

The Elliptic Algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and the Drinfeld Realization of the Elliptic Quantum Group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$

Takeo KOJIMA ^{*,†} and Hitoshi KONNO ^{**,†}

** Department of Mathematics, College of Science and Technology,
Nihon University, Chiyoda-ku, Tokyo 101-0062, Japan.
E-mail: kojima@math.cst.nihon-u.ac.jp*

*** Department of Mathematics, Faculty of Integrated Arts and Sciences,
Hiroshima University, Higashi-Hiroshima 739-8521, Japan.
E-mail: konno@mis.hiroshima-u.ac.jp*

*† Department of Mathematics, Heriot-Watt University,
Edinburgh EH14 4AS, UK.*

Abstract

By using the elliptic analogue of the Drinfeld currents in the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$, we construct a L -operator, which satisfies the RLL -relations characterizing the face type elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$. For this purpose, we introduce a set of new currents $K_j(v)$ ($1 \leq j \leq N$) in $U_{q,p}(\widehat{\mathfrak{sl}}_N)$. As in the $N = 2$ case, we find a structure of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ as a certain tensor product of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ and a Heisenberg algebra. In the level-one representation, we give a free field realization of the currents in $U_{q,p}(\widehat{\mathfrak{sl}}_N)$. Using the coalgebra structure of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ and the above tensor structure, we derive a free field realization of the $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ -analogue of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ -intertwining operators. The resultant operators coincide with those of the vertex operators in the $A_{N-1}^{(1)}$ -type face model.

1 Introduction

In recent papers[1, 2, 3, 4, 5], the notion of elliptic quantum groups has been proposed. There are two types of elliptic quantum groups, the vertex type $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$ and the face type $\mathcal{B}_{q,\lambda}(\mathfrak{g})$, where $\mathfrak{g}$ is a Kac-Moody algebra associated with a symmetrizable generalized Cartan matrix. The elliptic quantum groups have the structure of quasi-triangular quasi-Hopf algebras introduced by Drinfeld [6]. Since certain finite dimensional representations of the universal R -matrices of these elliptic quantum groups yield known elliptic Boltzmann weights including, for example, those of the eight vertex model[7] and the Andrews-Baxter-Forrester (ABF) face model[8], we expect that we can perform an algebraic analysis of both types of elliptic lattice models based on the corresponding elliptic quantum groups.

Here, algebraic analysis means, in a restricted sense, a method of studying two dimensional solvable lattice models based on the representation theory of infinite dimensional quantum groups [9]. It can be regarded as an off-critical extension of conformal field theory, where the representation theory of the Virasoro algebras and/or affine Lie algebras plays an essential role. In fact, quite a lot of, but not all, solvable lattice models allow us, in the thermodynamic limit, to identify the space of states of the models with the infinite dimensional modules of certain quantum groups. Then two types of intertwining operators, type I and type II, of such modules become important. The type I intertwiner provides a realization of local operators, such as spin operators for example, on the infinite dimensional modules of quantum groups. And the type II plays the role of creation operator of physical excitations. Due to the coalgebra structure of quantum groups, these intertwiners can be determined uniquely. Realizing these ingredients in certain forms, such as the free field realization for example, one can perform a calculation of correlation functions as well as form factors of the models.

Through experience of the analysis of trigonometric models, such as the six vertex model, or equivalently the XXZ spin chain model (see the references in [9]), we know that a formulation of quantum groups in terms of the Drinfeld currents[10] provides a convenient framework. This is because one can construct a free field realization of the type I and II intertwining operators starting from a free field realization of the Drinfeld currents. In addition, the Drinfeld currents have a formal, but deep, resemblance to the currents in affine Kac-Moody algebras so that we can easily compare the results with those in conformal field theory. Hence to perform an algebraic analysis of the elliptic lattice models, it is an important step to find a new realization of the both elliptic quantum groups $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$ and $\mathcal{B}_{q,\lambda}(\mathfrak{g})$, $\mathfrak{g}$ being an affine Lie algebra, in terms of the Drinfeld currents.

In [11], one of the authors has introduced an elliptic analogue of the Drinfeld currents of $U_q(\widehat{\mathfrak{sl}}_2)$ independently from the formulation of the elliptic quantum groups. The algebra of the currents is called the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. Later in [12], it has been shown that $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ can be regarded essentially as the Drinfeld currents which gives a new realization of the face type elliptic algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$. According to this result, the type I and type II vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$, the analogues of the intertwining operators of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$, have been realized by the free bosonic fields. The resultant expressions coincide with those of the vertex operators of the ABF model obtained by Lukyanov and Pugai[13]. Hence a representation theoretical foundation to Lukyanov and Pugai's free field approach to the ABF model has been established.

The purpose of this paper is to extend this result to the higher rank case. We investigate a higher rank elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$, and show that $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ provides a new realization of the face type elliptic algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ in terms of the elliptic Drinfeld currents.

Our strategy is parallel to the one in [12]. We first give a definition of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ introducing the new currents $K_j(v)$ ($1 \leq j \leq N$) (Section 3). This gives a completion of the definition of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ given in Appendix A of [12]. As an example, a realization of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ as a certain tensor product of the algebra $U_q(\widehat{\mathfrak{sl}}_N)$ and a Heisenberg algebra $\mathbb{C}\{\hat{\mathcal{H}}\}$ is given. Then we define the ‘‘half currents’’ of the generating functions (total currents) $E_j(v)$, $F_j(v)$, $K_j(v)$ of the algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ (Section 4). The half currents allows us to construct a L -operator as a Gauss decomposed form of an operator valued matrix (5.1). We then argue that the thus obtained L -operator satisfies the RLL -relation which characterizes the algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$, when the generators of the mentioned Heisenberg algebra are reduced to a set of parameters (dynamical parameters) by properly removing half of the conjugate variables (Section 5). Hence, one can regard the algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ as a tensor product of the algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ and the Heisenberg algebra $\mathbb{C}\{\hat{\mathcal{H}}\}$.

The L -operator and the coalgebra structure of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ allows us to construct a free field realization of the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$, which are extension of the type I and II intertwining operators of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ by adding elements of the Heisenberg algebra, acting on the $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ -modules. In the level-one representation, we derived such a realization starting from a free field realization of the total currents of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$. The resultant expressions coincide with those of the type I and II vertex operators obtained in [14] and [15]. We also show that they satisfy the required commutation relations. We thus give a representation theoretical meaning to the vertex operators of the $A_{N-1}^{(1)}$ type face model[16]. Conversely, as a composition of the type I and type II intertwiners, one can construct a L -operator which satisfies the RLL -relations [17, 18]. As a check of our free field realization, we investigate a connection between the two L -operators, the one constructed by a composition of the vertex operators and the other by the

half currents, in the level-one representation. We then give a proof of our argument in Section 5 at $c = 1$.

The article is organized as follows. In the next section, we review some basic facts on the face type elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$. In Section 3, we present a definition of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$. New currents $K_j(u)$ ($j = 1, 2, \dots, N$) are introduced there. A realization of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ using the Drinfeld currents of $U_q(\widehat{\mathfrak{sl}}_N)$ and a Heisenberg algebra is also given. In Section 4, we introduce a set of half currents defined from $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and derive their commutation relations. In Section 5, constructing a L -operator in terms of the half currents, we show that it satisfies the required RLL -relation for $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$. According to this result, in Section 6, we discuss a free field realization of the two types of vertex operators of the level one $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ -modules. In addition, we have four appendices. Appendix A is devoted to a list of operator product expansions used in the text. In Appendix B, we give a proof of some formulae of commutation relations of the half currents. In Appendix C, we give a derivation of some formulae contained in the RLL -relation. Finally, in Appendix D, we give a summary of the N dimensional evaluation representation of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$.

2 The Elliptic Quantum Group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$

In this section, we give a review on the face type elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ based on the results in [5].

2.1 Notations

Through this article, we fix a complex number $q \neq 0, |q| < 1$. We often use the parameters

$$p = q^{2r} = e^{-\frac{2\pi i}{\tau}}, \quad p^* = pq^{-2c} = q^{2r^*} = e^{-\frac{2\pi i}{\tau^*}} \quad (r^* = r - c; \quad r, r^* \in \mathbb{R}_{>0}, \quad r\tau = r^*\tau^*).$$

The following notation is standard:

$$\begin{aligned} \Theta_p(z) &= (z, p)_\infty (pz^{-1}; p)_\infty (p; p)_\infty, \\ (z; t_1, \dots, t_k)_\infty &= \prod_{n_1, \dots, n_k \geq 0} (1 - zt_1^{n_1} \dots t_k^{n_k}). \end{aligned}$$

We also use the Jacobi theta functions

$$[v] = q^{\frac{v^2}{r} - v} \frac{\Theta_p(q^{2v})}{(p; p)_\infty^3}, \quad [v]^* = q^{\frac{v^2}{r^*} - v} \frac{\Theta_{p^*}(q^{2v})}{(p^*; p^*)_ \infty^3},$$

which satisfy $[-v] = -[v]$ and the quasi-periodicity property

$$[v + r] = -[v], \quad [v + r\tau] = -e^{-\pi i \tau - \frac{2\pi i v}{r}} [v]. \quad (2.1)$$

We take the normalization of the theta function to be

$$\oint_{C_0} \frac{dz}{2\pi iz} \frac{1}{[-v]} = 1, \quad (2.2)$$

where C_0 is a simple closed curve in the v -plane encircling $v = 0$ anticlockwise. The same holds for $[v]^*$, with r replaced by r^* , except for the normalization

$$\oint_{C_0} \frac{dz}{2\pi iz} \frac{1}{[-v]^*} = \frac{[v]}{[v]^*} \Big|_{v \rightarrow 0}.$$

2.2 Definition of the elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$

Let $U_q = U_q(\widehat{\mathfrak{sl}}_N)$ be the standard affine quantum group. Namely, $U_q(\widehat{\mathfrak{sl}}_N)$ is a quasi-triangular Hopf algebra equipped with the standard coproduct Δ , counit ε , antipode S and universal R matrix $\mathcal{R}$. Our conventions on the coalgebra structure follows [5]. Let $\mathfrak{h}$ and $\bar{\mathfrak{h}}$ be the Cartan subalgebras of $\widehat{\mathfrak{sl}}_N$ and $\mathfrak{sl}_N$, respectively. We denote a basis and its dual basis of $\mathfrak{h}$ by $\{\hat{h}_l\}$ and $\{\hat{h}^l\}$, respectively. More explicitly, they are given by $\{\hat{h}_l\} = \{d, c, h_j\}$ and $\{\hat{h}^l\} = \{c, d, h^j\}$ ($1 \leq j \leq N-1$), where c and d are a central element and a derivation operator of $\widehat{\mathfrak{sl}}_N$, respectively, and $\{h_j\}$ and $\{h^j\}$ are a basis and a dual basis of $\bar{\mathfrak{h}}$.

The face type elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ is a quasi-Hopf deformation of $U_q(\widehat{\mathfrak{sl}}_N)$ by the face type twistor $F(\lambda)$ ($\lambda \in \mathfrak{h}$). The twistor $F(\lambda)$ is an invertible element in $U_q \otimes U_q$ satisfying

$$(\text{id} \otimes \varepsilon)F(\lambda) = 1 = F(\lambda)(\varepsilon \otimes \text{id}), \quad (2.3)$$

$$F^{(12)}(\lambda)(\Delta \otimes \text{id})F(\lambda) = F^{(23)}(\lambda + h^{(1)})(\text{id} \otimes \Delta)F(\lambda). \quad (2.4)$$

where $\lambda = \sum_l \lambda_l \hat{h}^l$ ($\lambda_l \in \mathbb{C}$), $\lambda + h^{(1)} = \sum_l (\lambda_l + \hat{h}_l^{(1)}) \hat{h}^l$ and $\hat{h}_l^{(1)} = \hat{h}_l \otimes 1 \otimes 1$. An explicit construction of the twistor $F(\lambda)$ is given in [5]. A quasi-Hopf deformation means that as an associative algebra, $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ is isomorphic to $U_q(\widehat{\mathfrak{sl}}_N)$, but the coalgebra structure is deformed. Namely, the coproduct is changed to the new one given by

$$\Delta_\lambda(x) = F(\lambda)\Delta(x)F(\lambda)^{-1} \quad \forall x \in U_q(\widehat{\mathfrak{sl}}_N). \quad (2.5)$$

Δ_λ satisfies a weaker coassociativity

$$(\text{id} \otimes \Delta_\lambda)\Delta_\lambda(x) = \Phi(\lambda)(\Delta_\lambda \otimes \text{id})\Delta_\lambda(x)\Phi(\lambda)^{-1} \quad \forall x \in U_q(\widehat{\mathfrak{sl}}_N), \quad (2.6)$$

$$\Phi(\lambda) = F^{(23)}(\lambda)F^{(23)}(\lambda + h^{(1)})^{-1}. \quad (2.7)$$

The universal R -matrix is also deformed to

$$\mathcal{R}(\lambda) = F^{(21)}(\lambda)\mathcal{R}F^{(12)}(\lambda)^{-1}. \quad (2.8)$$

Definition 2.1 (Elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$) [5] *The face type elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ is a quasi-triangular quasi-Hopf algebra $(\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N), \Delta_\lambda, \varepsilon, S, \Phi(\lambda), \alpha, \beta, \mathcal{R}(\lambda))$, where α, β are defined by*

$$\alpha = \sum_i S(k_i) l_i, \quad \beta = \sum_i m_i S(n_i). \quad (2.9)$$

Here we set $\sum_i k_i \otimes l_i = F(\lambda)^{-1}$, $\sum_i m_i \otimes n_i = F(\lambda)$.

A characteristic feature of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ is that the universal R matrix $\mathcal{R}(\lambda)$ satisfies the dynamical Yang-Baxter equation.

$$\mathcal{R}^{(12)}(\lambda + h^{(3)}) \mathcal{R}^{(13)}(\lambda) \mathcal{R}^{(23)}(\lambda + h^{(1)}) = \mathcal{R}^{(23)}(\lambda) \mathcal{R}^{(13)}(\lambda + h^{(2)}) \mathcal{R}^{(12)}(\lambda). \quad (2.10)$$

Let $(\pi_{V,z}, V_z)$, $V_z = V \otimes \mathbb{C}[z, z^{-1}]$ be a (finite dimensional) evaluation representation of U_q . Taking images of $\mathcal{R}$, we have a R -matrix $R_{VW}^+(z, \lambda)$ and a L -operator $L_V^+(z, \lambda)$ as follows.

$$R_{VW}^+(z_1/z_2, \lambda) = (\pi_{V,z_1} \otimes \pi_{W,z_2}) q^{c \otimes d + d \otimes c} \mathcal{R}(\lambda), \quad (2.11)$$

$$L_V^+(z, \lambda) = (\pi_{V,z} \otimes \text{id}) q^{c \otimes d + d \otimes c} \mathcal{R}(\lambda). \quad (2.12)$$

Then from (2.10), we have the following dynamical RLL -relation.

$$R_{VW}^+(z_1/z_2, \lambda + h) L_V^+(z_1, \lambda) L_W^+(z_2, \lambda + h^{(1)}) = L_W^+(z_2, \lambda) L_V^+(z_1, \lambda + h^{(2)}) R_{VW}^+(z_1/z_2, \lambda). \quad (2.13)$$

Note that in $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$, $L_V^+(z, \lambda)$ and $L_V^-(z, \lambda) = (\pi_{V,z} \otimes \text{id}) \mathcal{R}^{(21)}(\lambda)^{-1} q^{-c \otimes d - d \otimes c}$ are not independent operators (Proposition 4.3 in [5]). Hence just one dynamical RLL -relation (2.13) characterizes the algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ completely in the sense of Reshetikhin and Semenov-Tian-Shansky [19].

Hereafter we parametrize the dynamical variable λ as

$$\lambda = (r^* + N)d + s'c + \sum_{j=1}^{N-1} (s_j + 1)h^j \quad (s' \in \mathbb{C}, r^* \equiv r - c). \quad (2.14)$$

Under this, we set $F(r^*, \{s_j\}) \equiv F(\lambda)$ and $\mathcal{R}(r^*, \{s_j\}) \equiv \mathcal{R}(\lambda)$. Since c is central, no s' dependence should appear. The dynamical shift $\lambda \rightarrow \lambda + h$ with $h = cd + \sum_{j=1}^N h_j h^j$, changes the universal R -matrix $\mathcal{R}(r^*, \{s_j\})$ to $\mathcal{R}(r, \{s_j + h_j\}) \equiv \mathcal{R}(\lambda + h)$. Note $r^* = r - c$.

Let us now take $(\pi_{V,z}, V_z)$ to be the evaluation representation associated with the vector representation $V \cong \mathbb{C}^N$ of $U_q(\mathfrak{sl}_N)$ (see Appendix D). We set

$$\begin{aligned} R^+(v, s + h) &= (\pi_{V,z_1} \otimes \pi_{V,z_2}) q^{c \otimes d + d \otimes c} \mathcal{R}(r, \{s_j + h_j\}), \\ L^+(v, s) &= (\pi_{V,z} \otimes \text{id}) q^{c \otimes d + d \otimes c} \mathcal{R}(r^*, \{s_j\}), \end{aligned}$$

where $z_i = q^{2v_i}$ ($i = 1, 2$), $v = v_1 - v_2$. One can obtain the finite dimensional representation of the twistor $F(r, \{s_j\})$ by solving the difference equation for $(\pi_{V, z_1} \otimes \pi_{V, z_2})F(r, \{s_j + h_j\})$ (Eq.(2.30) in [5]) derived by using the explicit realization of $F(\lambda)$, under the parametrization (2.14). Then noting the relation $\mathcal{R}(r, \{s_j + h_j\}) = F^{(21)}(r, \{s_j + h_j\})\mathcal{R}F^{(12)}(r, \{s_j + h_j\})^{-1}$, we obtain the R -matrix $R^+(v, s + h)$, up to a certain gauge transformation, as

$$R^+(v, s + h) = \rho^+(v)\bar{R}(v, s + h), \quad (2.15)$$

$$\begin{aligned} \bar{R}(v, s + h) = & \sum_{j=1}^N E_{jj} \otimes E_{jj} + \sum_{1 \leq j < l \leq N} (b(v, s_{j,l} + h_{j,l})E_{jj} \otimes E_{ll} + \bar{b}(v)E_{ll} \otimes E_{jj}) \\ & + \sum_{1 \leq j < l \leq N} (c(v, s_{j,l} + h_{j,l})E_{jl} \otimes E_{lj} + \bar{c}(v, s_{j,l} + h_{j,l})E_{lj} \otimes E_{jl}), \end{aligned} \quad (2.16)$$

where $s_{j,l} = \sum_{m=j}^{l-1} s_j$, $h_{j,l} = \sum_{m=j}^{l-1} h_j$ ($1 \leq j < l \leq N$) and

$$b(u, s) = \frac{[s+1][s-1][u]}{[s]^2[u+1]}, \quad \bar{b}(u) = \frac{[u]}{[u+1]}, \quad (2.17)$$

$$c(u, s) = \frac{[1][s+u]}{[s][u+1]}, \quad \bar{c}(u, s) = \frac{[1][s-u]}{[s][u+1]}. \quad (2.18)$$

The function $\rho^+(v)$ is chosen as

$$\rho^+(v) = q^{\frac{N-1}{N}} z^{\frac{N-1}{rN}} \frac{\{pq^2z\}\{pq^{2N-2}z\}\{1/z\}\{q^{2N}/z\}}{\{pz\}\{pq^{2N}z\}\{q^2/z\}\{q^{2N-2}/z\}}, \quad (2.19)$$

where

$$\{z\} = (z; p, q^{2N})_\infty. \quad (2.20)$$

Up to a gauge transformation, the R -matrix $R^+(v, s + h)$ is nothing but the Boltzmann weight of the $A_{N-1}^{(1)}$ type face model introduced in [16]. The R -matrix $R^{+*}(v, s) = (\pi_{V, z_1} \otimes \pi_{V, z_2})\mathcal{R}(r^*, \{s_j\})$ is obtained from $R^+(v, s)$ by the replacements $r \rightarrow r^*$. Hence, under the parametrization (2.14), the dynamical RLL -relation takes the form

$$R^{+(12)}(v, s + h)L^{+(1)}(v_1, s)L^{+(2)}(v_2, s + h^{(1)}) = L^{+(2)}(v_2, s)L^{+(1)}(v_1, s + h^{(2)})R^{+*(12)}(v, s). \quad (2.21)$$

2.3 Intertwining operators

Let $\mathcal{F}, \mathcal{F}'$ be highest weight U_q -modules. We denote the type-I and type II intertwining operators of U_q -modules by $\Phi(z)$ and $\Psi^*(z)$, respectively.

$$\Phi(z) : \mathcal{F} \longrightarrow \mathcal{F}' \otimes W_z, \quad \Psi^*(z) : W_z \otimes \mathcal{F} \longrightarrow \mathcal{F}'. \quad (2.22)$$

Twisting these operators by $F(r^*, s)$, we obtain the corresponding intertwining operators $\Phi(v, s)$ and $\Psi^*(v, s)$ of $\mathcal{B}_{q,\lambda}$ -modules.

$$\Phi_W(v, s) = (\text{id} \otimes \pi_{W,z})F(r^*, \{s_j\})\Phi(z), \quad (2.23)$$

$$\Psi_W^*(v, s) = \Psi^*(z)(\pi_{W,z} \otimes \text{id})F(r^*, \{s_j\})^{-1}. \quad (2.24)$$

From the intertwining relation satisfied by $\Phi(z)$ and $\Psi^*(z)$, one can derive the following dynamical intertwining relation for the new intertwiners [5].

$$\Phi_W^{(3)}(v_2 + \frac{c}{2}, s)L_V^{+(1)}(v_1, s) = R_{VW}^{+(13)}(v, s + h)L_V^{+(1)}(v_1, s)\Phi_W^{(3)}(v_2 + \frac{c}{2}, s + h^{(1)}), \quad (2.25)$$

$$L_V^{+(1)}(v_1, s)\Psi_W^{*(2)}(z_2, s + h^{(1)}) = \Psi_W^{*(2)}(z_2, s)L_V^{+(1)}(v_1, s + h^{(2)})R_{VW}^{+*(12)}(v_1 - v_2, s). \quad (2.26)$$

Note that (2.25) and (2.26) are the relations for the operators $V_{z_1} \otimes \mathcal{F} \rightarrow V_{z_1} \otimes \mathcal{F} \otimes W_{z_2}$ and $V_{z_1} \otimes W_{z_2} \otimes \mathcal{F} \rightarrow V_{z_1} \otimes \mathcal{F}$, respectively.

3 The Elliptic Algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$

In this section, we give a definition of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$. To define the algebra, we follow mainly the idea given in Appendix A of [12]. Namely, we first introduce the elliptic currents $e_i(z, p)$, $f_i(z, p)$ and $\psi_i^\pm(z, p)$ of $U_q(\widehat{\mathfrak{sl}}_N)$ by modifying the Drinfeld currents of $U_q(\widehat{\mathfrak{sl}}_N)$. Then we extend them to the currents of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ by taking a tensor product with a Heisenberg algebra $\mathbb{C}\{\hat{\mathcal{H}}\}$ given in Section 3.4.1. Our definition is an extended version of the one given in [12] introducing new currents $K_j(v)$ ($1 \leq j \leq N$). The currents $\{K_j(v)\}$ play an essential role in the construction of the L -operators (Section 5).

3.1 Drinfeld currents of $U_q(\widehat{\mathfrak{sl}}_N)$

Let us first recall the Drinfeld currents of $U_q(\widehat{\mathfrak{sl}}_N)$ [10]. We use the standard symbol of q -integer

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \quad (3.1)$$

We also use the symbol $A = (A_{jk})$ to express the Cartan matrix of $\mathfrak{sl}_N$.

Definition 3.1 (Drinfeld currents) *The algebra $U_q(\widehat{\mathfrak{sl}}_N)$ is a $\mathbb{C}$ -algebra generated by the generators h_i , $a_{i,m}$, $x_{i,n}^\pm$ ($i = 1, \dots, N-1 : m \in \mathbb{Z}_{\neq 0}$, $n \in \mathbb{Z}$), c , d . In terms of the generating functions*

$$x_i^\pm(z) = \sum_{n \in \mathbb{Z}} x_{i,n}^\pm z^{-n}, \quad (3.2)$$

$$\psi_i(q^{\frac{c}{2}} z) = q^{h_i} \exp \left((q - q^{-1}) \sum_{m>0} a_{i,m} z^{-m} \right), \quad (3.3)$$

$$\varphi_i(q^{-\frac{c}{2}} z) = q^{-h_i} \exp \left(-(q - q^{-1}) \sum_{m>0} a_{i,-m} z^m \right) \quad (i = 1, \dots, N-1), \quad (3.4)$$

the defining relations of $U_q(\widehat{\mathfrak{sl}}_N)$ are given by

$$c : \text{central}, \quad (3.5)$$

$$[h_i, d] = [d, h_i] = [d, a_{i,m}] = [a_{i,m}, d] = 0, \quad (3.6)$$

$$[d, x_{i,n}^\pm] = n x_{i,n}^\pm, \quad [h_i, a_{j,m}] = [a_{j,m}, h_i] = 0, \quad (3.7)$$

$$[h_i, x_j^\pm(z)] = \pm A_{ij} x_j^\pm(z), \quad (3.8)$$

$$[a_{i,m}, a_{j,n}] = \frac{[A_{ij}m]_q [cm]_q}{m} q^{-c|m|} \delta_{n+m,0}, \quad (3.9)$$

$$[a_{i,m}, x_j^+(z)] = \frac{[A_{ij}m]_q}{m} q^{-c|m|} z^m x_j^+(z), \quad (3.10)$$

$$[a_{i,m}, x_j^-(z)] = -\frac{[A_{ij}m]_q}{m} z^m x_j^-(z), \quad (3.11)$$

$$(z_1 - q^{\pm A_{ij}} z_2) x_i^\pm(z_1) x_j^\pm(z_2) = (q^{\pm A_{ij}} z_1 - z_2) x_j^\pm(z_2) x_i^\pm(z_1), \quad (3.12)$$

$$[x_i^+(z_1), x_j^-(z_2)] = \frac{\delta_{i,j}}{q - q^{-1}} \left(\delta(q^{-c} z_1/z_2) \psi_i(q^{\frac{c}{2}} z_2) - \delta(q^c z_1/z_2) \varphi_i(q^{-\frac{c}{2}} z_2) \right), \quad (3.13)$$

$$\begin{aligned} & (x_i^\pm(z_1) x_i^\pm(z_2) x_j^\pm(z) - [2]_q x_i^\pm(z_1) x_j^\pm(z) x_i^\pm(z_2) + x_j^\pm(z) x_i^\pm(z_1) x_i^\pm(z_2)) \\ & + (x_i^\pm(z_2) x_i^\pm(z_1) x_j^\pm(z) - [2]_q x_i^\pm(z_2) x_j^\pm(z) x_i^\pm(z_1) + x_j^\pm(z) x_i^\pm(z_2) x_i^\pm(z_1)) = 0, \\ & \text{for } |i - j| = 1. \end{aligned} \quad (3.14)$$

Here $\delta(z)$ denotes the delta function $\delta(z) = \sum_{m \in \mathbb{Z}} z^m$. We call the generators h_j , $a_{j,m}$, $x_{j,n}^\pm$, c , d the Drinfeld generators of $U_q(\widehat{\mathfrak{sl}}_N)$ and the generating functions $x_i^\pm(z)$, $\psi_i(z)$ and $\varphi_i(z)$ the Drinfeld currents.

3.2 Elliptic currents of $U_q(\widehat{\mathfrak{sl}}_N)$

We next introduce an elliptic modification of the currents $x_i^\pm(z)$, $\psi_i(z)$ and $\varphi_i(z)$ according to [12].

Let us define the auxiliary currents $u_i^\pm(z, p)$ by

$$u_i^+(z, p) = \exp \left(\sum_{m>0} \frac{1}{[r^*m]_q} a_{i,-m} (q^r z)^m \right), \quad (3.15)$$

$$u_i^-(z, p) = \exp \left(- \sum_{m>0} \frac{1}{[rm]_q} a_{i,m} (q^{-r} z)^{-m} \right). \quad (3.16)$$

Proposition 3.1 *The following commutation relations hold.*

$$u_i^+(z_1, p) x_j^+(z_2) = \frac{(p^* q^{A_{ij}} z_1 / z_2; p^*)_\infty}{(p^* q^{-A_{ij}} z_1 / z_2; p^*)_\infty} x_j^+(z_2) u_i^+(z_1, p), \quad (3.17)$$

$$u_i^+(z_1, p) x_j^-(z_2) = \frac{(p^* q^{c-A_{ij}} z_1 / z_2; p^*)_\infty}{(p^* q^{c+A_{ij}} z_1 / z_2; p^*)_\infty} x_j^-(z_2) u_i^+(z_1, p), \quad (3.18)$$

$$u_i^-(z_1, p) x_j^+(z_2) = \frac{(pq^{-c-A_{ij}} z_2 / z_1; p)_\infty}{(pq^{-c+A_{ij}} z_2 / z_1; p)_\infty} x_j^+(z_2) u_i^-(z_1, p), \quad (3.19)$$

$$u_i^-(z_1, p) x_j^-(z_2) = \frac{(pq^{A_{ij}} z_2 / z_1; p)_\infty}{(pq^{-A_{ij}} z_2 / z_1; p)_\infty} x_j^-(z_2) u_i^-(z_1, p), \quad (3.20)$$

$$u_i^+(z_1, p) u_j^-(z_2, p) = \frac{(pq^{-c-A_{ij}} z_1 / z_2; p)_\infty}{(pq^{-c+A_{ij}} z_1 / z_2; p)_\infty} \frac{(p^* q^{c+A_{ij}} z_1 / z_2; p^*)_\infty}{(p^* q^{c-A_{ij}} z_1 / z_2; p^*)_\infty} u_j^-(z_2, p) u_i^+(z_1, p). \quad (3.21)$$

Definition 3.2 (Elliptic currents) *Let us define “dressed” currents $e_i(z, p)$, $f_i(z, p)$, $\psi_i^\pm(z, p)$, ($i = 1, \dots, N-1$) by*

$$e_i(z, p) = u_i^+(z, p) x_i^+(z), \quad (3.22)$$

$$f_i(z, p) = x_i^-(z) u_i^-(z, p), \quad (3.23)$$

$$\psi_i^+(z, p) = u_i^+(q^{\frac{c}{2}} z, p) \psi_i(z) u_i^-(q^{-\frac{c}{2}} z, p), \quad (3.24)$$

$$\psi_i^-(z, p) = u_i^+(q^{-\frac{c}{2}} z, p) \varphi_i(z) u_i^-(q^{\frac{c}{2}} z, p). \quad (3.25)$$

We call the currents $e_i(z, p)$, $f_i(z, p)$ and $\psi_i^\pm(z, p)$ the elliptic currents of $U_q(\widehat{\mathfrak{sl}}_N)$.

The reason why we call “elliptic” is because the dressing operation specified by $u_i^\pm(z, p)$ changes the commutation relation of the Drinfeld currents to the elliptic ones.

Proposition 3.2 *The elliptic currents satisfy the following relations.*

$$z_1 \Theta_{p^*}(q^{A_{ij}} z_2 / z_1) e_i(z_1, p) e_j(z_2, p) = -z_2 \Theta_{p^*}(q^{A_{ij}} z_1 / z_2) e_j(z_2, p) e_i(z_1, p), \quad (3.26)$$

$$z_1 \Theta_p(q^{A_{ij}} z_2 / z_1) f_i(z_1, p) f_j(z_2, p) = -z_2 \Theta_p(q^{A_{ij}} z_1 / z_2) f_j(z_2, p) f_i(z_1, p), \quad (3.27)$$

$$[e_i(z_1, p), f_j(z_2, p)] = \frac{\delta_{i,j}}{q - q^{-1}} \left(\delta(q^{-c} z_1 / z_2) \psi_j^+(q^{\frac{c}{2}} z_2, p) - \delta(q^c z_1 / z_2) \psi_j^-(q^{-\frac{c}{2}} z_2, p) \right), \quad (3.28)$$

$$q^{-h_j} \psi_j^+(q^{-r+\frac{c}{2}} z, p) = q^{h_j} \psi_j^-(q^{r-\frac{c}{2}} z, p), \quad (3.29)$$

$$\psi_i^+(z_1, p)\psi_j^+(z_2, p) = \frac{\Theta_p(q^{-A_{ij}}z_1/z_2)\Theta_{p^*}(q^{A_{ij}}z_1/z_2)}{\Theta_p(q^{A_{ij}}z_1/z_2)\Theta_{p^*}(q^{-A_{ij}}z_1/z_2)}\psi_j^+(z_2, p)\psi_i^+(z_1, p), \quad (3.30)$$

$$\psi_i^+(z_1, p)e_j(z_2, p) = \frac{\Theta_{p^*}(q^{A_{ij}+\frac{c}{2}}z_1/z_2)}{\Theta_{p^*}(q^{-A_{ij}+\frac{c}{2}}z_1/z_2)}e_j(z_2, p)\psi_i^+(z_1, p), \quad (3.31)$$

$$\psi_i^+(z_1, p)f_j(z_2, p) = \frac{\Theta_p(q^{-A_{ij}-\frac{c}{2}}z_1/z_2)}{\Theta_p(q^{A_{ij}-\frac{c}{2}}z_1/z_2)}f_j(z_2, p)\psi_i^+(z_1, p), \quad (3.32)$$

$$\begin{aligned} & \frac{(p^*q^2z_2/z_1; p^*)_\infty}{(p^*q^{-2}z_2/z_1; p^*)_\infty} \left\{ \frac{(p^*q^{-1}z/z_1; p^*)_\infty(p^*q^{-1}z/z_2; p^*)_\infty}{(p^*qz/z_1; p^*)_\infty(p^*qz/z_2; p^*)_\infty} e_i(z_1, p)e_i(z_2, p)e_j(z, p) \right. \\ & - [2]_q \frac{(p^*q^{-1}z/z_1; p^*)_\infty(p^*q^{-1}z_2/z; p^*)_\infty}{(p^*qz/z_1; p^*)_\infty(p^*qz_2/z; p^*)_\infty} e_i(z_1, p)e_j(z, p)e_i(z_2, p) \\ & \left. + \frac{(p^*q^{-1}z_1/z; p^*)_\infty(p^*q^{-1}z_2/z; p^*)_\infty}{(p^*qz_1/z; p^*)_\infty(p^*qz_2/z; p^*)_\infty} e_j(z, p)e_i(z_1, p)e_i(z_2, p) \right\} + (z_1 \leftrightarrow z_2) = 0, \end{aligned} \quad (3.33)$$

$$\begin{aligned} & \frac{(pq^{-2}z_2/z_1; p)_\infty}{(pq^2z_2/z_1; p)_\infty} \left\{ \frac{(pqz/z_1; p)_\infty(pqz/z_2; p)_\infty}{(pq^{-1}z/z_1; p)_\infty(pq^{-1}z/z_2; p)_\infty} f_i(z_1, p)f_i(z_2, p)f_j(z, p) \right. \\ & - [2]_q \frac{(pqz/z_1; p)_\infty(pqz_2/z; p)_\infty}{(pq^{-1}z/z_1; p)_\infty(pq^{-1}z_2/z; p)_\infty} f_i(z_1, p)f_j(z, p)f_i(z_2, p) \\ & \left. + \frac{(pqz_1/z; p)_\infty(pqz_2/z; p)_\infty}{(pq^{-1}z_1/z; p)_\infty(pq^{-1}z_2/z; p)_\infty} f_j(z, p)f_i(z_1, p)f_i(z_2, p) \right\} + (z_1 \leftrightarrow z_2) = 0, \\ & \text{for } |i - j| = 1. \end{aligned} \quad (3.34)$$

3.3 New currents $k_j(v)$

In this subsection, we consider a decomposition of the elliptic currents $\psi_j^\pm(z, p)$ ($1 \leq j \leq N-1$) corresponding to the decomposition (3.58). For this purpose, we introduce new currents $k_j(v)$ ($1 \leq j \leq N$).

We first note that the currents $\psi_j^\pm(z, p)$ are expressed by using the Drinfeld generators $a_{j,m}$ as follows.

$$\psi_j^\pm(q^{\mp(r-\frac{c}{2})}z, p) = q^{\pm h_j} : \exp \left\{ - \sum_{m \neq 0} \frac{1}{[r^*m]_q} b_{j,m} (q^{N-j}z)^{-m} \right\} :, \quad (3.35)$$

where we set

$$b_{j,m} = \begin{cases} \frac{[r^*m]_q}{[rm]_q} a_{j,m} & m > 0 \\ q^{c|m|} a_{j,m} & m < 0. \end{cases} \quad (3.36)$$

The colons in (3.35) denote the standard normal ordering.

Let us introduce new generators, B_m^j ($j = 1, \dots, N; m \in \mathbb{Z}$), according to the formula

$$-B_m^j + B_m^{j+1} = \frac{m}{[m]_q} b_{j,m} q^{(N-j)m}, \quad \sum_{j=1}^N q^{2jm} B_m^j = 0, \quad (3.37)$$

or more explicitly,

$$B_m^j = \frac{m}{[m]_q [Nm]_q} \left(\sum_{k=1}^{j-1} [km]_q b_{k,m} - q^{Nm} \sum_{k=j}^{N-1} [(N-k)m]_q b_{k,m} \right). \quad (3.38)$$

From this and (3.9)-(3.11), we derive the following commutation relations.

Proposition 3.3 *For $m, m' \in \mathbb{Z}_{\neq 0}$, $j, k = 1, \dots, N$, the following commutation relations hold.*

$$[B_m^j, B_{m'}^k] = m \delta_{m+m', 0} \frac{[r^*m]_q [cm]_q}{[rm]_q [m]_q [Nm]_q} \times \begin{cases} [(N-1)m]_q & (j = k) \\ -q^{-mN \text{sgn}(j-k)} [m]_q & (j \neq k), \end{cases} \quad (3.39)$$

$$[B_m^j, x_j^\pm(z)] = \mp q^{m(N+1-j-c)} z^m x_j^\pm(z) \times \begin{cases} \frac{[r^*m]_q}{[rm]_q} & (m > 0) \\ q^{cm} & (m < 0), \end{cases} \quad (3.40)$$

$$[B_m^{j+1}, x_j^\pm(z)] = \pm q^{m(N-1-j-c)} z^m x_j^\pm(z) \times \begin{cases} \frac{[r^*m]_q}{[rm]_q} & (m > 0) \\ q^{cm} & (m < 0), \end{cases} \quad (3.41)$$

$$[B_m^k, x_j^\pm(z)] = 0 \quad (k \neq j, j+1). \quad (3.42)$$

We now define new currents $k_j(z, p)$ ($1 \leq j \leq N$) by

$$k_j(z, p) =: \exp \left(\sum_{m \neq 0} \frac{[m]_q}{m [r^*m]_q} B_m^j z^{-m} \right) :. \quad (3.43)$$

Then, from (3.35) and (3.37), we have the following decomposition.

$$\begin{aligned} \psi_j^\pm(q^{\pm(r-\frac{c}{2})}z, p) &= \kappa q^{\pm h_j} k_j(q^{N-j}z, p) k_{j+1}(q^{N-j}z, p)^{-1}, \\ \kappa &= \frac{(p; p)_\infty (p^* q^2; p^*)_\infty}{(p^*; p^*)_\infty (pq^2; p)_\infty}. \end{aligned} \quad (3.44)$$

It is also easy to verify the following commutation relations.

Proposition 3.4

$$\begin{aligned} k_j(z_1, p) k_j(z_2, p) &= \left(\frac{z_1}{z_2} \right)^{\frac{N-1}{N} (\frac{1}{r} - \frac{1}{r^*})} \rho(v_1 - v_2) k_j(z_2, p) k_j(z_1, p), \\ k_{j_1}(z_1, p) k_{j_2}(z_2, p) &= \left(\frac{z_1}{z_2} \right)^{\frac{N-1}{N} (\frac{1}{r} - \frac{1}{r^*})} \rho(v_1 - v_2) \frac{\Theta_p(z_1/z_2)}{\Theta_p(q^{-2}z_1/z_2)} \frac{\Theta_{p^*}(q^{-2}z_1/z_2)}{\Theta_{p^*}(z_1/z_2)} k_{j_2}(z_2, p) k_{j_1}(z_1, p), \end{aligned} \quad (3.45)$$

$$(1 \leq j_1 < j_2 \leq N), \quad (3.46)$$

$$k_1(z, p)k_2(q^2 z, p) \cdots k_{N-1}(q^{2(N-2)} z, p)k_N(q^{2(N-1)} z, p) = c_N^{(N-1)}, \quad (3.47)$$

$$k_j(z_1, p)e_j(z_2, p) = \frac{\Theta_{p^*}(q^{j-N+r^*} z_1/z_2)}{\Theta_{p^*}(q^{j-N+r^*-2} z_1/z_2)} e_j(z_2, p)k_j(z_1, p), \quad (3.48)$$

$$k_{j+1}(z_1, p)e_j(z_2, p) = \frac{\Theta_{p^*}(q^{j-N+r^*} z_1/z_2)}{\Theta_{p^*}(q^{j-N+r^*+2} z_1/z_2)} e_j(z_2, p)k_{j+1}(z_1, p), \quad (3.49)$$

$$k_i(z_1, p)e_j(z_2, p) = e_j(z_2, p)k_i(z_1, p), \quad (i \neq j, j+1), \quad (3.50)$$

$$k_j(z_1, p)f_j(z_2, p) = \frac{\Theta_p(q^{j-N+r-2} z_1/z_2; p)}{\Theta_p(q^{j-N+r} z_1/z_2; p)} f_j(z_2, p)k_j(z_1, p), \quad (3.51)$$

$$k_{j+1}(z_1, p)f_j(z_2, p) = \frac{\Theta_p(q^{j-N+r+2} z_1/z_2)}{\Theta_p(q^{j-N+r} z_1/z_2)} f_j(z_2, p)k_{j+1}(z_1, p), \quad (3.52)$$

$$k_i(z_1, p)f_j(z_2, p) = f_j(z_2, p)k_i(z_1, p), \quad (i \neq j, j+1). \quad (3.53)$$

Here we set

$$\rho(v) = \frac{\rho^+(v)}{\rho^{+*}(v)}, \quad (3.54)$$

$$c_N = \frac{\{pq^{2N+4}\}\{pq^{2N}\}\{p^*q^{2N+2}\}^{*2}}{\{pq^{2N+2}\}^2\{p^*q^{2N+4}\}^*\{p^*q^{2N}\}^*} \quad (3.55)$$

with $\rho^+(v)$ given in (2.19) and $\rho^{+*}(v) = \rho^+(v)|_{r \rightarrow r^*}$.

3.4 Definition of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$

Now we give a definition of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ by considering a tensor product of the elliptic currents of $U_q(\widehat{\mathfrak{sl}}_N)$ with a Heisenberg algebra. In order to keep the defining relations of the algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ with the new currents $K_j(v)$ same as those given in Appendix A of [12], we need to make a central extension of the Heisenberg algebra.

3.4.1 The Heisenberg algebra $\mathbb{C}\{\mathcal{H}\}$ and its extension $\mathbb{C}\{\widehat{\mathcal{H}}\}$

Let ϵ_j ($1 \leq j \leq N$) be the orthonormal basis in $\mathbb{R}^N$ with the inner product $\langle \epsilon_j, \epsilon_k \rangle = \delta_{j,k}$. Setting

$$\bar{\epsilon}_j = \epsilon_j - \epsilon, \quad \epsilon = \frac{1}{N} \sum_{j=1}^N \epsilon_j, \quad (3.56)$$

we have the weight lattice P of $A_{N-1}^{(1)}$

$$P = \oplus_{j=1}^N \mathbb{Z} \bar{\epsilon}_j. \quad (3.57)$$

Then the simple roots α_j ($1 \leq j \leq N-1$) of $\mathfrak{sl}_N$ are given by

$$\alpha_j = -\bar{\epsilon}_j + \bar{\epsilon}_{j+1}. \quad (3.58)$$

Let us introduce operators h_α, β ($\alpha, \beta \in P$) by

$$[h_{\bar{\epsilon}_j}, \bar{\epsilon}_k] = \langle \bar{\epsilon}_j, \bar{\epsilon}_k \rangle, \quad [h_{\bar{\epsilon}_j}, h_{\bar{\epsilon}_k}] = 0 = [\bar{\epsilon}_j, \bar{\epsilon}_k], \quad (3.59)$$

$h_\alpha = \sum_j n_j h_{\bar{\epsilon}_j}$ for $\alpha = \sum_j n_j \bar{\epsilon}_j$ and $h_0 = 0$. Note that $\langle \bar{\epsilon}_j, \bar{\epsilon}_k \rangle = \delta_{j,k} - \frac{1}{N}$ and $[h_{\alpha_j}, \alpha_k] = 2\delta_{j,k} - \delta_{j,k+1} - \delta_{j,k-1} = A_{jk}$. Hence, we identify $h_{\alpha_j} = -h_{\bar{\epsilon}_j} + h_{\bar{\epsilon}_{j+1}}$ with h_j in the Drinfeld generators of $U_q(\widehat{\mathfrak{sl}}_N)$ (Section 3.1). Noting $\sum_{j=1}^N h_{\bar{\epsilon}_j} = 0$, one can solve a set of equation $h_j = -h_{\bar{\epsilon}_j} + h_{\bar{\epsilon}_{j+1}}$ ($1 \leq j \leq N-1$) for $h_{\bar{\epsilon}_j}$.

$$h_{\bar{\epsilon}_j} = \sum_{k=1}^{j-1} h_k - \frac{1}{N} \sum_{k=1}^{N-1} (N-k) h_k. \quad (3.60)$$

From this and (3.6)-(3.8), one can verify the following commutation relations with the Drinfeld generators $c, d, a_{j,m}, x_{j,m}^\pm$ of $U_q(\widehat{\mathfrak{sl}}_N)$.

$$[h_{\bar{\epsilon}_i}, a_{j,m}] = [h_{\bar{\epsilon}_i}, d] = [h_{\bar{\epsilon}_i}, c] = 0, \quad (3.61)$$

$$[h_{\bar{\epsilon}_i}, x_{j,m}^\pm] = \pm(-\delta_{i,j} + \delta_{i,j+1}) x_{j,m}^\pm. \quad (3.62)$$

Now let us introduce another Heisenberg algebra $\mathbb{C}\{\mathcal{H}\}$ generated by P_α and Q_β ($\alpha, \beta \in P$) satisfying the commutation relations

$$[P_{\bar{\epsilon}_j}, Q_{\bar{\epsilon}_k}] = \langle \bar{\epsilon}_j, \bar{\epsilon}_k \rangle, \quad [P_{\bar{\epsilon}_j}, P_{\bar{\epsilon}_k}] = 0 = [Q_{\bar{\epsilon}_j}, Q_{\bar{\epsilon}_k}], \quad (3.63)$$

where $P_\alpha = \sum_j n_j P_{\bar{\epsilon}_j}$ for $\alpha = \sum_j n_j \bar{\epsilon}_j$ and $P_0 = 0$. We also impose that $\mathbb{C}\{\mathcal{H}\}$ commutes with $U_q(\widehat{\mathfrak{sl}}_N)$.

$$[P_{\bar{\epsilon}_j}, \alpha] = [Q_{\bar{\epsilon}_j}, \alpha] = 0, \quad (3.64)$$

$$[P_{\bar{\epsilon}_j}, U_q(\widehat{\mathfrak{sl}}_N)] = [Q_{\bar{\epsilon}_j}, U_q(\widehat{\mathfrak{sl}}_N)] = 0. \quad (3.65)$$

Definition 3.5 We define an extension $\mathbb{C}\{\hat{\mathcal{H}}\}$ of the Heisenberg algebra $\mathbb{C}\{\mathcal{H}\}$ by introducing new generators η_j ($1 \leq j \leq N-1$) and modifying the relations (3.63) to the following ones.

$$[P_{\bar{\epsilon}_j}, Q_{\bar{\epsilon}_k}] = \langle \bar{\epsilon}_j, \bar{\epsilon}_k \rangle, \quad [P_{\bar{\epsilon}_j}, P_{\bar{\epsilon}_k}] = 0, \quad (3.66)$$

$$[Q_{\bar{\epsilon}_j}, Q_{\bar{\epsilon}_k}] = \left(\frac{1}{r} - \frac{1}{r^*} \right) \text{sgn}(j-k) \log q, \quad (3.67)$$

$$[Q_{\bar{\epsilon}_j}, \eta_k] = \frac{1}{r} \text{sgn}(j-k) \log q, \quad (3.68)$$

$$[\eta_j, \eta_k] = \frac{1}{r} \text{sgn}(j-k) \log q, \quad (3.69)$$

$$[P_{\bar{\epsilon}_j}, \eta_k] = 0, \quad \sum_{j=1}^N \eta_j = 0. \quad (3.70)$$

We also impose the following commutation relations.

$$[\eta_j, \alpha] = [\eta_j, U_q(\widehat{\mathfrak{sl}}_N)] = 0. \quad (3.71)$$

If we set $\bar{\alpha}_j = -\eta_j + \eta_{j+1}$, we have

Proposition 3.6

$$[Q_{\alpha_j}, Q_{\alpha_k}] = \left(\frac{1}{r} - \frac{1}{r^*} \right) (\delta_{j,k+1} - \delta_{j,k-1}) \log q, \quad (3.72)$$

$$[Q_{\bar{\epsilon}_j}, Q_{\alpha_k}] = - \left(\frac{1}{r} - \frac{1}{r^*} \right) (\delta_{j,k} + \delta_{j,k+1}) \log q, \quad (3.73)$$

$$[Q_{\bar{\epsilon}_j}, \bar{\alpha}_k] = -\frac{1}{r} (\delta_{j,k} + \delta_{j,k+1}) \log q, \quad (3.74)$$

$$[Q_{\alpha_j}, \bar{\alpha}_k] = \frac{1}{r} (\delta_{j,k+1} - \delta_{j,k-1}) \log q, \quad (3.75)$$

$$[\bar{\alpha}_j, \bar{\alpha}_k] = \frac{1}{r} (\delta_{j,k+1} - \delta_{j,k-1}) \log q, \quad (3.76)$$

$$[\bar{\alpha}_j, P_{\bar{\epsilon}_k}] = [\bar{\alpha}_j, U_q(\widehat{\mathfrak{sl}}_N)] = 0. \quad (3.77)$$

3.4.2 Definition of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$

Now we are ready to define the currents $E_j(v), F_j(v), H_j^\pm(v)$ ($1 \leq j \leq N-1$) and $K_j(v)$ ($1 \leq j \leq N$).

$$E_j(v) = e_j(z, p) e^{\bar{\alpha}_j} e^{-Q_{\alpha_j}} (q^{-j+N} z)^{-\frac{P_{\alpha_j}-1}{r^*}}, \quad (3.78)$$

$$F_j(v) = f_j(z, p) e^{-\bar{\alpha}_j} (q^{-j+N} z)^{\frac{P_{\alpha_j}-1}{r}} (q^{-j+N} z)^{\frac{h_j}{r}}, \quad (3.79)$$

$$H_j^\pm(v) = \psi_j^\pm(z, p) q^{\mp h_j} e^{-Q_{\alpha_j}} (q^{-j+N \pm (r-\frac{c}{2})} z)^{(-\frac{1}{r^*} + \frac{1}{r})(P_{\alpha_j}-1) + \frac{1}{r} h_j}, \quad (3.80)$$

$$K_j(v) = k_j(z, p) e^{Q_{\bar{\epsilon}_j} z (\frac{1}{r^*} - \frac{1}{r}) P_{\bar{\epsilon}_j}} z^{-\frac{1}{r} h_{\bar{\epsilon}_j} + (\frac{1}{r^*} - \frac{1}{r}) \frac{N-1}{2N}}. \quad (3.81)$$

Here the currents $e_j(z, p), f_j(z, p), \psi_j^\pm(z, p)$ and $k_j(z, p)$ are the elliptic currents of $U_q(\widehat{\mathfrak{sl}}_N)$ given in (3.22)-(3.25) and (3.43), whereas $\bar{\alpha}_j, P_\alpha, Q_\beta$ ($\alpha, \beta \in P$) are the elements in the Heisenberg algebra $\mathbb{C}\{\hat{\mathcal{H}}\}$. From (3.26)-(3.28), (3.45)-(3.53) and (3.66)-(3.71), we can verify the following relations.

Proposition 3.7

$$E_i(v_1) E_j(v_2) = \frac{[v_1 - v_2 + \frac{A_{ij}}{2}]^*}{[v_1 - v_2 - \frac{A_{ij}}{2}]^*} E_j(v_2) E_i(v_1), \quad (3.82)$$

$$F_i(v_1) F_j(v_2) = \frac{[v_1 - v_2 - \frac{A_{ij}}{2}]}{[v_1 - v_2 + \frac{A_{ij}}{2}]} F_j(v_2) F_i(v_1), \quad (3.83)$$

$$[E_i(v_1), F_j(v_2)] = \frac{\delta_{i,j}}{q - q^{-1}} \left(\delta(q^{-c} z_1/z_2) H_j^+ \left(v_2 + \frac{c}{4} \right) - \delta(q^c z_1/z_2) H_j^- \left(v_2 - \frac{c}{4} \right) \right), \quad (3.84)$$

$$H_j^\pm \left(v \mp \frac{1}{2} \left(r - \frac{c}{2} \right) \right) = \kappa K_j \left(v + \frac{N-j}{2} \right) K_{j+1} \left(v + \frac{N-j}{2} \right)^{-1}, \quad (3.85)$$

$$K_j(v_1) K_j(v_2) = \rho(v_1 - v_2) K_j(v_2) K_j(v_1), \quad (3.86)$$

$$K_{j_1}(v_1) K_{j_2}(v_2) = \rho(v_1 - v_2) \frac{[v_1 - v_2 - 1]^* [v_1 - v_2]}{[v_1 - v_2]^* [v_1 - v_2 - 1]} K_{j_2}(v_2) K_{j_1}(v_1) \quad (1 \leq j_1 < j_2 \leq N), \quad (3.87)$$

$$K_j(v_1) E_j(v_2) = \frac{[v_1 - v_2 + \frac{j+r^*-N}{2}]^*}{[v_1 - v_2 + \frac{j+r^*-N}{2} - 1]^*} E_j(v_2) K_j(v_1), \quad (3.88)$$

$$K_{j+1}(v_1) E_j(v_2) = \frac{[v_1 - v_2 + \frac{j+r^*-N}{2}]^*}{[v_1 - v_2 + \frac{j+r^*-N}{2} + 1]^*} E_j(v_2) K_{j+1}(v_1), \quad (3.89)$$

$$K_{j_1}(v_1) E_{j_2}(v_2) = E_{j_2}(v_2) K_{j_1}(v_1) \quad (j_1 \neq j_2, j_2 + 1), \quad (3.90)$$

$$K_j(v_1) F_j(v_2) = \frac{[v_1 - v_2 + \frac{j+r-N}{2} - 1]}{[v_1 - v_2 + \frac{j+r-N}{2}]} F_j(v_2) K_j(v_1), \quad (3.91)$$

$$K_{j+1}(v_1) F_j(v_2) = \frac{[v_1 - v_2 + \frac{j+r-N}{2} + 1]}{[v_1 - v_2 + \frac{j+r-N}{2}]} F_j(v_2) K_{j+1}(v_1), \quad (3.92)$$

$$K_{j_1}(v_1) F_{j_2}(v_2) = F_{j_2}(v_2) K_{j_1}(v_1) \quad (j_1 \neq j_2, j_2 + 1), \quad (3.93)$$

$$\begin{aligned} & z_1^{-\frac{1}{r^*}} \frac{(p^* q^2 z_2/z_1; p^*)_\infty}{(p^* q^{-2} z_2/z_1; p^*)_\infty} \left\{ (z_2/z_1)^{\frac{1}{r^*}} \frac{(p^* q^{-1} z/z_1; p^*)_\infty (p^* q^{-1} z/z_2; p^*)_\infty}{(p^* q z/z_1; p^*)_\infty (p^* q z/z_2; p^*)_\infty} E_i(v_1) E_i(v_2) E_j(v) \right. \\ & - [2]_q \frac{(p^* q^{-1} z/z_1; p^*)_\infty (p^* q^{-1} z_2/z; p^*)_\infty}{(p^* q z/z_1; p^*)_\infty (p^* q z_2/z; p^*)_\infty} E_i(v_1) E_j(v) E_i(v_2) \\ & \left. + (z/z_1)^{\frac{1}{r^*}} \frac{(p^* q^{-1} z_1/z; p^*)_\infty (p^* q^{-1} z_2/z; p^*)_\infty}{(p^* q z_1/z; p^*)_\infty (p^* q z_2/z; p^*)_\infty} E_j(v) E_i(v_1) E_i(v_2) \right\} + (z_1 \leftrightarrow z_2) = 0, \end{aligned} \quad (3.94)$$

$$\begin{aligned} & z_1^{\frac{1}{r}} \frac{(pq^{-2} z_2/z_1; p)_\infty}{(pq^2 z_2/z_1; p)_\infty} \left\{ (z/z_2)^{\frac{1}{r}} \frac{(pq z/z_1; p)_\infty (pq z/z_2; p)_\infty}{(pq^{-1} z/z_1; p)_\infty (pq^{-1} z/z_2; p)_\infty} F_i(v_1) F_i(v_2) F_j(v) \right. \\ & - [2]_q \frac{(pq z/z_1; p)_\infty (pq z_2/z; p)_\infty}{(pq^{-1} z/z_1; p)_\infty (pq^{-1} z_2/z; p)_\infty} F_i(v_1) F_j(v) F_i(v_2) \\ & \left. + (z_1/z)^{\frac{1}{r}} \frac{(pq z_1/z; p)_\infty (pq z_2/z; p)_\infty}{(pq^{-1} z_1/z; p)_\infty (pq^{-1} z_2/z; p)_\infty} F_j(v) F_i(v_1) F_i(v_2) \right\} + (z_1 \leftrightarrow z_2) = 0 \quad (|i - j| = 1). \end{aligned} \quad (3.95)$$

Here the constant κ and the function $\rho(v)$ are given in (3.44) and (3.54), respectively.

The following relations among $H_j^\pm(v)$ are also useful.

Proposition 3.8

$$H_j^+ \left(v - r + \frac{c}{4} \right) = H_j^- \left(v - \frac{c}{4} \right), \quad (3.96)$$

$$H_i^+(v_1)H_j^+(v_2) = \frac{[v_1 - v_2 - \frac{A_{ij}}{2}][v_1 - v_2 + \frac{A_{ij}}{2}]^*}{[v_1 - v_2 + \frac{A_{ij}}{2}][v_1 - v_2 - \frac{A_{ij}}{2}]^*} H_j^+(v_2)H_i^+(v_1), \quad (3.97)$$

$$H_i^+(v_1)E_j(v_2) = \frac{[v_1 - v_2 + \frac{A_{ij}}{2} + \frac{c}{4}]^*}{[v_1 - v_2 - \frac{A_{ij}}{2} + \frac{c}{4}]^*} E_j(v_2)H_i^+(v_1), \quad (3.98)$$

$$H_i^+(v_1)F_j(v_2) = \frac{[v_1 - v_2 - \frac{A_{ij}}{2} - \frac{c}{4}]}{[v_1 - v_2 + \frac{A_{ij}}{2} - \frac{c}{4}]} F_j(v_2)H_i^+(v_1), \quad (3.99)$$

Definition 3.9 (Elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$) We define the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ to be the associative algebra of the currents $E_j(v)$, $F_j(v)$ ($1 \leq j \leq N-1$) and $K_j(v)$ ($1 \leq j \leq N$) satisfying the relations (3.82)-(3.95).

Proposition 3.10 The construction of $E_j(v)$, $F_j(v)$ and $K_j(v)$ given in (3.78)-(3.81) is a realization of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ in terms of the Drinfeld generators of $U_q(\widehat{\mathfrak{sl}}_N)$, $h_{\bar{e}_j}$, B_m^j , $x_{i,n}^\pm$ ($i = 1, \dots, N-1 : m \in \mathbb{Z}_{\neq 0}, n \in \mathbb{Z}$), c , d and the Heisenberg algebra $\mathbb{C}\{\hat{\mathcal{H}}\}$ generated by $P_{\bar{e}_j}$, $Q_{\bar{e}_j}$ ($1 \leq j \leq N$) and $\bar{\alpha}_j$ ($1 \leq j \leq N-1$).

Remark. In Appendix A of [12], a realization of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ is given by using the Drinfeld currents of $U_q(\widehat{\mathfrak{sl}}_N)$ and the Heisenberg algebra generated by $\{P_j, Q_j\}$ satisfying

$$[P_i, Q_j] = -\frac{A_{ij}}{2}, \quad (3.100)$$

which has no central extension. The relation between $\{P_j, Q_j\}$ and $\{P_{\alpha_j}, Q_{\alpha_j}\}$ in $\mathbb{C}\{\mathcal{H}\}$ is $P_{\alpha_j} = P_j$ and $Q_{\alpha_j} = -2Q_j$. The role of the central extension and the additional elements η_j (3.67)-(3.69) is to suppress some extra q -fractional-power-factors in the relations in Proposition 3.7. As for the problem realizing the L -operators satisfying the dynamical RLL -relation (Section 5), such q -factors can be absorbed into a choice of the gauge expressing the R -matrix. Conversely, in a gauge expressing the R -matrix components as $b(u, s) = q^{-\frac{1}{r} \frac{[s+1][s-1][u]}{[s]^2[u+1]}}$, $\bar{b}(u) = q^{\frac{1}{r} \frac{[u]}{[u+1]}}$ and the others remaining the same as in (2.18), we need neither the central extension nor the addition of η_j .

4 Half Currents

In order to construct a L -operator, we here introduce the half currents $E_{i,j}^+(v)$, $F_{j,l}^+(v)$ and $K_j^+(v)$ and investigate their commutation relations. We follow the idea of [20, 4, 12].

We often use the abbreviations

$$P_{j,l} = -P_{\bar{\epsilon}_j} + P_{\bar{\epsilon}_l} = P_{\alpha_j} + P_{\alpha_{j+1}} + \cdots + P_{\alpha_{l-1}}, \quad (4.1)$$

$$h_{j,l} = -h_{\bar{\epsilon}_j} + h_{\bar{\epsilon}_l} = h_j + h_{j+1} + \cdots + h_{l-1} \quad (4.2)$$

for $j < l$. From the definition of $\mathbb{C}\{\hat{\mathcal{H}}\}$ and (3.78)-(3.81), we have

$$[K_j(v), P_{k,l}] = (\delta_{j,k} - \delta_{j,l})K_j(v) = [K_j(v), P_{k,l} + h_{k,l}], \quad (4.3)$$

$$[E_j(v), P_{k,l}] = (\delta_{j,k} + \delta_{j+1,l} - \delta_{j,l} - \delta_{j+1,k})E_j(v), \quad (4.4)$$

$$[F_j(v), P_{j,l} + h_{j,l}] = (\delta_{j,k} + \delta_{j+1,l} - \delta_{j,l} - \delta_{j+1,k})F_j(v), \quad (4.5)$$

$$[F_j(v), P_{k,l}] = 0 = [E_j(v), P_{k,l} + h_{k,l}]. \quad (4.6)$$

Now we define the half currents of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ as follows.

Definition 4.1 (Half currents) We define the half currents $F_{j,l}^+(v), E_{l,j}^+(v), (1 \leq j < l \leq N)$ and $K_j^+(v) (j = 1, \dots, N)$ by

$$K_j^+(v) = K_j \left(v + \frac{r+1}{2} \right) \quad (1 \leq j \leq N), \quad (4.7)$$

$$\begin{aligned} F_{j,l}^+(v) &= a_{j,l} \oint_{C(j,l)} \prod_{m=j}^{l-1} \frac{dz_m}{2\pi i z_m} F_{l-1}(v_{l-1}) F_{l-2}(v_{l-2}) \cdots F_j(v_j) \\ &\quad \times \frac{[v - v_{l-1} + P_{j,l} + h_{j,l} + \frac{l-N}{2} - 1][1]}{[v - v_{l-1} + \frac{l-N}{2}][P_{j,l} + h_{j,l} - 1]} \\ &\quad \times \prod_{m=j}^{l-2} \frac{[v_{m+1} - v_m + P_{j,m+1} + h_{j,m+1} - \frac{1}{2}][1]}{[v_{m+1} - v_m + \frac{1}{2}][P_{j,m+1} + h_{j,m+1}]}, \end{aligned} \quad (4.8)$$

$$\begin{aligned} E_{l,j}^+(v) &= a_{j,l}^* \oint_{C^*(j,l)} \prod_{m=j}^{l-1} \frac{dz_m}{2\pi i z_m} E_j(v_j) E_{j+1}(v_{j+1}) \cdots E_{l-1}(v_{l-1}) \\ &\quad \times \frac{[v - v_{l-1} - P_{j,l} + \frac{l-N}{2} + \frac{c}{2} + 1]^*[1]^*}{[v - v_{l-1} + \frac{l-N}{2} + \frac{c}{2}]^*[P_{j,l} - 1]^*} \\ &\quad \times \prod_{m=j}^{l-2} \frac{[v_{m+1} - v_m - P_{j,m+1} + \frac{1}{2}]^*[1]^*}{[v_{m+1} - v_m + \frac{1}{2}]^*[P_{j,m+1} - 1]^*}. \end{aligned} \quad (4.9)$$

Here the integration contour $C(j,l)$ and $C^*(j,l)$ are given by

$$\begin{aligned} C(j,l) : \quad &|pq^{l-N}z| < |z_{l-1}| < |q^{l-N}z|, \\ &|pqz_{k+1}| < |z_k| < |qz_{k+1}|, \end{aligned} \quad (4.10)$$

$$\begin{aligned} C^*(j,l) : \quad &|p^*q^{l-N+c}z| < |z_{l-1}| < |q^{l-N+c}z|, \\ &|p^*qz_{k+1}| < |z_k| < |qz_{k+1}|, \end{aligned} \quad (4.11)$$

where $k = j, j+1, \dots, l-2$. The constants $a_{j,l}$ and $a_{j,l}^*$ are chosen to satisfy

$$\frac{\kappa a_{j,l} a_{j,l}^* [1]}{q - q^{-1}} = 1. \quad (4.12)$$

Then we can verify the following commutation relations

Theorem 4.1 *The half currents $E_{l,j}^+(v), F_{j,l}^+(v)$ and $K_j(v)$ ($1 \leq j < l \leq N$) satisfy the following relations.*

$$K_j^+(v_1) K_j^+(v_2) = \rho(v) K_j^+(v_2) K_j^+(v_1) \quad (1 \leq j \leq N), \quad (4.13)$$

$$K_j^+(v_1) K_l^+(v_2) = \rho(v) \frac{[v-1]^* [v]}{[v]^* [v-1]} K_l^+(v_2) K_j^+(v_1) \quad (1 \leq j < l \leq N), \quad (4.14)$$

$$K_l^+(v_1)^{-1} E_{l,j}^+(v_2) K_l^+(v_1) = E_{l,j}^+(v_2) \frac{1}{\bar{b}^*(v)} - E_{l,j}^+(v_1) \frac{c^*(v, P_{j,l})}{\bar{b}^*(v)}, \quad (4.15)$$

$$K_l^+(v_1) F_{j,l}^+(v_2) K_l^+(v_1)^{-1} = \frac{1}{\bar{b}(v)} F_{j,l}^+(v_2) - \frac{\bar{c}(v, P_{j,l} + h_{j,l})}{\bar{b}(v)} F_{j,l}^+(v_1), \quad (4.16)$$

$$\begin{aligned} & \frac{[1-v]^*}{[v]^*} E_{l,j}^+(v_1) E_{l,j}^+(v_2) + \frac{[1+v]^*}{[v]^*} E_{l,j}^+(v_2) E_{l,j}^+(v_1) \\ &= E_{l,j}^+(v_1)^2 \frac{[1]^* [P_{j,l} - 2 + v]^*}{[P_{j,l} - 2]^* [v]^*} + E_{l,j}^+(v_2)^2 \frac{[1]^* [P_{j,l} - 2 - v]^*}{[P_{j,l} - 2]^* [v]^*}, \end{aligned} \quad (4.17)$$

$$\begin{aligned} & \frac{[1+v]}{[v]} F_{j,l}^+(v_1) F_{j,l}^+(v_2) + \frac{[1-v]}{[v]} F_{j,l}^+(v_2) F_{j,l}^+(v_1) \\ &= F_{j,l}^+(v_1)^2 \frac{[1] [P_{j,l} + h_{j,l} - 2 - v]}{[P_{j,l} + h_{j,l} - 2] [v]} + F_{j,l}^+(v_2)^2 \frac{[1] [P_{j,l} + h_{j,l} - 2 + v]}{[P_{j,l} + h_{j,l} - 2] [v]}, \end{aligned} \quad (4.18)$$

$$\begin{aligned} & K_l^+(v_2)^{-1} E_{l,k}^+(v_1) K_l^+(v_2) E_{l,j}^+(v_2) \\ &= K_l^+(v_1)^{-1} E_{l,j}^+(v_2) K_l^+(v_1) E_{l,k}^+(v_1) \bar{R}_{kj}^{*kj}(v, P_{j,k}) \\ &+ K_l^+(v_1)^{-1} E_{l,k}^+(v_2) K_l^+(v_1) E_{l,j}^+(v_1) \bar{R}_{jk}^{*kj}(v, P_{j,k}) \quad (j \neq k), \end{aligned} \quad (4.19)$$

$$\begin{aligned} & F_{k,l}^+(v_2) K_l^+(v_2) F_{j,l}^+(v_1) K_l^+(v_2)^{-1} \\ &= \bar{R}_{jk}^{jk}(v, P_{j,k} + h_{j,k}) F_{j,l}^+(v_1) K_l^+(v_1) F_{k,l}^+(v_2) K_l^+(v_1)^{-1} \\ &+ \bar{R}_{jk}^{kj}(v, P_{j,k} + h_{j,k}) F_{k,l}^+(v_1) K_l^+(v_1) F_{j,l}^+(v_2) K_l^+(v_1)^{-1} \quad (j \neq k), \end{aligned} \quad (4.20)$$

$$\begin{aligned} [E_{l,l-1}^+(v_1), F_{j,l}^+(v_2)] &= F_{j,l-1}^+(v_2) K_{l-1}^+(v_2) K_l^+(v_2)^{-1} \frac{[P_{l-1,l} - v - 1]^* [1]^*}{[v]^* [P_{l-1,l} - 1]^*} \\ &- F_{j,l-1}^+(v_1) K_l^+(v_1)^{-1} K_{l-1}^+(v_1) \frac{[P_{j,l} + h_{j,l} - v - 1] [1]}{[v_1 - v_2] [P_{j,l} + h_{j,l} - 1]}, \end{aligned} \quad (4.21)$$

$$\begin{aligned}
[E_{l,j}^+(v_1), F_{l-1,l}^+(v_2)] &= K_{l-1}^+(v_2) K_l^+(v_2)^{-1} E_{l-1,j}^+(v_2) \frac{[P_{j,l} - v - 1]^* [1]^*}{[v]^* [P_{j,l} - 1]^*} \\
&\quad - K_l^+(v_1)^{-1} K_{l-1}^+(v_1) E_{l-1,j}^+(v_1) \frac{[P_{l-1,l} + h_{l-1,l} - v - 1][1]}{[v][P_{l-1,l} + h_{l-1,l} - 1]}, \quad (4.22)
\end{aligned}$$

where $v = v_1 - v_2$.

Proof. The relations (4.13) and (4.14) are direct consequences of (3.86) and (3.87).

We show the relation (4.16). The relations (4.15) can be proved in the same way. Setting $\pi_{l,j} = P_{j,l} + h_{j,l}$, we have from (3.91)-(3.92) and (4.3),

$$\begin{aligned}
&K_l^+(v_1) F_{j,l}^+(v_2) K_l^+(v_1)^{-1} \\
&= a_{j,l} \oint_{C(j,l)} \prod_{k=j}^{l-1} \frac{dz'_k}{2\pi i z'_k} F_{l-1}(v'_{l-1}) F_{l-2}(v'_{l-2}) \cdots F_j(v'_j) \\
&\quad \times \frac{[v_1 - v'_{l-1} + \frac{l-N}{2} + 1][v_2 - v'_{l-1} + \pi_{l,j} + \frac{l-N}{2} - 2][1]}{[v_1 - v'_{l-1} + \frac{l-N}{2}][v_2 - v'_{l-1} + \frac{l-N}{2}][\pi_{l,j} - 2]} \mathcal{A}(v_{l-1}, \dots, v_j; \pi_{l-1,j}, \dots, \pi_{j+1,j}),
\end{aligned}$$

where we set

$$\mathcal{A}(v'_{l-1}, \dots, v'_j; \pi_{l-1,j}, \dots, \pi_{j+1,j}) = \prod_{k=j}^{l-2} \frac{[v'_{k+1} - v'_k + \pi_{k+1,j} - \frac{1}{2}][1]}{[v'_{k+1} - v'_k + \frac{1}{2}][\pi_{k+1,j}]}. \quad (4.23)$$

Then the relation (4.16) follows from the theta function identity

$$\frac{[u_1 + t][u_2 + s]}{[u_1][u_2][s]} = \frac{[u_1 - u_2 + t][u_2 + s + t]}{[u_1 - u_2][u_2][s + t]} + \frac{[u_2 - u_1 + s][u_1 + s + t][t]}{[u_2 - u_1][u_1][s][s + t]} \quad (4.24)$$

with the replacement $u_i = v_i - v'_{l-1} + \frac{l-N}{2}$ ($i = 1, 2$), $s = \pi_{l,j} - 2$, $t = 1$.

Proofs of (4.17)-(4.18) and (4.19)-(4.20) are lengthy. We put them in Appendix B.

Next let us consider the relation (4.21). Integrating the delta function appearing from (3.84), we have

$$\begin{aligned}
&(a_{j,l} a_{j,l}^*)^{-1} (q - q^{-1}) [E_{l,l-1}^+(v_1), F_{j,l}^+(v_2)] \\
&= \left\{ \oint_{C_{l-1}^+} \frac{dz'_{l-1}}{2\pi i z'_{l-1}} \cdots \oint \frac{dz'_j}{2\pi i z'_j} H_{l-1}^+ \left(v'_{l-1} + \frac{c}{4} \right) F_{l-1}^+(v'_{l-1}) \cdots F_j^+(v'_j) \frac{[u_1 - \pi_{l,l-1} + 1]^* [1]^*}{[u_1]^* [\pi_{l,l-1} - 1]^*} \right. \\
&\quad \left. - \oint_{C_{l-1}^-} \frac{dz'_{l-1}}{2\pi i z'_{l-1}} \cdots \oint \frac{dz'_j}{2\pi i z'_j} H_{l-1}^- \left(v'_{l-1} - \frac{c}{4} \right) F_{l-1}^+(v'_{l-1}) \cdots F_j^+(v'_j) \frac{[u_1 - \pi_{l,l-1} + 1 + c]^* [1]^*}{[u_1 + c]^* [\pi_{l,l-1} - 1]^*} \right\} \\
&\quad \times \mathcal{A}(v'_{l-1}, \dots, v'_j; \pi_{l-1,j}, \dots, \pi_{j+1,j}).
\end{aligned}$$

Here the contours $C_{l-1}^\pm$ are now

C_{l-1}^+ encloses $z_1 p^{*n}, z_2 p^n$ ($n = 1, 2, \dots$),

C_{l-1}^- encloses $z_1 q^{2c} p^{*n}, z_2 p^n$ ($n = 1, 2, \dots$).

Then in the second term, changing the variable $z'_{l-1} \rightarrow pz'_{l-1}$ and using the relation $H^-(v' + r - c/4) = H^+(v' + c/4)$, we have the same integrand as the first term but the integration contour C_{l-1}^- becomes

$$C_{l-1}' \text{ encloses } z_1 p^{*n}, z_2 p^n \quad (n = 0, 1, 2, \dots).$$

Therefore taking the residue at $z'_{l-1} = z_1, z_2$ and using the relation (3.85), we get (4.21).

Q.E.D.

5 The L -operator of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and Relation to $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$

In this section, we construct a L -operator $\widehat{L}^+(u)$ by using the half currents and show that it satisfies the dynamical RLL -relation (2.21), which characterizes the algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$. We then clarify the relation between the two elliptic algebras $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$.

5.1 L -operator

Definition 5.1 (L -operator) *By using the half currents, we define the L -operator $\widehat{L}^+(v) \in \text{End}(\mathbb{C}^N) \otimes U_{q,p}(\widehat{\mathfrak{sl}}_N)$ as follows.*

$$\begin{aligned} \widehat{L}^+(u) = & \begin{pmatrix} 1 & F_{1,2}^+(u) & F_{1,3}^+(u) & \cdots & F_{1,N}^+(u) \\ 0 & 1 & F_{2,3}^+(u) & \cdots & F_{2,N}^+(u) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & F_{N-1,N}^+(u) \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} K_1^+(u) & 0 & \cdots & 0 \\ 0 & K_2^+(u) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K_N^+(u) \end{pmatrix} \\ & \times \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ E_{2,1}^+(u) & 1 & \ddots & & \vdots \\ E_{3,1}^+(u) & E_{3,2}^+(u) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ E_{N,1}^+(u) & E_{N,2}^+(u) & \cdots & E_{N,N-1}^+(u) & 1 \end{pmatrix}. \quad (5.1) \end{aligned}$$

Here $E_{i,j}^+(v), F_{j,l}^+(v)$ and $K_j^+(v)$ are the half currents given in Section 4.

Let (π_z, V_z) , $V_z = V \otimes \mathbb{C}[z, z^{-1}]$ be the evaluation representation of $U_q(\widehat{\mathfrak{sl}}_N)$ based on the vector representation $V \cong \mathbb{C}^N$ (see Appendix D). The image of the universal R -matrix $\mathcal{R}(r, \{s_j\})$ of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ in the evaluation representation $(\pi_{V,z} \otimes \pi_{V,1})$ is given by the R -matrix $R^+(v, P)$ in (2.15). Then from a direct comparison with the relations of the half currents in Theorem 4.1, we conjecture the following property of the L -operator.

Conjecture 5.1 *The L -operator $\widehat{L}^+(v)$ satisfies the following $RLL = LLR^*$ relation.*

$$R^{+(12)}(v_1 - v_2, P + h) \widehat{L}^{+(1)}(v_1) \widehat{L}^{+(2)}(v_2) = \widehat{L}^{+(2)}(v_2) \widehat{L}^{+(1)}(v_1) R^{+*(12)}(v_1 - v_2, P). \quad (5.2)$$

In Appendix C, we give a derivation of some of the relations among the half currents involved in (5.2) and discuss their direct comparison with those in Theorem 4.1. In Section 6.3, we give a proof of this statement in the case $c = 1$.

5.2 $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$

Based on the conjecture, we give a relation between $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$. We argue that the RLL relation (5.2) is equivalent to the dynamical RLL relation of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$. Hence we can regard the elliptic currents in $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ as an elliptic analogue of the Drinfeld currents in $U_q(\widehat{\mathfrak{sl}}_N)$ providing a new realization of the elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$.

In order to show this, we consider the realization of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ given in (3.78)-(3.81) and modify the half currents in such a way that they have no $Q_{\bar{\epsilon}_j}$, η_j ($1 \leq j \leq N$) dependence. Let us define the modified half currents $k_j^+(v, P)$ ($1 \leq j \leq N$) and $e_{j,l}^+(v, P)$, $f_{l,j}^+(v, P)$ ($1 \leq j < l \leq N - 1$) as follows.

$$k_j^+(v, P) = K_j^+(v) e^{-Q_{\bar{\epsilon}_j}}, \quad (5.3)$$

$$e_{l,j}^+(v, P) = e^{Q_{\bar{\epsilon}_l} - \eta_l} E_{l,j}^+(v) e^{-Q_{\bar{\epsilon}_j} + \eta_j}, \quad (5.4)$$

$$f_{j,l}^+(v, P) = e^{-\eta_j} F_{j,l}^+(v) e^{\eta_l}. \quad (5.5)$$

Then it is easy to see from (3.78)-(3.81) and (4.7)-(4.9) that the modified half currents depend on neither $Q_{\bar{\epsilon}_j}$ nor η_j and commute with $P_{\bar{\epsilon}_j} \forall j$. We hence regard them as the currents in $U_q(\widehat{\mathfrak{sl}}_N)$ with parameters $P_{\bar{\epsilon}_j}$ and r .

Now we define a modified L -operator $L^+(v, P)$ by

$$L^+(u, P) = \begin{pmatrix} 1 & f_{1,2}^+(u, P) & f_{1,3}^+(u, P) & \cdots & f_{1,N}^+(u, P) \\ 0 & 1 & f_{2,3}^+(u, P) & \cdots & f_{2,N}^+(u, P) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & f_{N-1,N}^+(u, P) \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \\ \times \begin{pmatrix} k_1^+(u, P) & 0 & \cdots & 0 \\ 0 & k_2^+(u, P) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & k_N^+(u, P) \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ e_{2,1}^+(u, P) & 1 & \ddots & & \vdots \\ e_{3,1}^+(u, P) & e_{3,2}^+(u, P) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ e_{N,1}^+(u, P) & e_{N,2}^+(u, P) & \cdots & e_{N,N-1}^+(u, P) & 1 \end{pmatrix}.$$

(5.6)

Then the L -operator $\widehat{L}^+(v)$ and the modified one $L^+(v, P)$ are related by

$$L^+(v, P) = \widehat{L}^+(v) \begin{pmatrix} e^{-Q_{\bar{\epsilon}_1}} & 0 & \cdots & 0 \\ 0 & e^{-Q_{\bar{\epsilon}_2}} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & e^{-Q_{\bar{\epsilon}_N}} \end{pmatrix} = \widehat{L}^+(v) \exp \left\{ \sum_{m=1}^N h_{\epsilon_m}^{(1)} Q_{\bar{\epsilon}_m} \right\}. \quad (5.7)$$

Here $h_{\epsilon_j}^{(1)} = h_{\epsilon_j} \otimes 1$, $h_{\epsilon_m} \equiv -E_{mm}$ (a $N \times N$ matrix unit). The reader should not confuse h_{ϵ_m} with $h_{\bar{\epsilon}_m}$, but note $h_j = -h_{\bar{\epsilon}_j} + h_{\bar{\epsilon}_{j+1}} = -h_{\epsilon_j} + h_{\epsilon_{j+1}}$ on V .

Substituting (5.7) into (5.2) and noting the commutation relations

$$P_{j,l} \exp \left\{ \sum_{m=1}^N h_{\epsilon_m}^{(k)} Q_{\bar{\epsilon}_m} \right\} = \exp \left\{ \sum_{m=1}^N h_{\epsilon_m}^{(k)} Q_{\bar{\epsilon}_m} \right\} (P_{j,l} + h_{j,l}^{(k)}) \quad (5.8)$$

and

$$\left[\sum_{m=1}^N (h_{\epsilon_m}^{(1)} + h_{\epsilon_m}^{(2)}) Q_{\bar{\epsilon}_m}, R^{+*(12)}(v, P) \right] = 0, \quad (5.9)$$

or equivalently

$$[Q_{\bar{\epsilon}_j} + Q_{\bar{\epsilon}_l}, P_{j,l}] = 0, \quad (5.10)$$

we can move each factor $\exp \left\{ -\sum_{m=1}^N h_{\epsilon_m}^{(k)} Q_{\bar{\epsilon}_m} \right\}$ ($k = 1, 2$) to the right end in the both sides.

We then obtain the following statement.

Corollary 5.2 *The modified L -operator $L^+(v, P)$ satisfies the dynamical RLL relation*

$$R^{+(12)}(v, P + h) L^{+(1)}(v_1, P) L^{+(2)}(v_2, P + h^{(1)}) = L^{+(2)}(v_2, P) L^{+(1)}(v_1, P + h^{(2)}) R^{+*(12)}(v, P), \quad (5.11)$$

where $v = v_1 - v_2$.

Comparing this with (2.21), we identify our $L^+(v, P)$ with $L^+(v, s)$ in (2.21) and s_j with P_{α_j} . Note the parametrization (2.14). As a consequence of this result, we regard the elliptic currents $E_j(v)$, $F_j(v)$ ($1 \leq j \leq N-1$) and $K_j(v)$ ($1 \leq j \leq N$) in $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ as the Drinfeld currents of the elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ up to tensoring with the Heisenberg algebra. Conversely, this indicates that $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ is an extension of the algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ by tensoring the Heisenberg algebra $\mathbb{C}\{\hat{\mathcal{H}}\}$ generated by $\{P_{\bar{\epsilon}_j}, Q_{\bar{\epsilon}_j}, \eta_j\}$. Namely, $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ is obtained from $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$, first by tensoring the half of the generators $e^{nQ_{\bar{\epsilon}_j}} e^{m\eta_l}$ ($1 \leq j, l \leq N; n, m \in \mathbb{Z}$), then regarding $s_j = P_{\alpha_j}$ and imposing the commutation relations (3.66)-(3.71). Hence

$$U_{q,p}(\widehat{\mathfrak{sl}}_N) = \mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N) \otimes_{\mathbb{C}\{P_{\bar{\epsilon}_1}, P_{\bar{\epsilon}_2}, \dots, P_{\bar{\epsilon}_{N-1}}\}} \mathbb{C}\{\hat{\mathcal{H}}\}. \quad (5.12)$$

6 Vertex Operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$

Tensoring the Heisenberg algebra breaks down the coalgebra structure of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ [12]. But we can still define $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ counterparts of the intertwining operators of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$. We call such operators the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$. In this section, we study such vertex operators and compare them with those of the $A_{N-1}^{(1)}$ -type face model obtained in the papers [14, 15].

6.1 Intertwining relations

Here we derive $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ counterparts of the dynamical intertwining relations (2.25)-(2.26). In the next subsection, we use such relations to derive a free field realization of the vertex operators.

Let us first define an extension of the U_q modules by

$$\widehat{\mathcal{F}} = \bigoplus_{\mu_1, \dots, \mu_N \in \mathbb{Z}} \mathcal{F} \otimes e^{\mu_1 Q_{\bar{\epsilon}_1} + \dots + \mu_N Q_{\bar{\epsilon}_N}}.$$

Let $\Phi_W(z, P)$ and $\Psi_W^*(z, P)$ be the type I and type II intertwining operators of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ (2.23) - (2.24). We define type I and type II vertex operators $\widehat{\Phi}_W(v)$, $\widehat{\Psi}_W^*(v)$ of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ as the following extensions of the corresponding intertwining operators of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$.

$$\widehat{\Phi}_W(v) = \Phi_W(q^c z, P) : \widehat{\mathcal{F}} \longrightarrow \widehat{\mathcal{F}}' \otimes W_z, \quad (6.1)$$

$$\widehat{\Psi}_W^*(v) = \Psi_W^*(z, P) \exp \left\{ \sum_{j=1}^N h_{\epsilon_j} Q_{\bar{\epsilon}_j} \right\} : W_z \otimes \widehat{\mathcal{F}} \longrightarrow \widehat{\mathcal{F}}'. \quad (6.2)$$

From the relations (5.7) and (2.25)-(2.26), the new operators $\widehat{\Phi}_W(v)$ and $\widehat{\Psi}_W^*(v)$ satisfy the following “intertwining relations”.

$$\widehat{\Phi}_W^{(3)}(v_2) \widehat{L}_V^{+(1)}(v_1) = R_{VW}^{+(13)}(v_1 - v_2, P + h) \widehat{L}_V^{+(1)}(v_1) \widehat{\Phi}_W^{(3)}(v_2), \quad (6.3)$$

$$\widehat{L}_V^{+(1)}(v_1) \widehat{\Psi}_W^{*(2)}(v_2) = \widehat{\Psi}_W^{*(2)}(v_2) \widehat{L}_V^{+(1)}(v_1) R_{VW}^{+*(12)}(v_1 - v_2, P - h^{(1)} - h^{(2)}). \quad (6.4)$$

Now we restrict ourselves to the vector representation V and investigate the relations (6.3)-(6.4) in detail. We denote a basis of V by $\{\mathbf{v}_m\}_{m=1}^N$. In this representation, the R -matrix $R_{VV}^+(v, P)$ is given by $R^+(v, P)$ in (2.15) and the L -operator $\widehat{L}_V^+(v)$ by $\widehat{L}^+(v)$ in (5.1).

We define the components of the vertex operators by

$$\widehat{\Phi}_V \left(u - \frac{1}{2} \right) = \sum_{m=1}^N \Phi_m(u) \otimes \mathbf{v}_m, \quad \widehat{\Psi}_V^* \left(u - \frac{c+1}{2} \right) (\mathbf{v}_m \otimes \cdot) = \Psi_m^*(u), \quad (6.5)$$

and the matrix elements of the L -operator $\widehat{L}^+(u)$ by

$$\widehat{L}^+(u) \mathbf{v}_j = \sum_{1 \leq m \leq N} \mathbf{v}_m L^+(u)_{mj}. \quad (6.6)$$

Using these components, the equation (6.3) is read as follows.

$$\Phi_m(v_2) L_{mj}^+(v_1) = \rho^+(v_1 - v_2 + 1/2) L_{mj}^+(v_1) \Phi_m(v_2), \quad (6.7)$$

$$\begin{aligned} \rho^+(v_1 - v_2 + 1/2)^{-1} \Phi_m(v_2) L_{lj}^+(v_1) \\ = b(v_1 - v_2 + 1/2, P_{l,m} + h_{l,m}) L_{lj}^+(v_1) \Phi_m(v_2) \\ + c(v_1 - v_2 + 1/2, P_{l,m} + h_{l,m}) L_{mj}^+(v_1) \Phi_l(v_2), \end{aligned} \quad (6.8)$$

$$\begin{aligned} \rho^+(v_1 - v_2 + 1/2)^{-1} \Phi_l(v_2) L_{mj}^+(v_1) \\ = \bar{b}(v_1 - v_2 + 1/2) L_{mj}^+(v_1) \Phi_l(v_2) + \bar{c}(v_1 - v_2 + 1/2, P_{l,m} + h_{l,m}) L_{lj}^+(v_1) \Phi_m(v_2), \end{aligned} \quad (6.9)$$

for $1 \leq l < m \leq N$ and $1 \leq j \leq N$. For the type II, we have the following set of the equations arising from the equation (6.4)

$$L_{jm}^+(v_1) \Psi_m^*(v_2) = \rho^{+*}(v_1 - v_2 + 1) \Psi_m^*(v_2) L_{jm}^+(v_1), \quad (6.10)$$

$$\begin{aligned} \rho^{+*}(v_1 - v_2 + 1)^{-1} L_{jl}^+(v_1) \Psi_m^*(v_2) \\ = \Psi_m^*(v_2) L_{jl}^+(v_1) b^*(v_1 - v_2 + 1, P_{l,m}) + \Psi_l^*(v_2) L_{jm}^+(v_1) \bar{c}^*(v_1 - v_2 + 1, P_{l,m}), \end{aligned} \quad (6.11)$$

$$\begin{aligned} \rho^{+*}(v_1 - v_2 + 1)^{-1} L_{jm}^+(v_1) \Psi_l^*(v_2) \\ = \Psi_l^*(v_2) L_{jm}^+(v_1) \bar{b}^*(v_1 - v_2 + 1) + \Psi_m^*(v_2) L_{jl}^+(v_1) c^*(v_1 - v_2 + 1, P_{l,m}), \end{aligned} \quad (6.12)$$

for $1 \leq l < m \leq N$ and $1 \leq j \leq N$.

Let us investigate equations (6.7)-(6.9) in detail. From the component $j = m = N$ of equation (6.7), we have

$$\Phi_N(v_2) K_N^+(v_1) = \rho^+ \left(v_1 - v_2 + \frac{1}{2} \right) K_N^+(v_1) \Phi_N(v_2). \quad (6.13)$$

Setting $1 \leq j < m = N$ in (6.7), we have

$$\Phi_N(v_2) E_{N,j}^+(v_1) = E_{N,j}^+(v_1) \Phi_N(v_2). \quad (6.14)$$

The following relations turn out to be sufficient conditions for (6.14) to hold.

$$\Phi_N(v_2) E_j(v_1) = E_j(v_1) \Phi_N(v_2) \quad (1 \leq j \leq N-1). \quad (6.15)$$

Next let us consider the component $l < m = j = N$ of equation (6.8). We set

$$\rho^+(v) = -\frac{[v+1]}{\varphi(v)}, \quad (6.16)$$

$$\varphi(v) = (q^r z)^{\frac{N+1}{rN}} [v-1] \frac{\{pz\} \{pq^{2N}z\} \{q^{2N+2}/z\} \{q^{-2}/z\}}{\{pq^{2N+2}z\} \{pq^{-2}z\} \{1/z\} \{q^{2N}/z\}}. \quad (6.17)$$

Then (6.8) with $m = j = N$ can be written as

$$\begin{aligned}
& \varphi(v_1 - v_2 + 1/2) \Phi_N(v_2) F_{l,N}^+(v_1) K_N^+(v_1) \\
&= \frac{[P_{l,N} + h_{l,N} - 1][P_{l,N} + h_{l,N} + 1][v_1 - v_2 + 1/2]}{[P_{l,N} + h_{l,N}]^2} F_{l,N}^+(v_1) K_N^+(v_1) \Phi_m(v_2) \\
&+ \frac{[P_{l,N} + h_{l,N} + v_1 - v_2 + 1/2][1]}{[P_{l,N} + h_{l,N}]} K_N^+(v_1) \Phi_l(v_2). \tag{6.18}
\end{aligned}$$

In order to solve (6.18), let us assume that the operator product $K_N^+(v_1) \Phi_N(v_2)$ does not have a pole at $v_1 - v_2 + 3/2 + r = 0$. Later we will check that, for $c = 1$, this assumption is satisfied in a free field realization. Then from relations (6.13) and (6.16), we conclude that the product $\Phi_N(v_2) K_N^+(v_1)$ in the LHS of (6.18) has zero at $v_1 - v_2 + 3/2 + r = 0$. Therefore, setting $v_1 - v_2 + 3/2 + r = 0$ in (6.18), we have

$$\begin{aligned}
\Phi_l(v_2) &= K_N^+(v_1)^{-1} \frac{[P_{j,N} + h_{j,N} + 1]}{[P_{j,N} + h_{j,N}]} F_{l,N}^+(v_1) K_N^+(v_1) \Phi_N(v_2) \\
&= F_{l,N}^+(v_2 - 1/2 - r) \Phi_N(v_2) \quad (1 \leq l \leq N - 1). \tag{6.19}
\end{aligned}$$

Note that the shift of v by r in $F_{l,N}^+(v)$ yields a change of contour (see (6.42)). Substituting (6.13) and (6.19) into (6.8) for $l < m = j = N$, and using Riemann's theta identity, we find that (6.19) and the following relations are sufficient conditions for (6.8) with $l < m = j = N$.

$$F_{N-1}(v_1) \Phi_N(v_2) = \frac{[v_1 - v_2 + \frac{1}{2}]}{[v_1 - v_2 - \frac{1}{2}]} \Phi_N(v_2) F_{N-1}(v_1), \tag{6.20}$$

$$F_l(v_1) \Phi_N(v_2) = \Phi_N(v_2) F_l(v_1) \quad (1 \leq l \leq N - 2), \tag{6.21}$$

$$[\Phi_N(v), P_{j,k} + h_{j,k}] = -\delta_{k,N} \Phi_N(v) \quad (j < k). \tag{6.22}$$

In the next section, we construct a free field realization of the type I vertex operators using relations (6.13), (6.15) and (6.19)-(6.22) for $c = 1$. We then check that the resulting vertex operators satisfy the remaining relations in (6.8) and (6.9).

Similarly, from the $j = m = N$ component of (6.10), we have for the type-II vertex operator

$$K_N^+(v_1) \Psi_N^*(v_2) = \rho^{+*} (v_1 - v_2 + 1) \Psi_N^*(v_2) K_N^+(v_1) \tag{6.23}$$

and from the $1 \leq j < m = N$ component of (6.10),

$$F_{j,N}^+(v_1) \Psi_N^*(v_2) = \Psi_N^*(v_2) F_{j,N}^+(v_1) \quad (1 \leq j \leq N - 1). \tag{6.24}$$

We find the following as sufficient conditions for (6.24).

$$F_j(v_1) \Psi_N^*(v_2) = \Psi_N^*(v_2) F_j(v_1) \quad (1 \leq j \leq N - 1). \tag{6.25}$$

To solve equation (6.11) with $l < j = m = N$, we assume that the product $\Psi_N^*(v_2)K_N^+(v_1)$ has no pole at $v_1 - v_2 + 2 + r^* = 0$. Then the product $K_N^+(v_1)\Psi_N^*(v_2)$ in the LHS has a zero at $v_1 - v_2 + 2 + r^* = 0$ for the same reason as the type I case. Therefore, from (6.11) with $l < j = m = N$ and setting $v_1 - v_2 + 2 + r^* = 0$, we have

$$\Psi_l^*(v) = \Psi_N^*(v)E_{N,l}^+ \left(v - \frac{c+1}{2} - r^* \right) \quad (1 \leq l \leq N-1). \quad (6.26)$$

Then (6.26) and the following relations turns out to be the sufficient conditions for (6.11) and (6.12).

$$E_{N-1}(v_1)\Psi_N^*(v_2) = \frac{[v_1 - v_2 - \frac{1}{2}]^*}{[v_1 - v_2 + \frac{1}{2}]^*} \Psi_N^*(v_2)E_{N-1}(v_1), \quad (6.27)$$

$$E_j(v_1)\Psi_N^*(v_2) = \Psi_N^*(v_2)E_j(v_1) \quad (1 \leq j \leq N-2), \quad (6.28)$$

$$[\Psi_N^*(v), P_{j,k}] = \delta_{k,N}\Psi_N^*(v) \quad (j < k). \quad (6.29)$$

6.2 Free field realizations

Now we construct a free field realization of the vertex operators fixing the representation level $c = 1$. For this purpose, we first consider the simple root operator α_j introduced in Section 3.4.1. We make the following standard central extension.

$$[\alpha_j, \alpha_k] = i\pi A_{jk}. \quad (6.30)$$

Setting $\hat{\alpha}_j = \alpha_j + \bar{\alpha}_j$ where $\bar{\alpha}_j$ is an element of the Heisenberg algebra $\mathbb{C}\{\hat{H}\}$, we have

$$[\hat{\alpha}_j, \hat{\alpha}_k] = i\pi A_{jk} + \frac{1}{r}(\delta_{j,k+1} - \delta_{j,k-1}) \log q, \quad (6.31)$$

$$[h_{\bar{\epsilon}_j}, \hat{\alpha}_k] = -\delta_{j,k} + \delta_{j,k+1}, \quad (6.32)$$

$$[Q_{\bar{\epsilon}_j}, \hat{\alpha}_k] = -\frac{1}{r}(\delta_{j,k} + \delta_{j,k+1}) \log q, \quad (6.33)$$

$$[Q_{\alpha_j}, \hat{\alpha}_k] = \frac{1}{r}(\delta_{j,k+1} - \delta_{j,k-1}) \log q, \quad (6.34)$$

$$[\hat{\alpha}_j, P_{\bar{\epsilon}_k}] = 0. \quad (6.35)$$

Then the following statement holds.

Proposition 6.1 *The currents $E_j(v)$ and $F_j(v)$ given by*

$$E_j(v) = : \exp \left(- \sum_{m \neq 0} \frac{[rm]_q}{m[r^*m]_q} (-B_m^j + B_m^{j+1})(q^{N-j}z)^{-m} \right) : e^{\hat{\alpha}_j} z^{h_j} e^{-Q_{\alpha_j}} (q^{-j+N}z)^{-\frac{P_{\alpha_j}-1}{r^*}}, \quad (6.36)$$

$$F_j(v) = : \exp \left(\sum_{m \neq 0} \frac{1}{m} (-B_m^j + B_m^{j+1})(q^{N-j}z)^{-m} \right) : e^{-\hat{\alpha}_j} z^{-h_j} (q^{-j+N}z)^{\frac{P_{\alpha_j}-1}{r} + \frac{h_j}{r}}, \quad (6.37)$$

together with $H_j^\pm(v), K_j(v)$ given in (3.80)-(3.81) satisfy the commutation relations in Proposition 3.7 for level $c = 1$. Hence they give a free field realization of the level one elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$.

Now substituting the free field realization of $E_j(v), F_j(v), K_j(v)$ into (4.7)-(4.9), we obtain a realization of the half currents $E_j^+(v), F_j^+(v), K_j^+(v)$ as well as the L -operator $\widehat{L}^+(v)$ satisfying the RLL -relation (5.2) for $c = 1$. Using such a L -operator in the “intertwining relations”, (6.13)-(6.22) for type I and (6.23)-(6.29) for the type II, one can solve them for the vertex operators. The results are stated as follows.

Theorem 6.2 *The highest components of the type I and the type II vertex operators $\Phi_N(v), \Psi_N^*(v)$ are realized in terms of a free field by*

$$\Phi_N(v) = : \exp \left(- \sum_{m \neq 0} \frac{1}{m} B_m^N z^{-m} \right) : e^{\bar{\Lambda}_{N-1} z^{(1-\frac{1}{r})h_{\bar{\epsilon}_N}} z^{-\frac{1}{r}P_{\bar{\epsilon}_N}} z^{(1-\frac{1}{r})\frac{N-1}{2N}}}, \quad (6.38)$$

$$\Psi_N^*(v) = : \exp \left(\sum_{m \neq 0} \frac{[rm]_q}{m[r^*m]_q} B_m^N z^{-m} \right) : e^{-\bar{\Lambda}_{N-1} z^{-h_{\bar{\epsilon}_N}} z^{Q_{\bar{\epsilon}_N}} z^{\frac{1}{r^*}P_{\bar{\epsilon}_N}} z^{(1+\frac{1}{r^*})\frac{N-1}{2N}} q^{(\frac{1}{r}-\frac{1}{2})\frac{N-1}{N}}}, \quad (6.39)$$

where

$$\bar{\Lambda}_{N-1} = \frac{1}{N}(\hat{\alpha}_1 + 2\hat{\alpha}_2 + \cdots + (N-1)\hat{\alpha}_{N-1}). \quad (6.40)$$

For the other components of the type I vertex operator $\Phi_j(v)$ ($j = 1, \dots, N$), we obtain from (6.19)

$$\begin{aligned} \Phi_j(v) &= a_{j,N} \oint_C \prod_{m=j}^{N-1} \frac{dz_m}{2\pi i z_m} \Phi_N(v) F_{N-1}(v_{N-1}) \cdots F_j(v_j) \\ &\quad \times \prod_{m=j}^{N-1} \frac{[v_{m+1} - v_m + P_{j,m+1} + h_{j,m+1} - \frac{1}{2}][1]}{[v_{m+1} - v_m + \frac{1}{2}][P_{j,m+1} + h_{j,m+1}]} \\ &= a_{j,N} \oint_C \prod_{m=j}^{N-1} \frac{dz_m}{2\pi i z_m} F_j(v_j) \cdots F_{N-1}(v_{N-1}) \Phi_N(v) \\ &\quad \times \prod_{m=j}^{N-1} \frac{[v_{m+1} - v_m + P_{j,m+1} + h_{j,m+1} - \frac{1}{2}][1]}{[v_{m+1} - v_m - \frac{1}{2}][P_{j,m+1} + h_{j,m+1}]}, \end{aligned} \quad (6.41)$$

where $v = v_N$ and the integration contour C is specified by the condition

$$\begin{aligned} |q^{-1}z| &< |z_{N-1}| < |p^{-1}q^{-1}z|, \\ |pqz_{m+1}| &< |z_m| < |qz_{m+1}| \quad (j \leq m \leq N-2). \end{aligned} \quad (6.42)$$

For the type II vertex $\Psi_j^*(v)$ ($j = 1, \dots, N$), we obtain from (6.26)

$$\begin{aligned}
\Psi_j^*(v) &= a_{j,N}^* \oint_{C^*} \prod_{m=j}^{N-1} \frac{dz_m}{2\pi i z_m} E_j(v_j) \cdots E_{N-1}(v_{N-1}) \Psi_N^*(v_N) \\
&\quad \times \prod_{m=j}^{N-1} \frac{[v_{m+1} - v_m - P_{j,m+1} + \frac{1}{2}]^* [1]^*}{[v_{m+1} - v_m + \frac{1}{2}]^* [P_{j,m+1} - 1]^*} \\
&= a_{j,N}^* \oint_{C^*} \prod_{m=j}^{N-1} \frac{dz_m}{2\pi i z_m} \Psi_N^*(v_N) E_{N-1}(v_{N-1}) \cdots E_j(v_j) \\
&\quad \times \prod_{m=j}^{N-1} \frac{[v_{m+1} - v_m - P_{j,m+1} + \frac{1}{2}]^* [1]^*}{[v_{m+1} - v_m - \frac{1}{2}]^* [P_{j,m+1} - 1]^*}. \tag{6.43}
\end{aligned}$$

The integration contour C^* is specified as follows.

$$|p^* q^{-1} z_{m+1}|, |q^{-1} z_{m+1}| < |z_m| < |q z_{m+1}|, |p^{*-1} q z_{m+1}| \quad (j \leq m \leq N-1). \tag{6.44}$$

Here the integration variable z_m ($j \leq m \leq N-1$) should encircle the poles $p^* q^{-1} z_{m+1}, q^{-1} z_{m+1}$ but not the poles $p^{*-1} q z_{m+1}, q z_{m+1}$.

In addition, we have the following commutation relations.

Proposition 6.3 *The highest components $\Phi_N(v)$ and $\Psi_N^*(v)$ satisfy*

$$[\Phi_N(v), P_{j_1, j_2}] = [\Psi_N^*(v), P_{j_1, j_2} + h_{j_1, j_2}] = 0, \tag{6.45}$$

$$\Phi_N(v_1) \Psi_N^*(v_2) = \chi(v_1 - v_2) \Psi_N^*(v_2) \Phi_N(v_1), \tag{6.46}$$

$$\chi(v) = \frac{\Theta_{q^{2N}}(qz)}{\Theta_{q^{2N}}(q/z)}. \tag{6.47}$$

Remark The free field realizations of the vertex operators in Theorem 6.2 are essentially the same as those of the $A_{N-1}^{(1)}$ -type face model obtained in [14, 15]. There are two differences between ours and those in [14, 15]; the choice of the gauge expressing the R -matrices and the zero-mode operators. Due to our gauge, we have the extra factors $\prod_{m=j}^{N-1} \frac{[1]}{[P_{j,m+1} + h_{j,m+1}]}$ and $\prod_{m=j}^{N-1} \frac{[1]^*}{[P_{j,m+1} - 1]^*}$ in the type I and the type II vertex operators, respectively. As for the zero-modes, the correspondence between ours $P_{\bar{\epsilon}_j}, Q_{\bar{\epsilon}_j}, h_j, \hat{\alpha}_j$ and those in [14, 15], $P_{\alpha_j}, P_{\omega_N}, Q_{\alpha_j}, Q_{\omega_N}$ is given by

$$\begin{pmatrix} \sqrt{\frac{r^*}{r}} P_{\alpha_j} \\ \sqrt{\frac{r^*}{r}} P_{\omega_N} \\ \sqrt{\frac{r}{r^*}} P_{\alpha_j} \\ \sqrt{\frac{r}{r^*}} P_{\omega_N} \end{pmatrix} \leftrightarrow \begin{pmatrix} \frac{r^*}{r} h_j - \frac{1}{r} P_{\alpha_j} \\ \frac{r^*}{r} h_{\bar{\epsilon}_N} - \frac{1}{r} P_{\bar{\epsilon}_N} \\ h_j - \frac{1}{r^*} P_{\alpha_j} \\ h_{\bar{\epsilon}_N} - \frac{1}{r^*} P_{\bar{\epsilon}_N} \end{pmatrix}, \quad \begin{pmatrix} i\sqrt{\frac{r^*}{r}} Q_{\alpha_j} \\ i\sqrt{\frac{r^*}{r}} Q_{\omega_N} \\ i\sqrt{\frac{r}{r^*}} Q_{\alpha_j} \\ i\sqrt{\frac{r}{r^*}} Q_{\omega_N} \end{pmatrix} \leftrightarrow \begin{pmatrix} \hat{\alpha}_j \\ \bar{\Lambda}_{N-1} \\ \hat{\alpha}_j - Q_{\alpha_j} \\ \bar{\Lambda}_{N-1} - Q_{\bar{\epsilon}_N} \end{pmatrix}. \tag{6.48}$$

One should note that to define the currents $K_j(v)$ by factoring the operators $H_j^\pm(v)$, the use of two sets of the Heisenberg operators $\{P_{\bar{\epsilon}_j}, Q_{\bar{\epsilon}_j}\}$ and $\{h_j, \hat{\alpha}_j\}$ are essential.

6.3 Commutation relations

We next investigate commutation relations of the vertex operators and show that our realization satisfies the full intertwining relations for $c = 1$.

Theorem 6.4 *The free field realizations of the type-I vertex operator $\Phi_\mu(v)$ (6.41) and the type-II vertex operator $\Psi_\mu^*(v)$ (6.43) satisfy the following commutation relations.*

$$\Phi_{j_2}(v_2)\Phi_{j_1}(v_1) = \sum_{j'_1, j'_2=1}^N R_{j_1 j_2}^{j'_1 j'_2}(v_1 - v_2, P + h) \Phi_{j'_1}(v_1)\Phi_{j'_2}(v_2), \quad (6.49)$$

$$\Psi_{j_1}^*(v_1)\Psi_{j_2}^*(v_2) = \sum_{j'_1, j'_2=1}^N \Psi_{j'_2}^*(v_2)\Psi_{j'_1}^*(v_1) R_{j'_1 j'_2}^{*j_1 j_2}(v_1 - v_2, P), \quad (6.50)$$

$$\Phi_j(v_1)\Psi_k^*(v_2) = \chi(v_1 - v_2) \Psi_k^*(v_2)\Phi_j(v_1). \quad (6.51)$$

Here we set

$$R(v, P + h) = \mu(v)\bar{R}(v, P + h), \quad R^*(v, P) = \mu^*(v)\bar{R}^*(v, P), \quad (6.52)$$

with

$$\mu(v) = z^{(\frac{1}{r}-1)\frac{N-1}{N}} \frac{\{pq^{2N-2}z\}\{q^2z\}\{p/z\}\{q^{2N}/z\}}{\{pz\}\{q^{2N}z\}\{pq^{2N-2}/z\}\{q^2/z\}}, \quad (6.53)$$

$$(6.54)$$

and $\mu^*(v) = \mu(v)|_{r \rightarrow r^*}$.

Proof. Using the formulae (6.19)-(6.22) and (6.26)-(6.29), the commutation relations (6.49) and (6.50) are reduced to the relations among the half currents (B.1), (B.2), (4.19) and (4.20). Then the proofs of the latter relations are given in Appendix B.

Let us consider the relation (6.51). The case $j = N$ or $k = N$ is a direct consequence of (6.14), (6.24) and (6.46). The simplest non-trivial case is $j = k = N - 1$. From (6.19), (6.26) and (3.84), we have the following equation after integrating the delta functions.

$$\begin{aligned} & \Phi_{N-1}(v_1)\Psi_{N-1}^*(v_2) - \chi(v_1 - v_2) \Psi_{N-1}^*(v_2)\Phi_{N-1}(v_1) \\ = & -\frac{a_{N-1,N}a_{N-1,N}^*}{q - q^{-1}}\Phi_N(v_1)\Psi_N^*(v_2) \\ \times & \left(\oint_{C_+} \frac{dz'}{2\pi iz'} H_{N-1}^+ \left(v' + \frac{1}{4} \right) \frac{[v_2 - v' - P_{N-1,N}]^*[1]^*[v_1 - v' + P_{N-1,N} + h_{N-1,N} - \frac{1}{2}][1]}{[v_2 - v' - 1]^*[P_{N-1,N} - 1]^*[v_1 - v' + \frac{1}{2}][P_{N-1,N} + h_{N-1,N}]} \right. \\ - & \left. \oint_{C_-} \frac{dz'}{2\pi iz'} H_{N-1}^- \left(v' - \frac{1}{4} \right) \frac{[v_2 - v' - P_{N-1,N} + 1]^*[1]^*[v_1 - v' + P_{N-1,N} + h_{N-1,N} - \frac{1}{2}][1]}{[v_2 - v']^*[P_{N-1,N} - 1]^*[v_1 - v' + \frac{1}{2}][P_{N-1,N} + h_{N-1,N}]} \right). \end{aligned} \quad (6.55)$$

The contours are specified by

$$C_+ : |qz_1|, |q^{-2}z_2| < |z'| < |p^{-1}qz_1|, |p^{*-1}q^{-2}z_2|, |p^{-1}q^{-1}z_1|, |p^{*-1}z_2|, \quad (6.56)$$

$$C_- : |qz_1|, |z_2| < |z'| < |p^{-1}qz_1|, |p^{*-1}z_2|, |q^{-1}z_1|, |q^2z_2|. \quad (6.57)$$

Here the conditions $|z'| < |p^{-1}q^{-1}z_1|, |p^{*-1}z_2|$ for C_+ and $|z'| < |q^{-1}z_1|, |q^2z_2|$ for C_- are added because of the convergence of the operator product $\Phi_N(v_1)\Psi_N^*(v_2)H_{N-1}^+(v' + 1/4)$ and $\Phi_N(v_1)\Psi_N^*(v_2)H_{N-1}^-(v' - 1/4)$, respectively. Changing the integration variable $z' \rightarrow pz'$ in the second term and using the periodicity of $[v], [v]^*$ and the relation $H_{N-1}^-(v' + r - \frac{1}{4}) = H_{N-1}^+(v' + \frac{1}{4})$, we see that the integrand in the second term coincides with the one in the first term but the contour in the second term is changed to

$$\tilde{C}_- : |p^{-1}qz_1|, |p^{*-1}q^{-2}z_2| < |z'| < |p^{-2}qz_1|, |p^{*-2}q^{-2}z_2|, |p^{-1}q^{-1}z_1|, |p^{*-1}z_2|.$$

Here $\tilde{C}_-$ encircles the same poles as C_+ . In addition, $\tilde{C}_-$ would encircle two extra poles at $z' = p^{-1}qz_1, p^{*-1}q^{-2}z_2$, if the operator product $\Phi_N(v_1)\Psi_N^*(v_2)H_{N-1}^+(v' + \frac{1}{4})$ had no zeros which cancel these extra poles. In fact, the operator product does have zeros at the required points. Therefore the RHS of (6.55) vanishes. The proof for the general $1 \leq j_1, j_2 \leq N - 1$ case is similar.

Q.E.D.

Now let us investigate the intertwining relation for level $c = 1$. For this purpose, we remind the reader of the fact that in the trigonometric case, i.e. $U_q(\widehat{\mathfrak{sl}}_N)$, the L -operator can be constructed as a composition of type I and II vertex operators [17, 18]. The following theorem is an elliptic analogue of such a construction.

Theorem 6.5 *For $c = 1$, the L -operator $\widehat{L}^+(v)$ is given by a product of the type-I and type-II vertex operators.*

$$\widehat{L}_{jk}^+(v) = \frac{1}{g_N} \Psi_k^*(v + r) \Phi_j \left(v + r + \frac{1}{2} \right) \quad (1 \leq j, k \leq N). \quad (6.58)$$

Here we set

$$g_N = \frac{(q^2; q^{2N})_\infty}{(q^{2N}; q^{2N})_\infty}. \quad (6.59)$$

Proof. For the special component $j = k = N$ of (6.58), we have

$$K_N^+(v) = g_N^{-1} \Psi_N^*(v + r) \Phi_N \left(v + r + \frac{1}{2} \right). \quad (6.60)$$

This is a direct consequence of (6.38) and (6.39). Let us consider the $j < k = N$ component of (6.58). After a few calculations using (6.24) and (6.60), we can reduce this to relation (6.19). Similarly, the $N = j > k$ component of (6.58) is reduced to relation (6.26).

Next, let us study the simplest non-trivial component $j = k = N - 1$. From (6.19), (6.24)-(6.28), (3.89)-(3.90) and (3.84), we have the following equation after integrating the delta functions.

$$\begin{aligned}
& g_N^{-1} \Psi_{N-1}^*(v+r) \Phi_{N-1}(v+r+1/2) - F_{N-1,N}^+(v) K_N^+(v) E_{N,N-1}^+(v) \\
&= \frac{a_{N-1,N} a_{N-1,N}^*}{q - q^{-1}} \\
&\times \left(\oint_{C_+} \frac{dz'}{2\pi i z'} H_{N-1}^+ \left(v' + \frac{1}{4} \right) K_N^+(v) \frac{[v-v'-P_{N-1,N}+1]^*[1]^*[v-v'+P_{N-1,N}+h_{N-1,N}][1]}{[v-v'+1]^*[P_{N-1,N}-1]^*[v-v'] [P_{N-1,N}+h_{N-1,N}]} \right. \\
&- \left. \oint_{C_-} \frac{dz'}{2\pi i z'} H_{N-1}^- \left(v' - \frac{1}{4} \right) K_N^+(v) \frac{[v-v'-P_{N-1,N}+2]^*[1]^*[v-v'+P_{N-1,N}+h_{N-1,N}][1]}{[v-v'+2]^*[P_{N-1,N}-1]^*[v-v'] [P_{N-1,N}+h_{N-1,N}]} \right). \tag{6.61}
\end{aligned}$$

Here the contours are

$$C_+ : |pz|, |p^*z| < |z'| < |z|, \tag{6.62}$$

$$C_- : |pz| < |z'| < |z|, |q^2z|. \tag{6.63}$$

Changing the integration variable $z' \rightarrow pz'$ in the second term and using the periodicity of $[v], [v]^*$ and the relation $H_{N-1}^-(v'+r-1/4) = H_{N-1}^+(v'+1/4)$, we see that the integrand in the second term coincides with the first term, but the contour in the second term is changed to

$$\tilde{C}_- : |z| < |z'| < |p^{-1}z|, |p^{*-1}z|. \tag{6.64}$$

The contour $\tilde{C}_-$ encircles the same poles as C_+ together with one additional pole at $z' = z$. Hence the RHS of (6.61) becomes the residue at $z' = z$. We thus obtain

$$g_N^{-1} \Psi_{N-1}^*(v+r) \Phi_{N-1}(v+r+1/2) = F_{N-1,N}^+(v) K_N^+(v) E_{N,N-1}^+(v) + K_{N-1}^+(v). \tag{6.65}$$

The RHS coincides with the $(N-1, N-1)$ component of $\hat{L}^+(v)$. The proof of the general case $1 \leq j, k \leq N-1$ is similar.

Q.E.D.

Corollary 6.6 *For $c = 1$, the L -operator $\hat{L}^+(v)$ satisfies the $RLL = LLR^*$ relation (5.2).*

Proof. Let us substitute the expression (6.58) into the LHS of (5.2). Then using the commutation relations of the vertex operators (6.49)-(6.51) and the formula

$$\frac{\rho^+(v)}{\rho^{+*}(v)} = \frac{\tilde{\rho}^+(v)}{\tilde{\rho}^{+*}(v)} \frac{\chi(\frac{1}{2} - v)}{\chi(\frac{1}{2} + v)}, \quad (6.66)$$

one gets the desired result.

Q.E.D.

In the same way, we have

Corollary 6.7 *For $c = 1$, the type-I and the type II vertex operators $\widehat{\Phi}_V(v)$, $\widehat{\Psi}_V^*(v)$ satisfy the full intertwining relations (6.3) and (6.4) with $V = W \cong \mathbb{C}^N$.*

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A Operator Product Expansions

Here we list formulae of operator product expansions (OPE) used in Section 3.3 and 6.2. For operators $A(z)$, $B(z)$, we write

$$A(z_1)B(z_2) = \langle A(z_1)B(z_2) \rangle : A(z_1)B(z_2) : .$$

(I) In Section 3.3, we used the OPE's of the currents $\psi_j^\pm(z, p)$ (3.24)-(3.25) and $k_j(z, p)$ (3.43) for generic c :

$$\langle k_j(z_1, p)k_j(z_2, p) \rangle = \frac{\{pq^2 z_2/z_1\}\{pq^{2N-2} z_2/z_1\}\{p^* q^{2N} z_2/z_1\}^* \{p^* z_2/z_1\}^*}{\{pq^{2N} z_2/z_1\}\{pz_2/z_1\}\{p^* q^2 z_2/z_1\}^* \{p^* q^{2N-2} z_2/z_1\}^*}, \quad (A.1)$$

$$\langle k_{j_1}(z_1, p)k_{j_2}(z_2, p) \rangle = \frac{\{pq^{2N+2} z_2/z_1\}\{pq^{2N-2} z_2/z_1\}\{p^* q^{2N} z_2/z_1\}^*}{\{pq^{2N} z_2/z_1\}^2 \{p^* q^{2N+2} z_2/z_1\}^* \{p^* q^{2N-2} z_2/z_1\}^*} \quad (j_1 < j_2), \quad (A.2)$$

$$\langle k_{j_2}(z_1, p)k_{j_1}(z_2, p) \rangle = \frac{\{pq^2 z_2/z_1\}\{pq^{-2} z_2/z_1\}\{p^* z_2/z_1\}^*}{\{pz_2/z_1\}^2 \{p^* q^2 z_2/z_1\}^* \{p^* q^{-2} z_2/z_1\}^*} \quad (j_1 < j_2). \quad (A.3)$$

$$\langle \psi_j^+(z_1, p) \psi_j^+(z_2, p) \rangle = \frac{(pq^2 z_2/z_1; p)_\infty (p^* q^{-2} z_2/z_1; p^*)_\infty}{(pq^{-2} z_2/z_1; p)_\infty (p^* q^2 z_2/z_1; p^*)_\infty}, \quad (\text{A.4})$$

$$\langle \psi_j^+(z_1, p) \psi_{j+1}^+(z_2, p) \rangle = \frac{(pq^{-1} z_2/z_1; p)_\infty (p^* q z_2/z_1; p^*)_\infty}{(pq z_2/z_1; p)_\infty (p^* q^{-1} z_2/z_1; p^*)_\infty}, \quad (\text{A.5})$$

$$\langle \psi_{j+1}^+(z_1, p) \psi_j^+(z_2, p) \rangle = \frac{(pq^{-1} z_2/z_1; p)_\infty (p^* q z_2/z_1; p^*)_\infty}{(pq z_2/z_1; p)_\infty (p^* q^{-1} z_2/z_1; p^*)_\infty}. \quad (\text{A.6})$$

(II) In Section 6.2, we used the OPE's of the currents $\Phi_N(v)$, $\Psi_N^*(v)$, $E_j(v)$, $F_j(v)$, $K_j^+(v)$ for $c = 1$. We here list their boson part only. Namely, let us define the boson part of them by

$$\phi_N(v) =: \exp \left(- \sum_{m \neq 0} \frac{1}{m} B_m^N z^{-m} \right) :, \quad (\text{A.7})$$

$$\psi_N^*(v) =: \exp \left(\sum_{m \neq 0} \frac{[rm]_q}{m[r^*m]_q} B_m^N z^{-m} \right) :, \quad (\text{A.8})$$

$$e_j(v) =: \exp \left(- \sum_{m \neq 0} \frac{[rm]_q}{m[r^*m]_q} (-B_m^j + B_m^{j+1}) (q^{N-j} z)^{-m} \right) :, \quad (\text{A.9})$$

$$f_j(v) =: \exp \left(\sum_{m \neq 0} \frac{1}{m} (-B_m^j + B_m^{j+1}) (q^{N-j} z)^{-m} \right) :. \quad (\text{A.10})$$

Then the OPE's of them are given by

$$\langle K_j^+(v_1) f_j(v_2) \rangle = \frac{(q^{N-j+1} z_2/z_1; p)_\infty}{(q^{N-j-1} z_2/z_1; p)_\infty}, \quad (\text{A.11})$$

$$\langle K_{j+1}^+(v_1) f_j(v_2) \rangle = \frac{(q^{N-j-3} z_2/z_1; p)_\infty}{(q^{N-j-1} z_2/z_1; p)_\infty}, \quad (\text{A.12})$$

$$\langle f_j(v_1) K_j^+(v_2) \rangle = \frac{(q^{-N+j-1} z_2/z_1; p)_\infty}{(q^{-N+j-3} z_2/z_1; p)_\infty}, \quad (\text{A.13})$$

$$\langle f_j(v_1) K_{j+1}^+(v_2) \rangle = \frac{(q^{-N+j-1} z_2/z_1; p)_\infty}{(q^{-N+j+1} z_2/z_1; p)_\infty}, \quad (\text{A.14})$$

$$\langle K_j^+(v_1) e_j(v_2) \rangle = \frac{(q^{N-j-2} z_2/z_1; p^*)_\infty}{(q^{N-j} z_2/z_1; p^*)_\infty}, \quad (\text{A.15})$$

$$\langle K_{j+1}^+(v_1) e_j(v_2) \rangle = \frac{(q^{N-j-2} z_2/z_1; p^*)_\infty}{(q^{N-j-4} z_2/z_1; p^*)_\infty}, \quad (\text{A.16})$$

$$\langle e_j(v_1) K_j^+(v_2) \rangle = \frac{(q^{-N+j-4} z_2/z_1; p^*)_\infty}{(q^{-N+j-2} z_2/z_1; p^*)_\infty}, \quad (\text{A.17})$$

$$\langle e_j(v_1) K_{j+1}^+(v_2) \rangle = \frac{(q^{-N+j} z_2/z_1; p^*)_\infty}{(q^{-N+j-2} z_2/z_1; p^*)_\infty}, \quad (\text{A.18})$$

$$\langle \phi_N(v_1) K_N^+(v_2) \rangle = \frac{\{pq^3 z_2/z_1\} \{pq^{2N-1} z_2/z_1\}}{\{pq z_2/z_1\} \{pq^{2N+1} z_2/z_1\}}, \quad (\text{A.19})$$

$$\langle K_N^+(v_2)\phi_N(v_1) \rangle = \frac{\{qz_1/z_2\}\{q^{2N-3}z_1/z_2\}}{\{q^{-1}z_1/z_2\}\{q^{2N-1}z_1/z_2\}}, \quad (\text{A.20})$$

$$\langle \phi_N(v_1)K_j^+(v_2) \rangle = \frac{\{pq^3z_2/z_1\}\{pq^{-1}z_2/z_1\}}{\{pqz_2/z_1\}^2}, \quad (\text{A.21})$$

$$\langle K_j^+(v_2)\phi_N(v_1) \rangle = \frac{\{q^{2N+1}z_1/z_2\}\{q^{2N-3}z_1/z_2\}}{\{q^{2N-1}z_1/z_2\}^2}, \quad (\text{A.22})$$

$$\langle \psi_N^*(v_1)K_N^+(v_2) \rangle = \frac{\{p^*q^2z_2/z_1\}^*\{p^*q^{2N+2}z_2/z_1\}^*}{\{p^*q^4z_2/z_1\}^*\{p^*q^{2N}z_2/z_1\}^*}, \quad (\text{A.23})$$

$$\langle K_N^+(v_2)\psi_N^*(v_1) \rangle = \frac{\{q^{-2}z_1/z_2\}^*\{q^{2N-2}z_1/z_2\}^*}{\{z_1/z_2\}^*\{q^{2N-4}z_1/z_2\}^*}, \quad (\text{A.24})$$

$$\langle \psi_N^*(v_1)K_j^+(v_2) \rangle = \frac{\{p^*q^2z_2/z_1\}^{*2}}{\{p^*q^4z_2/z_1\}^*\{p^*z_2/z_1\}^*}, \quad (\text{A.25})$$

$$\langle K_j^+(v_2)\psi_N^*(v_1) \rangle = \frac{\{q^{2N-2}z_1/z_2\}^{*2}}{\{q^{2N}z_1/z_2\}^*\{q^{2N-4}z_1/z_2\}^*}, \quad (\text{A.26})$$

$$\langle f_{N-1}(v_1)\phi_N(v_2) \rangle = \frac{(pq^{-1}z_2/z_1; p)_\infty}{(qz_2/z_1; p)_\infty}, \quad (\text{A.27})$$

$$\langle \phi_N(v_1)f_{N-1}(v_2) \rangle = \frac{(pq^{-1}z_2/z_1; p)_\infty}{(qz_2/z_1; p)_\infty}, \quad (\text{A.28})$$

$$\langle e_{N-1}(v_1)\psi_N^*(v_2) \rangle = \frac{(p^*qz_2/z_1; p^*)_\infty}{(q^{-1}z_2/z_1; p^*)_\infty}, \quad (\text{A.29})$$

$$\langle \psi_N^*(v_1)e_{N-1}(v_2) \rangle = \frac{(p^*qz_2/z_1; p^*)_\infty}{(q^{-1}z_2/z_1; p^*)_\infty}, \quad (\text{A.30})$$

$$\langle \phi_N(v_1)\phi_N(v_2) \rangle = \frac{\{pq^{2N-2}z_2/z_1\}\{q^2z_2/z_1\}}{\{q^{2N}z_2/z_1\}\{pz_2/z_1\}}, \quad (\text{A.31})$$

$$\langle \psi_N^*(v_2)\psi_N^*(v_1) \rangle = \frac{\{p^*q^{2N}z_2/z_1\}^*\{z_2/z_1\}^*}{\{p^*q^2z_2/z_1\}^*\{q^{2N-2}z_2/z_1\}^*}, \quad (\text{A.32})$$

$$\langle \phi_N(v_1)\psi_N^*(v_2) \rangle = \frac{(q^{2N-1}z_2/z_1; q^{2N})_\infty}{(qz_2/z_1; q^{2N})_\infty}, \quad (\text{A.33})$$

$$\langle \psi_N^*(v_1)\phi_N(v_2) \rangle = \frac{(q^{2N-1}z_2/z_1; q^{2N})_\infty}{(qz_2/z_1; q^{2N})_\infty}, \quad (\text{A.34})$$

$$\langle e_j(v_1)e_j(v_2) \rangle = \frac{(z_2/z_1; p^*)_\infty(q^{-2}z_2/z_1; p^*)_\infty}{(p^*q^2z_2/z_1; p^*)_\infty(p^*z_2/z_1; p^*)_\infty}, \quad (\text{A.35})$$

$$\langle e_j(v_1)e_{j+1}(v_2) \rangle = \frac{(p^*qz_2/z_1; p^*)_\infty}{(q^{-1}z_2/z_1; p^*)_\infty}, \quad (\text{A.36})$$

$$\langle e_{j+1}(v_1)e_j(v_2) \rangle = \frac{(p^*qz_2/z_1; p^*)_\infty}{(q^{-1}z_2/z_1; p^*)_\infty}, \quad (\text{A.37})$$

$$\langle f_j(v_1)f_j(v_2) \rangle = \frac{(z_2/z_1; p)_\infty(q^2z_2/z_1; p)_\infty}{(pz_2/z_1; p)_\infty(pq^{-2}z_2/z_1; p)_\infty}, \quad (\text{A.38})$$

$$\langle f_j(v_1)f_{j+1}(v_2) \rangle = \frac{(pq^{-1}z_2/z_1;p)_\infty}{(qz_2/z_1;p)_\infty}, \quad (\text{A.39})$$

$$\langle f_{j+1}(v_1)f_j(v_2) \rangle = \frac{(pq^{-1}z_2/z_1;p)_\infty}{(qz_2/z_1;p)_\infty}. \quad (\text{A.40})$$

B Proof of the Relations (4.17)-(4.18) and (4.19)-(4.20)

Let us consider the relations

$$K_l^+(v_2)^{-1}E_{l,j}^+(v_1)K_l^+(v_2)E_{l,j}^+(v_2) = K_l^+(v_1)^{-1}E_{l,j}^+(v_2)K_l^+(v_1)E_{l,j}^+(v_1), \quad (\text{B.1})$$

$$F_{j,l}^+(v_1)K_l^+(v_1)F_{j,l}^+(v_2)K_l^+(v_1)^{-1} = F_{j,l}^+(v_2)K_l^+(v_2)F_{j,l}^+(v_1)K_l^+(v_2)^{-1}, \quad (\text{B.2})$$

for $1 \leq j < l \leq N$. Then the relations (4.17) and (4.18) follow from these relations and (4.15), (4.16). In this Appendix, we give proofs of the relations (B.2) and (4.20). The proof of the other cases (B.1) and (4.19) are similar.

Let us set

$$f(v, w) = \frac{[v + \frac{1}{2} - w]}{[v - \frac{1}{2}]}, \quad (\text{B.3})$$

$$h(v) = \frac{[v - 1]}{[v + 1]}. \quad (\text{B.4})$$

Recall that the half current $F_{j,l}^+(v)$ is given by

$$\begin{aligned} F_{j,l}^+(v) &= a_{j,l} \oint_{C(j,l)} \prod_{m=j}^{l-1} \frac{dz_k}{2\pi i z_k} F_{l-1}(v_{l-1}) F_{l-2}(v_{l-2}) \cdots F_j(v_j) \\ &\quad \times \prod_{m=j}^{l-1} f(v_m - v_{m+1}, \pi_{m+1,j}) \frac{[1]}{[\pi_{m+1,j} - \delta_{m,l-1}]}, \end{aligned} \quad (\text{B.5})$$

where we set $v_l = v + \frac{l-N-1}{2}$. Recall also $z_m = q^{2v_m}$ and $\pi_{l,j} = P_{j,l} + h_{j,l}$. We call $F_{l-1}(v_{l-1}) F_{l-2}(v_{l-2}) \cdots F_j(v_j)$ the operator part, and $\prod_{m=j}^{l-1} f(v_m - v_{m+1}, \pi_{m+1,j}) \frac{[1]}{[\pi_{m+1,j} - \delta_{m,l-1}]}$ the coefficient part. We keep coefficient parts in the right of operator parts. In the coefficient part, We represent $\prod_{m=j}^{l-1} f(v_m - v_{m+1}, \pi_{m+1,j})$ by the diagram

$$v_j \xrightarrow{\pi_{j+1,j}} v_{j+1} \xrightarrow{\pi_{j+2,j}} \cdots \xrightarrow{\pi_{l-1,j}} v_{l-1} \xrightarrow{\pi_{l,j}} v_l.$$

According to the relation (3.83) with $i = j$, we have the equality

$$\oint \frac{dz_j}{2\pi i z_j} \frac{dz'_j}{2\pi i z'_j} F_j(v_j) F_j(v'_j) A(v_j, v'_j) = \oint \frac{dz_j}{2\pi i z_j} \frac{dz'_j}{2\pi i z'_j} F_j(v_j) F_j(v'_j) h(v'_j - v_j) A(v'_j, v_j),$$

when the integration contours for z_j and z'_j are the same. We define ‘weak equality’ in the following sense[14]. The two coefficient functions $A(v_j, v'_j)$ and $B(v_j, v'_j)$ coupled to $F_j(v_j)F_j(v'_j)$ in integrals are equal in weak sense if

$$A(v_j, v'_j) + h(v'_j - v_j)A(v'_j, v_j) = B(v_j, v'_j) + h(v'_j - v_j)B(v'_j, v_j).$$

We write the weak equality as

$$A(v_j, v'_j) \sim B(v_j, v'_j).$$

To prove the equality (B.2) and (4.20), it is enough to show the equalities of coefficient parts in weak sense.

Let us recall the following two lemmas[14].

Lemma B.1 *The coefficient function*

$$\begin{array}{ccccccc} v_j & \xrightarrow{\pi_{j+1,j-1}} & v_{j+1} & \xrightarrow{\pi_{j+2,j-1}} & \cdots & \xrightarrow{\pi_{l-2,j-1}} & v_{l-2} & \xrightarrow{\pi_{l-1,j-1}} & v_{l-1} \\ & \searrow 0 & & \searrow 0 & & \searrow 0 & & \searrow 0 & \\ v'_j & \xrightarrow{\pi_{j+1,j}} & v'_{j+1} & \xrightarrow{\pi_{j+2,j}} & \cdots & \xrightarrow{\pi_{l-2,\mu}} & v'_{l-2} & \xrightarrow{\pi_{l-1,\mu}} & v'_{l-1} \end{array} \quad (\text{B.6})$$

is invariant in weak sense when v_{l-1} and v'_{l-1} are exchanged.

Lemma B.2

$$\begin{aligned} & \begin{array}{ccccccc} v'_{k-1} & \xrightarrow{\pi_{k,j}+1} & v_k & \xrightarrow{\pi_{k+1,j}} & v_{k+1} & \xrightarrow{\pi_{k+2,j}} & \cdots & \xrightarrow{\pi_{l-2,j}} & v_{l-2} & \xrightarrow{\pi_{l-1,j}} & v_{l-1} \\ & \searrow 0 & & \searrow 0 & & \searrow 0 & & \searrow 0 & & \searrow 0 & \\ & & v'_k & \xrightarrow{\pi_{k+1,k}} & v'_{k+1} & \xrightarrow{\pi_{k+2,k}} & \cdots & \xrightarrow{\pi_{l-2,k}} & v'_{l-2} & \xrightarrow{\pi_{l-1,k}} & v'_{l-1} \end{array} \\ & \sim \frac{1}{\beta(v_{l-1} - v'_{l-1}, \pi_{k,j})} \\ & \times \begin{array}{ccccccc} v'_{k-1} & \xrightarrow{\pi_{k,j}} & v'_k & \xrightarrow{\pi_{k+1,j}} & v'_{k+1} & \xrightarrow{\pi_{k+2,j}} & \cdots & \xrightarrow{\pi_{l-2,j}} & v'_{l-2} & \xrightarrow{\pi_{l-1,j}} & v_{l-1} \\ & & \nearrow 0 & & \nearrow 0 & & \nearrow 0 & & \nearrow 0 & & \\ v_k & \xrightarrow{\pi_{k+1,k}} & v_{k+1} & \xrightarrow{\pi_{k+2,k}} & \cdots & \xrightarrow{\pi_{l-2,k}} & v_{l-2} & \xrightarrow{\pi_{l-1,k}} & v'_{l-1} \end{array} \\ & - \frac{\gamma(v_{l-1} - v'_{l-1}, \pi_{k,j})}{\beta(v_{l-1} - v'_{l-1}, \pi_{k,j})} \\ & \times \begin{array}{ccccccc} v'_{k-1} & \xrightarrow{\pi_{k,j}} & v'_k & \xrightarrow{\pi_{k+1,j}} & v'_{k+1} & \xrightarrow{\pi_{k+2,j}} & \cdots & \xrightarrow{\pi_{l-2,j}} & v'_{l-2} & \xrightarrow{\pi_{l-1,j}} & v'_{l-1} \\ & & \nearrow 0 & & \nearrow 0 & & \nearrow 0 & & \nearrow 0 & & \\ v_k & \xrightarrow{\pi_{k+1,k}} & v_{k+1} & \xrightarrow{\pi_{k+2,k}} & \cdots & \xrightarrow{\pi_{l-2,k}} & v_{l-2} & \xrightarrow{\pi_{l-1,k}} & v_{l-1} \end{array} \end{aligned} \quad (\text{B.7})$$

where

$$\beta(v, w) = \frac{[v][w-1]}{[v+1][w]}, \quad \gamma(v, w) = \frac{[v+w][1]}{[v+1][w]}.$$

Now let us show the relation (B.2). By using (3.91)-(3.93), (3.83), (4.3) and (4.5) in the LHS of (B.2), the operator part can be arranged to $F_{l-1}(v_{l-1})F_{l-1}(v'_{l-1})F_{l-2}(v_{l-2})F_{l-2}(v'_{l-2}) \cdots F_j(v_j)F_j(v'_j)$. Then the coefficient part is given by the product of the factors $\prod_{m=j}^{l-1} \frac{[1]^2}{[\pi_{m+1,j}-1-2\delta_{k,l-1}][\pi_{m+1,j}-2\delta_{k,l-1}]}$ and the one represented by the diagram

$$\begin{array}{ccccccccccc} v_j & \xrightarrow{\pi_{j+1,j-1}} & v_{j+1} & \xrightarrow{\pi_{j+2,j-1}} & \cdots & \xrightarrow{\pi_{l-2,j-1}} & v_{l-2} & \xrightarrow{\pi_{l-1,j-1}} & v_{l-1} & \xrightarrow{\pi_{l,j-2}} & v_l \\ & \searrow 0 & & \searrow 0 & & \searrow 0 & & \searrow 0 & & \nearrow 2 & \\ v'_j & \xrightarrow{\pi_{j+1,j}} & v'_{j+1} & \xrightarrow{\pi_{j+2,j}} & \cdots & \xrightarrow{\pi_{l-2,\mu}} & v'_{l-2} & \xrightarrow{\pi_{l-1,\mu}} & v'_{l-1} & \xrightarrow{\pi_{l,\mu-1}} & v'_l. \end{array} \quad (\text{B.8})$$

The relation (B.2) denotes that (B.8) is invariant, at least in weak sense, when v_l and v'_l are exchanged. Applying the Lemma B.1 to the corresponding part of (B.8), it is enough to show the weak equality for the rest part

$$\begin{aligned} & f(v_{l-1} - v_l, \pi_{l,j} - 2) f(v'_{l-1} - v'_l, \pi_{l,j} - 1) f(v'_{l-1} - v_l, 2) \\ & \sim f(v_{l-1} - v'_l, \pi_{l,j} - 2) f(v'_{l-1} - v_l, \pi_{l,j} - 1) f(v'_{l-1} - v'_l, 2). \end{aligned} \quad (\text{B.9})$$

Let us set $v = v_{l-1}$, $v' = v'_{l-1}$, $w = \pi_{l,j}$ and denote the LHS and the RHS by $A(v, v')$ and $B(v, v')$, respectively. Then from the theta function identity such as (4.24), we have

$$A(v, v') - B(v, v') = \frac{[v - v' + 1][v + v' - v_l - v'_l - w][v_l - v'_l][w - 2]}{[v - v_l - \frac{1}{2}][v' - v'_l - \frac{1}{2}][v - v'_l - \frac{1}{2}][v' - v_l - \frac{1}{2}]}.$$

Then it is easy to show

$$h(v' - v)(A(v', v) - B(v', v)) = -(A(v, v') - B(v, v')).$$

Therefore we get the weak equality (B.9).

Next we prove (4.20) ($j < k < l$). The equality follows from the weak equality (A)+(B)+(C) ~ 0 , where

$$\begin{aligned} (A) = & \begin{array}{ccccccccccc} v'_{k-1} & \xrightarrow{\pi_{k,j}} & v'_k & \xrightarrow{\pi_{k+1,j}} & v'_{k+1} & \xrightarrow{\pi_{k+2,j}} & \cdots & \xrightarrow{\pi_{l-2,j}} & v'_{l-2} & \xrightarrow{\pi_{l-1,j}} & v'_{l-1} & \xrightarrow{\pi_{l,j-1}} & v'_l \\ & & \nearrow 0 & & \nearrow 0 & & \nearrow 0 & & \nearrow 0 & & \nearrow 0 & & \searrow 2 \\ v_k & \xrightarrow{\pi_{k+1,k}} & v_{k+1} & \xrightarrow{\pi_{k+2,k}} & \cdots & \xrightarrow{\pi_{l-2,k}} & v_{l-2} & \xrightarrow{\pi_{l-1,k}} & v_{l-1} & \xrightarrow{\pi_{l,k}-1} & v_l, \end{array} \\ (B) = & -b(v'_l - v_l, \pi_{k,j}) \frac{[\pi_{k,j}]}{[\pi_{k,j} + 1]} \times \end{aligned}$$

$$\begin{array}{ccccccccccc}
v'_{k-1} & \xrightarrow{\pi_{k,j}+1} & v_k & \xrightarrow{\pi_{k+1,j}} & v_{k+1} & \xrightarrow{\pi_{k+2,j}} & \cdots & \xrightarrow{\pi_{l-2,j}} & v_{l-2} & \xrightarrow{\pi_{l-1,j}} & v_{l-1} & \xrightarrow{\pi_{l,j}-1} & v'_l \\
& \searrow 0 & & \searrow 0 & & \searrow 0 & & \searrow 0 & & \searrow 0 & & \nearrow 2 & \\
& & v'_k & \xrightarrow{\pi_{k+1,k}} & v'_{k+1} & \xrightarrow{\pi_{k+2,k}} & \cdots & \xrightarrow{\pi_{l-2,k}} & v'_{l-2} & \xrightarrow{\pi_{l-1,k}} & v'_{l-1} & \xrightarrow{\pi_{l,k}-1} & v_l
\end{array}$$

$(C) = -c(v'_l - v_l, \pi_{k,j}) \times$

$$\begin{array}{ccccccccccc}
v'_{k-1} & \xrightarrow{\pi_{k,j}} & v'_k & \xrightarrow{\pi_{k+1,j}} & v'_{k+1} & \xrightarrow{\pi_{k+2,j}} & \cdots & \xrightarrow{\pi_{l-2,j}} & v'_{l-2} & \xrightarrow{\pi_{l-1,j}} & v'_{l-1} & \xrightarrow{\pi_{l,j}-1} & v_l \\
& \nearrow 0 & & \nearrow 0 & & \nearrow 0 & & \nearrow 0 & & \nearrow 0 & & \searrow 2 & \\
v_k & \xrightarrow{\pi_{k+1,k}} & v_{k+1} & \xrightarrow{\pi_{k+2,k}} & \cdots & \xrightarrow{\pi_{l-2,k}} & v_{l-2} & \xrightarrow{\pi_{l-1,k}} & v_{l-1} & \xrightarrow{\pi_{l,k}-1} & v'_l
\end{array}$$

(B.10)

Using the weak equality in Lemma B.2, we modify (B) to $(A') + (C')$ where

$$\begin{aligned}
(A') &= -\frac{b(v'_l - v_l, \pi_{k,j})}{\beta(v_{l-1} - v'_{l-1}, \pi_{k,j})} \frac{[\pi_{k,j}]}{[\pi_{k,j} + 1]} \times \\
&\begin{array}{ccccccccccc}
v'_{k-1} & \xrightarrow{\pi_{k,j}} & v'_k & \xrightarrow{\pi_{k+1,j}} & v'_{k+1} & \xrightarrow{\pi_{k+2,j}} & \cdots & \xrightarrow{\pi_{l-2,j}} & v'_{l-2} & \xrightarrow{\pi_{l-1,j}} & v_{l-1} & \xrightarrow{\pi_{l,j}-1} & v'_l \\
& \nearrow 0 & & \nearrow 0 & & \nearrow 0 & & \nearrow 0 & & \nearrow 0 & & \nearrow 2 & \\
v_k & \xrightarrow{\pi_{k+1,k}} & v_{k+1} & \xrightarrow{\pi_{k+2,k}} & \cdots & \xrightarrow{\pi_{l-2,k}} & v_{l-2} & \xrightarrow{\pi_{l-1,k}} & v'_{l-1} & \xrightarrow{\pi_{l,k}-1} & v_l
\end{array} \\
(C') &= \frac{b(v'_l - v_l, \pi_{k,j})\gamma(v_{l-1} - v'_{l-1}, \pi_{k,j})}{\beta(v_{l-1} - v'_{l-1}, \pi_{k,j})} \frac{[\pi_{k,j}]}{[\pi_{k,j} + 1]} \times \\
&\begin{array}{ccccccccccc}
v'_{k-1} & \xrightarrow{\pi_{k,j}} & v'_k & \xrightarrow{\pi_{k+1,j}} & v'_{k+1} & \xrightarrow{\pi_{k+2,j}} & \cdots & \xrightarrow{\pi_{l-2,j}} & v'_{l-2} & \xrightarrow{\pi_{l-1,j}} & v'_{l-1} & \xrightarrow{\pi_{l,k}-1} & v_l \\
& \nearrow 0 & & \nearrow 0 & & \nearrow 0 & & \nearrow 0 & & \nearrow 0 & & \searrow 2 & \\
v_k & \xrightarrow{\pi_{k+1,k}} & v_{k+1} & \xrightarrow{\pi_{k+2,k}} & \cdots & \xrightarrow{\pi_{l-2,k}} & v_{l-2} & \xrightarrow{\pi_{l-1,k}} & v_{l-1} & \xrightarrow{\pi_{l,j}-1} & v'_l
\end{array}
\end{aligned}$$

Noting that $h(v_{l-1} - v'_{l-1})\beta(v'_{l-1} - v_{l-1}, w) = \beta(v_{l-1} - v'_{l-1}, w)$, we can exchange v_{l-1} and v'_{l-1} in (A') . Let (A'') be the term we thus obtain. Note that $(A') \sim (A'')$. Using the equality

$$\begin{aligned}
&f(v'_{l-1} - v_l, 2) - \frac{b(v'_l - v_l, w)}{\beta(v_{l-1} - v'_{l-1}, w)} \frac{[w]}{[w+1]} f(v_{l-1} - v'_l, 2) \\
&= \frac{[1][v_l - v'_l + v_{l-1} - v'_{l-1}][v_l - v_{l-1} + \frac{1}{2}][v'_l - v'_{l-1} + \frac{3}{2}]}{[v_{l-1} - v'_{l-1}][v'_l - v_l + 1][v'_{l-1} - v_l - \frac{1}{2}][v_{l-1} - v'_l - \frac{1}{2}]}
\end{aligned}$$

we have

$$\begin{aligned}
(A) + (A'') &= \frac{[1][v_l - v'_l + v_{l-1} - v'_{l-1}][v_l - v_{l-1} + \frac{1}{2}][v'_l - v'_{l-1} + \frac{3}{2}]}{[v_{l-1} - v'_{l-1}][v'_l - v_l + 1][v'_{l-1} - v_l - \frac{1}{2}][v_{l-1} - v'_l - \frac{1}{2}]} \times \\
&\begin{array}{ccccccc}
v'_{k-1} & \xrightarrow{\pi_{k,j}} & v'_k & \xrightarrow{\pi_{k+1,j}} & \cdots & \xrightarrow{\pi_{l-2,j}} & v'_{l-2} & \xrightarrow{\pi_{l-1,j}} & v'_{l-1} & \xrightarrow{\pi_{l,j}-1} & v'_l \\
& \nearrow 0 & & \nearrow 0 & & \nearrow 0 & & \nearrow 0 & & & \\
v_k & \xrightarrow{\pi_{k+1,k}} & \cdots & \xrightarrow{\pi_{l-2,k}} & v_{l-2} & \xrightarrow{\pi_{l-1,k}} & v_{l-1} & \xrightarrow{\pi_{l,k}-1} & v_l
\end{array}
\end{aligned}$$

(B.11)

On the other hand, to calculate $(C) + (C')$, we use the equality

$$\begin{aligned} & \frac{c(v_{l-1} - v'_{l-1}, w_1)b(v'_l - v_l, w_1)}{\beta(v_{l-1} - v'_{l-1}, w_1)} \frac{[w_1]}{[w_1 + 1]} f(v'_{l-1} - v_l, w_2 - 1) f(v_{l-1} - v'_l, w_1 + w_2 - 1) \\ & - c(v'_l - v_l, w_1) f(v_{l-1} - v'_l, w_2 - 1) f(v'_{l-1} - v_l, w_1 + w_2 - 1) \\ & = - \frac{[1][v_l - v'_l + v_{l-1} - v'_{l-1}][v_{l-1} - v_l + \frac{3}{2} - w_2][v'_{l-1} - v'_l + \frac{3}{2} - w_1 - w_2]}{[v_{l-1} - v'_{l-1}][v'_l - v_l + 1][v'_{l-1} - v_l - \frac{1}{2}][v_{l-1} - v'_l - \frac{1}{2}]}. \end{aligned}$$

Then we have

$$(C) + (C') = - \frac{[1][v_l - v'_l + v_{l-1} - v'_{l-1}][v_{l-1} - v_l + \frac{3}{2} - w_2][v'_{l-1} - v'_l + \frac{3}{2} - w_1 - w_2]}{[v_{l-1} - v'_{l-1}][v'_l - v_l + 1][v'_{l-1} - v_l - \frac{1}{2}][v_{l-1} - v'_l - \frac{1}{2}]} \times$$

$$\begin{array}{ccccccc} v'_{k-1} & \xrightarrow{\pi_{k,j}} & v'_k & \xrightarrow{\pi_{k+1,j}} & \cdots & \xrightarrow{\pi_{l-2,j}} & v'_{l-2} \xrightarrow{\pi_{l-1,j}} v'_{l-1} \xrightarrow{2} v'_l \\ & & \nearrow 0 & & \nearrow 0 & & \nearrow 0 \\ v_k & \xrightarrow{\pi_{k+1,k}} & \cdots & \xrightarrow{\pi_{l-2,k}} & v_{l-2} & \xrightarrow{\pi_{l-1,k}} & v_{l-1}. \end{array} \quad (\text{B.12})$$

Comparing (B.11) and (B.12), we have $(A) + (A') + (C) + (C') = 0$.

C $RLL = LLR^*$ Relation

We here derive some of the relations of the half currents involved in the RLL -relation (5.2), and compare them with those in Theorem 4.1.

From the definition (5.1), the components of the L -operator $\hat{L}^+(v)$ are given by

$$\hat{L}_l^+(v) = K_l^+(v) + \sum_{m=1}^N F_{l,m}^+(v) K_m^+(v) E_{m,l}^+, \quad (\text{C.1})$$

$$\hat{L}_{kl}^+(v) = \begin{cases} F_{k,l}^+(v) K_l^+(v) + \sum_{m=l+1}^N F_{k,m}^+(v) K_m^+(v) E_{m,l}^+ & (k < l), \\ K_k^+(v) E_{k,l}^+(v) + \sum_{m=k+1}^N F_{k,m}^+(v) K_m^+(v) E_{m,l}^+ & (k > l). \end{cases} \quad (\text{C.2})$$

It is convenient to introduce the reduced R -matrix and L -operators, $R^+(v, s|j)$ and $\hat{L}^+(v|j)$ ($1 \leq j \leq N$), by

$$R^+(v, s|j) = (R_{kl}^{+mn}(v, s))_{j \leq k, l, m, n \leq N}, \quad (\text{C.3})$$

$$\hat{L}^+(v|j) = (\hat{L}_{kl}^+(v))_{j \leq k, l \leq N}. \quad (\text{C.4})$$

Then the inverse of $L^+(v|j)$ is given by

$$L^+(v|j)^{-1} = \begin{pmatrix} K_j^{+-1} & -K_j^{+-1} F_{j,j+1}^+ & K_j^{-1} x_j & * \\ -E_{j+1,j}^+ K_j^{K^{+-1}} & E_{j+1,j}^+ K_j^{K^{+-1}} F_{j,j+1}^+ + K_{j+1}^{+-1} & -E_{j+1,j}^+ K_j^{K^{+-1}} x_j - K_{j+1}^{K^{+-1}} F_{j+1,j+2}^+ & * \\ y_j K_j^{-1} & -y_j K_j^{K^{+-1}} F_{j,j+1}^+ - E_{j+2,j+1}^+ K_{j+1}^{K^{+-1}} & * & * \\ * & * & * & * \end{pmatrix}. \quad (\text{C.5})$$

Here we omitted the argument v and set

$$x_j(v) = F_{j,j+1}^+(v)F_{j+1,j+2}^+(v) - F_{j,j+2}^+(v), \quad (\text{C.6})$$

$$y_j(v) = E_{j+2,j+1}^+(v)E_{j+1,j}^+(v) - E_{j+2,j}^+(v). \quad (\text{C.7})$$

Due to the speciality of the form of the R -matrix (2.16), we have the reduced relation

$$R^{+(1,2)}(v, P + h|j)L^{+(1)}(v_1|j)L^{+(2)}(v_2|j) = L^{+(2)}(v_2|j)L^{+(1)}(v_1|j)R^{*+}(v, P|j). \quad (\text{C.8})$$

In the below, we use this rather in its inverted form

$$L^{+(1)}(v_1|j)^{-1}L^{+(2)}(v_2|j)^{-1}R^{+(1,2)}(v, P + h|j) = R^{*+(1,2)}(v, P|j)L^{+(2)}(v_2|j)^{-1}L^{+(1)}(v_1|j)^{-1}, \quad (\text{C.9})$$

$$L^{+(2)}(v_2|j)^{-1}R^{+(1,2)}(v, P + h|j)L^{+(1)}(v_1|j) = L^{+(1)}(v_1|j)R^{*+(1,2)}(v, P|j)L^{+(2)}(v_2|j)^{-1}. \quad (\text{C.10})$$

C.1 Relations among $K_j^+(v)$'s

Now some of the relations among $K_j^+(v)$ ($1 \leq j \leq N$) are derived as follows. The $(N, N), (N, N)$ component of the $RLL = LLR^*$ relation (5.2) yields

$$K_N^+(v_1)K_N^+(v_2) = \rho(v_1 - v_2)K_N^+(v_2)K_N^+(v_1). \quad (\text{C.11})$$

Similarly, the $(j, j), (j, j)$ component of the $L_j^{-1}L_j^{-1}R = R^*L_j^{-1}L_j^{-1}$ relation (C.9) ($1 \leq j \leq N-1$) yields

$$K_j^+(v_1)K_j^+(v_2) = \rho(v_1 - v_2)K_j^+(v_2)K_j^+(v_1) \quad (\text{C.12})$$

and the $(N, j), (N, j)$ component of the $L_j^{-1}RL_j = L_jR^*L_j^{-1}$ relation (C.9) ($1 \leq j \leq N-1$) yields

$$K_j^+(v_1)K_N^+(v_2) = \rho(v_1 - v_2) \frac{[v_1 - v_2 - 1]^*[v_1 - v_2]}{[v_1 - v_2]^*[v_1 - v_2 - 1]} K_N^+(v_2)K_j^+(v_1). \quad (\text{C.13})$$

These relations coincide with the relations (4.13) and (4.14).

C.2 Relations between $K_N^+(v)$ and $E_{N,j}^+(v)$

The $(N, N), (N, j)$ components of the $RLL = LLR^*$ relation (5.2) ($1 \leq j \leq N-1$) yields

$$K_N^+(v_1)^{-1}E_{N,j}^+(v_2)K_N^+(v_1) = E_{N,j}^+(v_2) \frac{1}{b^*(v_1 - v_2)} - E_{N,j}^+(v_1) \frac{c^*(v_1 - v_2, P_{j,N})}{b^*(v_1 - v_2)}. \quad (\text{C.14})$$

This coincides with the case $l = N$ of (4.15).

C.3 Relations between $K_N^+(v)$ and $F_{j,N}^+(v)$

The $(N, j), (N, N)$ components of the $RLL = LLR^*$ relation (5.2) ($1 \leq j \leq N-1$) yields

$$K_N^+(v_1)F_{j,N}^+(v_2)K_N^+(v_1)^{-1} = \frac{1}{\bar{b}(v_1 - v_2)}F_{j,N}^+(v_2) - \frac{\bar{c}(v_1 - v_2, P_{j,N} + h_{j,N})}{\bar{b}(v_1 - v_2)}F_{j,N}^+(v_1). \quad (\text{C.15})$$

This coincides with the case $l = N$ of (4.16).

C.4 Relations among $E_{l,j}^+(v)$'s

The $(N, N), (j, j)$ component of the $RLL = LLR^*$ relation (5.2) ($1 \leq j \leq N-1$) yields

$$K_N^+(v_1)E_{N,j}^+(v_1)K_N^+(v_2)E_{N,j}^+(v_2) = \rho(v_1 - v_2)K_N^+(v_2)E_{N,j}^+(v_2)K_N^+(v_1)E_{N,j}^+(v_1). \quad (\text{C.16})$$

The $(N, N), (k, j)$ component of the $RLL = LLR^*$ relation (5.2) ($1 \leq j, k \leq N-1, j \neq k$) yields

$$\begin{aligned} & \rho^+(v_1 - v_2)K_N^+(v_1)E_{N,k}^+(v_1)K_N^+(v_2)E_{N,j}^+(v_2) \\ &= K_N^+(v_2)E_{N,j}^+(v_2)K_N^+(v_1)E_{N,j}^+(v_1)R_{kj}^{*kj}(v_1 - v_2, P_{j,k}) \\ &+ K_N^+(v_2)E_{N,j}^+(v_2)K_N^+(v_1)E_{N,k}^+(v_1)R_{jk}^{*kj}(v_1 - v_2, P_{j,k}). \end{aligned} \quad (\text{C.17})$$

After a little calculation using (4.13), (C.16) and (C.17) coincide with the case $l = N$ in (B.1) and (4.19), respectively.

C.5 Relations among $F_{j,l}^+(v)$'s

The $(j, j), (N, N)$ component of the $RLL = LLR^*$ relation (5.2) ($1 \leq j \leq N-1$) yields

$$F_{j,N}^+(v_1)K_N^+(v_1)F_{j,N}^+(v_2)K_N^+(v_2) = \rho(v_1 - v_2)F_{j,N}^+(v_2)K_N^+(v_2)F_{j,N}^+(v_1)K_N^+(v_1). \quad (\text{C.18})$$

The $(j, k), (N, N)$ component of the $RLL = LLR^*$ relation (5.2) ($1 \leq j, k \leq N-1, j \neq k$) yields

$$\begin{aligned} & \rho^{+*}(v_1 - v_2)F_{k,N}^+(v_2)K_N^+(v_2)F_{j,N}^+(v_1)K_N^+(v_1) \\ &= R_{jk}^{jk}(v, P_{j,k} + h_{j,k})F_{j,N}^+(v_1)K_N^+(v_1)F_{k,N}^+(v_2)K_N^+(v_2) \\ &+ R_{jk}^{kj}(v, P_{j,k} + h_{j,k})F_{k,N}^+(v_1)K_N^+(v_1)F_{j,N}^+(v_2)K_N^+(v_2). \end{aligned} \quad (\text{C.19})$$

After a little calculation using (4.13), (C.18) and (C.19) coincide with the case $l = N$ in (B.2) and (4.20), respectively.

C.6 The relations between $E_{l,j}^+$'s and $F_{k,l}^+$'s

The $(j, N), (N, N-1)$ component together with the $(j, N), (N, N)$ and $(N, N), (N, N-1)$ components of the $RLL = LLR^*$ relation (5.2) ($1 \leq j \leq N-2$) yield

$$\begin{aligned} [E_{N,N-1}^+(v_2), F_{j,N}^+(v_1)] &= K_N^+(v_2)^{-1} \frac{c(v_1 - v_2, P_{j,N} + h_{j,N})}{\bar{b}(v_1 - v_2)} F_{j,N-1}^+(v_2) K_{N-1}^+(v_2) \\ &\quad - F_{j,N-1}^+(v_1) K_{N-1}^+(v_1) \frac{c^*(v_1 - v_2, P_{N-1,N})}{\bar{b}^*(v_1 - v_2)} K_N^+(v_1)^{-1}. \end{aligned} \quad (\text{C.20})$$

The $(j+1, j), (j, j+1)$ component together with the $(j+1, j), (j, j)$ and $(j, j), (j, j+1)$ components of the $L_j^{-1} L_j^{-1} R = R^* L_j^{-1} L_j^{-1}$ relation (C.9) ($1 \leq j \leq N-1$) yield

$$\begin{aligned} [E_{j+1,j}^+(v_1), F_{j,j+1}^+(v_2)] &= K_j^+(v_2) \frac{\bar{c}^*(v_1 - v_2, P_{j,j+1})}{\bar{b}^*(v_1 - v_2)} K_{j+1}^+(v_2)^{-1} \\ &\quad - K_{j+1}^+(v_1)^{-1} \frac{\bar{c}(v_1 - v_2, P_{j,j+1} + h_{j,j+1})}{\bar{b}(v_1 - v_2)} K_j^+(v_1). \end{aligned} \quad (\text{C.21})$$

These equations (C.20) and (C.21) coincide with the cases $l = N$ and $l = j+1$ of (4.21), respectively.

Similarly, the $(N-1, N), (N, j)$ component together with the $(N-1, N), (N, N)$ and $(N, N), (N, j)$ components of the $RLL = LLR^*$ relation (5.2) ($1 \leq j \leq N-2$) yield

$$\begin{aligned} [E_{N,j}^+(v_2), F_{N-1,N}^+(v_1)] &= K_N^+(v_2)^{-1} \frac{c(v_1 - v_2, P_{N-1,N} + h_{N-1,N})}{\bar{b}(v_1 - v_2)} K_{N-1}^+(v_2) E_{N-1,j}^+(v_2) \\ &\quad - K_{N-1}^+(v_1) E_{N-1,j}^+(v_1) \frac{c^*(v_1 - v_2, P_{j,N})}{\bar{b}^*(v_1 - v_2)} K_N^+(v_1)^{-1}. \end{aligned} \quad (\text{C.22})$$

The $(j, j+1), (j+1, j)$ component together with the $(j, j), (j+1, j)$ and $(j, j+1), (j, j)$ components of the $L_j^{-1} L_j^{-1} R = R^* L_j^{-1} L_j^{-1}$ relation (C.9) ($1 \leq j \leq N-1$) yield

$$\begin{aligned} [E_{j+1,j}^+(v_2), F_{j,j+1}^+(v_1)] &= K_{j+1}^+(v_2)^{-1} \frac{c(v_1 - v_2, P_{j,j+1} + h_{j,j+1})}{\bar{b}(v_1 - v_2)} K_j^+(v_2) \\ &\quad - K_j^+(v_1) \frac{c^*(v_1 - v_2, P_{j,j+1})}{\bar{b}^*(v_1 - v_2)} K_{j+1}^+(v_1)^{-1}. \end{aligned} \quad (\text{C.23})$$

These equations (C.22) and (C.23) coincide with the cases $l = N$ and $l = j+1$ of (4.22), respectively.

Finally, the following relations with $j \leq N-2$ are examples of those which we have not yet checked for our half currents.

$$\begin{aligned} &[E_{N,j}^+(v_2), F_{j,N}^+(v_1)] \\ &= K_N^+(v_2)^{-1} \frac{c(v_1 - v_2, P_{j,N} + h_{j,N})}{\bar{b}(v_1 - v_2)} K_j^+(v_2) - K_j^+(v_1) \frac{c^*(v_1 - v_2, P_{j,N})}{\bar{b}^*(v_1 - v_2)} K_N^+(v_1)^{-1} \end{aligned}$$

$$+ \sum_{k=j+1}^{N-1} \left(K_N^+(v_2)^{-1} \frac{c(v_1 - v_2, P_{j,N} + h_{j,N})}{\bar{b}(v_1 - v_2)} F_{j,k}^+(v_2) K_k^+(v_2) E_{k,j}^+(v_2) \right. \\ \left. - F_{j,k}^+(v_1) K_k^+(v_1) E_{k,j}^+(v_1) \frac{c^*(v_1 - v_2, P_{j,N})}{\bar{b}^*(v_1 - v_2)} K_N^+(v_1)^{-1} \right). \quad (\text{C.24})$$

These are derived from the $(j, N), (N, j)$ components together with the $(j, N), (N, N)$ and $(N, N), (N, j)$ components of the $RLL = LLR^*$ relation (5.2) ($1 \leq j \leq N-1$).

D Evaluation Module

We here summarize the evaluation module $(\pi_{V,z}, V_z = V[z, z^{-1}])$ of $U_q(\widehat{\mathfrak{sl}}_N)$ associated with the vector representation $V = \mathbb{C}^N$.

The evaluation module (π_z, V_z) in terms of the Drinfeld generators, is defined by the following formulae.

$$\pi_z(c) = 0, \quad \pi_z(d) = z \frac{d}{dz}, \quad (\text{D.1})$$

$$\pi_z(a_{j,n}) = \frac{[n]}{n} (q^{j-N+1} z)^n (q^{-n} E_{jj} - q^n E_{j+1j+1}), \quad (\text{D.2})$$

$$\pi_z(x_{j,n}^+) = (q^{j-N+1} z)^n E_{jj+1}, \quad (\text{D.3})$$

$$\pi_z(x_{j,n}^-) = (q^{j-N+1} z)^n E_{j+1j}, \quad (\text{D.4})$$

$$\pi_z(h_j) = E_{jj} - E_{j+1j+1}, \quad \pi_z(h_{\bar{\epsilon}_j}) = -E_{jj}. \quad (\text{D.5})$$

Then the elliptic currents $k_j(w, p), \psi_j^\pm(w, p), e_j(w, p), f_j(w, p)$ of $U_q(\widehat{\mathfrak{sl}}_N)$ defined in (3.22)-(3.25) are represented by

$$\pi_z(k_j(w, p)) = \frac{\{q^{r+2N+1}z/w\}\{q^{r+1}z/w\}\{q^{r-1}w/z\}\{q^{r-2N+3}w/z\}}{\{q^{r+2N-1}z/w\}\{q^{r+3}z/w\}\{q^{r+1}w/z\}\{q^{r-2N+1}w/z\}} \\ \times \left(\frac{\Theta_p(q^{r+1}z/w)}{\Theta_p(q^{r-1}z/w)} \sum_{k=1}^{j-1} E_{kk} + E_{jj} + \frac{\Theta_p(q^{r+3}z/w)}{\Theta_p(q^{r+1}z/w)} \sum_{k=j+1}^N E_{kk} \right), \quad (\text{D.6})$$

$$\pi_z(\psi_j^\pm(q^{\mp r}w, p)) = q^{\pm h_j} \frac{\Theta_p(q^{r-j+2h_j+N-1}w/z)}{\Theta_p(q^{r-j+N-1}w/z)}, \quad (\text{D.7})$$

$$\pi_z(e_j(w, p)) = E_{jj+1} \frac{(pq^2; p)_\infty}{(p; p)_\infty} \delta(q^{j-N+1}z/w), \quad (\text{D.8})$$

$$\pi_z(f_j(w, p)) = E_{j+1j} \frac{(pq^{-2}; p)_\infty}{(p; p)_\infty} \delta(q^{j-N+1}z/w), \quad (\text{D.9})$$

where $\{z\} = (z; p, q^{2N})_\infty$. Especially, the auxiliary currents $u_j^\pm(w, p)$ are represented by

$$\pi_z(u_j^+(w, p)) = \frac{(pq^{-j+2h_j+N-1}w/z; p)_\infty}{(pq^{-j+N-1}w/z; p)_\infty}, \quad \pi_z(u_j^-(w, p)) = \frac{(pq^{j-2h_j-N+1}z/w; p)_\infty}{(pq^{j-N+1}z/w; p)_\infty}. \quad (\text{D.10})$$

Due to this representation, we can obtain the representation of the half currents. After getting rid of some unpleasant fractional power factors of q and z by a certain gauge transformation, we have the following result.

$$\begin{aligned} \pi_{v_2}(K_j^+(v_1)e^{-Q_{\epsilon_j}}) &= \rho^+(v_1 - v_2) \\ &\times \left(\frac{[v_1 - v_2]}{[v_1 - v_2 + 1]} \sum_{k=1}^{j-1} E_{kk} + E_{jj} + \frac{[v_1 - v_2 - 1]}{[v_1 - v_2]} \sum_{k=j+1}^N E_{kk} \right), \end{aligned} \quad (\text{D.11})$$

$$\pi_{v_2}(e^{-\eta_j} F_{j,l}^+(v_1)e^{\eta_j}) = E_{lj} \frac{[v_1 - v_2 + P_{j,l} - 1][1]}{[v_1 - v_2][P_{j,l} - 1]}, \quad (\text{D.12})$$

$$\pi_{v_2}(e^{Q_{\epsilon_l} - \eta_l} E_{l,j}^+(v_1)e^{-Q_{\epsilon_j} + \eta_j}) = -E_{jl} \frac{[v_1 - v_2 - P_{j,l}][1]}{[v_1 - v_2][P_{j,l}]}, \quad (\text{D.13})$$

where $z = q^{2v}$. It is easy to check that these quantities satisfy the commutation relations of the half currents $K_j^+(v)$, $F_{j,l}^+(v)$ and $E_{l,j}^+(v)$.

Finally, let us check the results by calculating the R -matrix as the image of the L -operator $\hat{L}^+(v)$ in (5.1).

$$R^+(v_1 - v_2, P) = (\pi_{v_2} \otimes id) \hat{L}^+(v_1). \quad (\text{D.14})$$

Using (4.3) and Riemann's theta identity, we obtain the following.

$$R^+(v, P) = \rho^+(v) \begin{pmatrix} R_{11}(v, P) & \cdots & \cdots & R_{1N}(v, P) \\ R_{21}(v, P) & \cdots & \cdots & R_{2N}(v, P) \\ \vdots & & \ddots & \vdots \\ R_{N1}(v, P) & \cdots & \cdots & R_{NN}(v, P) \end{pmatrix}, \quad (\text{D.15})$$

where

$$R_{jj}(v, P) = \sum_{k=1}^{j-1} \bar{b}(v) E_{kk} + E_{jj} + \sum_{k=j+1}^N b(v, P_{j,k}) E_{kk} \quad (1 \leq j \leq N), \quad (\text{D.16})$$

$$R_{jl}(v, P) = c(v, P_{j,l}) E_{lj} \quad , \quad (\text{D.17})$$

$$R_{lj}(v, P) = \bar{c}(v, P_{j,l}) E_{jl} \quad (1 \leq j < l \leq N). \quad (\text{D.18})$$

This expression coincides with the R -matrix given by (2.15).

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