constants not important here.

The basic operators $\Lambda_j(z)$ ($j = 1, 2, \ldots, N - 1$) coincides with those in the deformed $W_N$ algebra[7, 8]. The expressions for $\tilde{T}_n$ ($1 \not\subseteq n \not\subseteq N$) are almost same as those of the generating “currents” of the deformed $W_N$ algebra, but the unit of the $q$-shift in the arguments in $\Lambda_j(z)$ is different. In an identification of the parameters $p_W = q^{-2}$, $q_W = p = q^{2r}$, where $p_W$ and $q_W$ are $p$ and $q$ in [7, 8], respectively, the unit of the $q$-shift in [7, 8] is given by $p_W$, whereas it is
\[ p^* = q^{2(r-1)} \] in our \( \tilde{T}_n(z) \). As a consequence, we have

\[
\tilde{T}_N(z) = \Lambda_1(z_2q^{(N-1)r^*})\Lambda_2(z_2q^{(N-3)r^*}) \cdots \Lambda_N(z_2q^{-(N-1)r^*}) \neq 1. \tag{4.4}
\]

Therefore, our deformed \( W \) algebra generated by \( \tilde{T}_n(1 \square n \square N) \) is \( \mathfrak{g}_N \) type instead of \( \mathfrak{sl}_N \) type.

On the other hand, since the type II vertex operator \( \Psi^*(z) \) and its dual \( \Psi(z) \) are the creation operators of the physical excited particle and anti-particle, it is natural to identify the operators \( \tilde{T}_n(z) (1 \square n \square N) \) with the creation operator of their bound states. The \( S \)-matrix of the bound state particles are calculated as follows.

\[
\tilde{T}_n(z)\tilde{T}_m(w) = S_{n,m}(w/z) \tilde{T}_m(w)\tilde{T}_n(z), \tag{4.5}
\]

\[
S_{n,m}(z) = \prod_{k=1}^{n} \prod_{l=1}^{m} \varphi_N \left( zq^{-r^*(n-m+2(l-k))} \right), \tag{4.6}
\]

\[
\varphi_N(z) = \frac{\Theta_{q^{2N}}(q^2z)\Theta_{q^{2N}}(p^z\Theta_{q^{2N}}(p^{r-1}q^{-2}z))}{\Theta_{q^{2N}}(q^{-2}z)\Theta_{q^{2N}}(p^{r-1}z)\Theta_{q^{2N}}(p^{r}q^{2}z)}. \tag{4.7}
\]

Again, this \( S \)-matrix is different from the one obtained by Feigin and Frenkel (sec.7.2 in [7]) only by the choice of the unit of the \( q \)-shift.

The scaling limit of the \( \mathfrak{sl}_N \) RSOS model is expected to be the RSOS restriction of the affine Toda field theory with imaginary coupling constant. It is interesting to compare the scaling limit of our \( S \)-matrices \( R^*(v,P) \) for the excited particle and \( S_{n,m}(z) \) for the bound states with the bootstrap results [16, 17].

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**References**


