

Quadratic relations of the deformed W -algebra

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Abstract The deformed W -algebra is a quantum deformation of the W -algebra $\mathcal{W}_\beta(\mathfrak{g})$ in conformal field theory. Using the free field construction, we obtain a closed set of quadratic relations of the W -currents of the deformed W -algebra. This allows us to define the deformed W -algebra by generators and relations. In this review, we study two types of deformed W -algebra. One is the deformed W -algebra $\mathcal{W}_{x,r}(A_{2N}^{(2)})$, and the other is the q -deformed corner vertex algebra $q\text{-}Y_{L_1, L_2, L_3}$ that is a generalization of the deformed W -algebra $\mathcal{W}_{x,r}(A(M, N)^{(1)})$ via the quantum toroidal algebra.

1 Introduction

The deformed W -algebra $\mathcal{W}_{x,r}(\mathfrak{g})$ is both a two-parameter deformation of the classical W -algebra $\mathcal{W}(\mathfrak{g})$ in soliton theory and a one-parameter deformation of the W -algebra $\mathcal{W}_\beta(\mathfrak{g})$ in conformal field theory. The deformation theory of the W -algebra $\mathcal{W}_\beta(\mathfrak{g})$ has been studied in papers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. In comparison with the conformal case, the deformation theory of the W -algebra is still not fully understood. Except in low-rank cases such as the Virasoro algebra and the \mathcal{W}_3 -algebra, it isn't easy to handle the W -algebras $\mathcal{W}_\beta(\mathfrak{g})$ in a computational way [15]. In the case of the deformed W -algebra, it is sometimes possible to perform concrete calculations relatively easily. For instance, quadratic relations of the deformed W -algebra $\mathcal{W}_{x,r}(\mathfrak{g})$ have already been known in the cases of $\mathfrak{g} = A_N^{(1)}$ and $A_2^{(2)}$ [1, 2, 3, 4, 7]. In the case of $\mathcal{W}_{x,r}(A_1^{(1)})$, the basic W -current $\mathbf{T}_1(z)$ satisfy the following quadratic relation [1]

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$$g\left(\frac{z_2}{z_1}\right)\mathbf{T}_1(z_1)\mathbf{T}_1(z_2) - g\left(\frac{z_1}{z_2}\right)\mathbf{T}_1(z_2)\mathbf{T}_1(z_1) = c\left(\delta\left(\frac{x^{-2}z_2}{z_1}\right) - \delta\left(\frac{x^2z_2}{z_1}\right)\right)$$

with an appropriate constant c and a function $g(z)$. In the case of $\mathcal{W}_{x,r}(A_2^{(2)})$, the basic W -current $\mathbf{T}_1(z)$ satisfy the following quadratic relation [4]

$$\begin{aligned} & f\left(\frac{z_2}{z_1}\right)\mathbf{T}_1(z_1)\mathbf{T}_1(z_2) - f\left(\frac{z_1}{z_2}\right)\mathbf{T}_1(z_2)\mathbf{T}_1(z_1) \\ &= \delta\left(\frac{x^{-2}z_2}{z_1}\right)\mathbf{T}_1(x^{-1}z_2) - \delta\left(\frac{x^2z_2}{z_1}\right)\mathbf{T}_1(xz_2) + c\left(\delta\left(\frac{x^{-3}z_2}{z_1}\right) - \delta\left(\frac{x^3z_2}{z_1}\right)\right) \end{aligned}$$

with an appropriate constant c and a function $f(z)$. In this review, the author would like to report the quadratic relations in the cases of the twisted algebra $\mathcal{W}_{x,r}(A_{2N}^{(2)})$ [12] and the q -deformed corner vertex algebra $q\text{-}Y_{L_1, L_2, L_3}$ that is a generalization of the deformed W -algebra $\mathcal{W}_{x,r}(A(M, N)^{(1)})$ via the quantum toroidal algebra [8, 10, 11, 13]. These relations allow us to define the deformed W -algebras by generators and relations.

The text is organized as follows. In Section 2, we review the quantum toroidal algebra \mathcal{E} associated to \mathfrak{gl}_1 and the quantum algebra \mathcal{K} . In Section 3, we review the free field constructions of the basic W -currents $\mathbf{T}_1(z)$ both for $\mathcal{W}_{x,r}(A_{2N}^{(2)})$ and $q\text{-}Y_{L_1, L_2, L_3}$. We introduce the higher W -currents $\mathbf{T}_i(z)$, $i = 2, 3, 4, \dots$, by fusion procedure. We present a closed set of quadratic relations. Using these relations, we define the deformed W -algebras by generators and relations.

2 Quantum toroidal algebra \mathcal{E} associated to \mathfrak{gl}_1

2.1 Notation

Throughout the text we fix three complex parameters $q_1, q_2, q_3 \in \mathbf{C}^\times$ such that $q_1 q_2 q_3 = 1$. We assume $q_1^l q_2^m q_3^n = 1$ ($l, m, n \in \mathbf{Z}$) implies $l = m = n = 0$. We use the notation $s_c = q^{\frac{1}{c}}$ ($c = 1, 2, 3$), $\kappa_r = \prod_{c=1}^3 (1 - q_c^r)$ ($r \in \mathbf{Z}$). For any integer n , define q -integer

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

for complex number $q \neq 0$. We use symbols or infinite products

$$(a; p)_\infty = \prod_{k=0}^{\infty} (1 - ap^k), \quad (a_1, a_2, \dots, a_N; p)_\infty = \prod_{i=1}^N (a_i; p)_\infty$$

for $|p| < 1$ and $a, a_1, \dots, a_N \in \mathbf{C}$. Define $\delta(z)$ by the formal series

$$\delta(z) = \sum_{m \in \mathbf{Z}} z^m.$$

2.2 Quantum toroidal algebra \mathcal{E} associated to \mathfrak{gl}_1

In this section, we review the quantum toroidal algebra \mathcal{E} associated to \mathfrak{gl}_1 in Refs. [8, 18, 19]. We set

$$g(z, w) = \prod_{j=1}^3 (z - q_j w), \quad \bar{g}(z, w) = \prod_{j=1}^3 (z - q_j^{-1} w).$$

The quantum toroidal algebra \mathcal{E} associated to \mathfrak{gl}_1 is an associative algebra with parameters q_1, q_2, q_3 generated by e_n, f_n ($n \in \mathbf{Z}$), h_r ($r \in \mathbf{Z}_{r \neq 0}$) and invertible central element C . We set the currents

$$e(z) = \sum_{n \in \mathbf{Z}} e_n z^{-n}, \quad f(z) = \sum_{n \in \mathbf{Z}} f_n z^{-n}, \quad \psi^\pm(z) = \exp \left(\sum_{r > 0} \kappa_r h_{\pm r} z^{\mp r} \right).$$

The defining relations are given as follows.

$$\begin{aligned} [h_r, h_s] &= \delta_{r+s, 0} \frac{1}{\kappa_r} \frac{C^r - C^{-r}}{r}, \\ g(z, w) \psi^+(C^{-1}z) e(w) &= \bar{g}(z, w) e(w) \psi^+(C^{-1}z), \\ g(z, w) \psi^-(z) e(w) &= \bar{g}(z, w) e(w) \psi^-(z), \\ \bar{g}(z, w) \psi^+(z) f(w) &= g(z, w) f(w) \psi^+(z), \\ \bar{g}(z, w) \psi^-(C^{-1}z) f(w) &= g(z, w) f(w) \psi^-(C^{-1}z), \\ [e(z), f(w)] &= \frac{1}{\kappa_1} \left(\delta \left(\frac{Cw}{z} \right) \psi^+(w) - \delta \left(\frac{Cz}{w} \right) \psi^-(z) \right), \\ g(z, w) e(z) e(w) &= \bar{g}(z, w) e(w) e(z), \quad \bar{g}(z, w) f(z) f(w) = g(z, w) f(w) f(z), \\ \text{Sym}_{z_1, z_2, z_3} \frac{z_2}{z_3} [e(z_1), [e(z_2), e(z_3)]] &= 0, \quad \text{Sym}_{z_1, z_2, z_3} \frac{z_2}{z_3} [f(z_1), [f(z_2), f(z_3)]] = 0, \end{aligned}$$

where we used $\text{Sym}_{z_1, z_2, z_3} F(z_1, z_2, z_3) = \frac{1}{3!} \sum_{\sigma \in S_3} F(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)})$. The quantum toroidal algebra \mathcal{E} is endowed with a topological Hopf algebra structure $(\mathcal{E}, \Delta, \varepsilon, S)$. We define the topological coproduct $\Delta : \mathcal{E} \rightarrow \mathcal{E} \tilde{\otimes} \mathcal{E}$, the counit $\varepsilon : \mathcal{E} \rightarrow \mathbf{C}$, and the antipode $S : \mathcal{E} \rightarrow \mathcal{E}$ as follows.

$$\begin{aligned} \Delta e(z) &= e(C_2^{-1}z) \otimes \psi^+(C_2^{-1}z) + 1 \otimes e(z), \\ \Delta f(z) &= f(z) \otimes 1 + \psi^-(C_1^{-1}z) \otimes f(C_1^{-1}z), \\ \Delta \psi^+(z) &= \psi^+(z) \otimes \psi^+(C_1 z), \end{aligned}$$

$$\Delta\psi^-(z) = \psi^-(C_2z) \otimes \psi^-(z), \quad \Delta(C) = C \otimes C,$$

where $C_1 = C \otimes 1$, $C_2 = 1 \otimes C$.

$$\begin{aligned} \varepsilon(e(z)) &= 0, \quad \varepsilon(f(z)) = 0, \quad \varepsilon(\psi^\pm(z)) = 1, \quad \varepsilon(C) = 1, \\ \tilde{e}(z) &= S(e(z)) = -e(Cz)\psi^+(z)^{-1}, \\ \tilde{f}(z) &= S(f(z)) = -\psi^-(z)^{-1}f(Cz), \\ S(\psi^\pm(z)) &= \psi^\pm(C^{-1}z), \quad S(C) = C^{-1}. \end{aligned}$$

The quantum toroidal algebra \mathcal{E} has three families of Fock representations $\mathcal{F}_c(u)$, where $c = 1, 2, 3$ and $u \in \mathbf{C}^\times$. We call c the color. The Fock module $\mathcal{F}_c(u)$ has level s_c . The Fock modules $\mathcal{F}_c(u)$ are irreducible with respect to the Heisenberg algebra of \mathcal{E} generated by $\{h_r\}_{r \in \mathbf{Z} \setminus \{0\}}$ with relations $[h_r, h_s] = \delta_{r+s,0} \frac{1}{\kappa_r} \frac{C^r - C^{-r}}{r}$. Let $v_c \neq 0$ be the Fock vacuum of $\mathcal{F}_c(u)$, we have the identification of vector spaces

$$\mathcal{F}_c(u) = \mathbf{C}[h_{-r}]_{r>0} v_c, \quad h_r v_c = 0 \quad (r > 0), \quad C v_c = s_c v_c.$$

The generators $e(z)$ and $\tilde{f}(z)$ are realized by vertex operators

$$e(z) \rightarrow b_c : V_c(z; u) :, \quad \tilde{f}(z) \rightarrow b_c : V_c(z; u)^{-1} :, \quad C \rightarrow s_c,$$

where $b_c = -(s_c - s_c^{-1})/\kappa_1$ and

$$V_c(z; u) = u \exp\left(\sum_{r>0} \frac{\kappa_r h_{-r}}{1 - q_c^r} z^r\right) \exp\left(\sum_{r>0} \frac{\kappa_r h_r}{1 - q_c^r} q_c^{\frac{r}{2}} z^{-r}\right).$$

2.3 Quantum algebra \mathcal{K}

The quantum algebra \mathcal{K} introduced in Ref.[8] is an associative algebra with parameters q_1, q_2, q_3 generated by E_n ($n \in \mathbf{Z}$) and H_r ($r \in \mathbf{Z} \setminus \{0\}$), and an invertible central element C . We set the currents $E(z)$, $K^\pm(z)$, and $K(z)$ as follows.

$$E(z) = \sum_{n \in \mathbf{Z}} E_n z^{-n}, \quad K^\pm(z) = \exp\left(\sum_{\pm r > 0} H_r z^{-r}\right), \quad K(z) = K^-(z)K^+(C^2 z).$$

The defining relations are as follows.

$$\begin{aligned} [H_r, H_s] &= -\delta_{r+s,0} \kappa_r \frac{1 + C^{2r}}{r}, \\ g(z, w)E(z)E(w) &+ g(w, z)E(w)E(z) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\kappa_1} \left(g(z, w) \delta \left(C^2 \frac{z}{w} \right) K(z) + g(w, z) \delta \left(C^2 \frac{w}{z} \right) K(w) \right), \\
 &g(z, w) K^\pm(z) E(w) = \bar{g}(z, w) E(w) K^\pm(z), \\
 &\text{Sym}_{z_1, z_2, z_3} \frac{z_2}{z_3} [E(z_1), [E(z_2), E(z_3)]] \\
 &= \text{Sym}_{z_1, z_2, z_3} X(z_1, z_2, z_3) \kappa_1^{-1} \delta \left(C^2 \frac{z_1}{z_3} \right) K^-(z_1) E(z_2) K^+(z_3),
 \end{aligned}$$

where

$$\begin{aligned}
 X(z_1, z_2, z_3) &= \frac{(z_1 + z_2)(z_3^2 - z_1 z_2)}{z_1 z_2 z_3} G(z_2/z_3) + \frac{(z_2 + z_3)(z_1^2 - z_2 z_3)}{z_1 z_2 z_3} G(z_1/z_2) \\
 &\quad + \frac{(z_3 + z_1)(z_2^2 - z_3 z_1)}{z_1 z_2 z_3}
 \end{aligned}$$

and $G(w/z)$ stands for the power series expansion of $\bar{g}(z, w)/g(z, w)$ in w/z . The algebra \mathcal{K} is a comodule over the quantum toroidal algebra \mathcal{E} . We define the map $\Delta : \mathcal{K} \rightarrow \mathcal{E} \tilde{\otimes} \mathcal{K}$ as follows.

$$\begin{aligned}
 \Delta E(z) &= e(C_2^{-1} z) \otimes K^+(z) + 1 \otimes E(z) + \tilde{f}(C_2 z) \otimes K^-(z), \\
 \Delta K^+(z) &= \psi^+(C_1^{-1} C_2^{-1} z) \otimes K^+(z), \\
 \Delta K^-(z) &= \psi^-(C_2 z)^{-1} \otimes K^-(z), \quad \Delta C = C \otimes C,
 \end{aligned}$$

where $C_1 = C \otimes 1, C_2 = 1 \otimes C$.

We introduce three families of the Fock modules \mathcal{F}_c^B of the quantum algebra \mathcal{K} , which we call the boundary Fock modules. We call c the color. For a complex number $s_c \in \mathbf{C}^\times$, let $\mathcal{H}_{s_c^{1/2}}$ be the Heisenberg generated by $\{H_r\}_{r \in \mathbf{Z} \setminus \{0\}}$ with relations $[H_r, H_s] = -\delta_{r+s, 0} \kappa_r \frac{1+s_c^r}{r}$. For $c = 1, 2, 3$, we denote \mathcal{F}_c^B the corresponding Fock modules of the Heisenberg algebra $\mathcal{H}_{s_c^{1/2}}$. For $c = 1, 2, 3$, the generating function $E(z)$ is realized by vertex operators

$$E(z) \rightarrow k_c^B : \tilde{K}_c^-(z) \tilde{K}_c^+(s_c z) :, \quad C \rightarrow s_c^{1/2},$$

where $k_c^B = (1 + s_c)(s_d - s_b)/\kappa_1$ with $(c, d, b) = \text{cycl}(1, 2, 3)$, and

$$\tilde{K}_c^\pm(z) = \exp \left(\sum_{\pm r > 0} \frac{1}{1 + s_c^{-r}} H_r z^{-r} \right).$$

3 Quadratic relations of $\mathcal{W}_{x,r}(A_{2N}^{(2)})$

In this section, we fix a real number $r > 1$ and $0 < |x| < 1$. We fix the rank $N = 1, 2, 3, \dots$. Throughout this section we set

$$q_1 = x^{2r}, \quad q_2 = x^{-2}, \quad q_3 = x^{2(1-r)}.$$

3.1 Basic W -current

Consider a \mathcal{K} module defined as a tensor product of N Fock modules $\mathcal{F}_2(u_i)$ of \mathcal{E} with a boundary Fock module \mathcal{F}_2^B :

$$\mathcal{F}_2(u_1) \otimes \mathcal{F}_2(u_2) \otimes \cdots \otimes \mathcal{F}_2(u_N) \otimes \mathcal{F}_2^B.$$

The total level is $C = x^{-N-\frac{1}{2}}$. The current $E(z)$ acts as a sum of vertex operators in $N+1$ bosons of the form

$$\Delta^{(N)}E(z) = b_2 \sum_{k=1}^N \Lambda_k(z) + k_2^B \Lambda_0(z) + b_2 \sum_{k=1}^N \Lambda_{\bar{k}}(z).$$

Here, for $k = 1, 2, \dots, N$ we set

$$\begin{aligned} \Lambda_k(z) &= 1 \otimes \cdots \otimes V_2(a_k z; u_k) \otimes \psi^+(s_2^{-1} a_{k+1} z) \otimes \cdots \otimes \psi^+(s_2^{-1} a_N z) \otimes K^+(z), \\ \Lambda_0(z) &= 1 \otimes \cdots \otimes 1 \otimes \tilde{K}_2(z), \\ \Lambda_{\bar{k}}(z) &= 1 \otimes \cdots \otimes V_2^{-1}(a_{\bar{k}}^{-1} z; u_k) \otimes \psi^-(a_{k+1}^{-1} z)^{-1} \otimes \cdots \otimes \psi^-(a_N^{-1} z)^{-1} \otimes K^-(z), \end{aligned}$$

where a_k are given by $a_k = x^{N-k+\frac{1}{2}}$. Define the dressed current $\Lambda_i(z)$ depending on $\mu = -x^{-2N-1}$ by

$$\Lambda_i(z) = \Lambda_i(z) \Delta^{(N)} K_\mu^+(z)^{-1}, \quad K_\mu^+(z) = \prod_{s=0}^{\infty} K^+(\mu^{-s} z), \quad i = 1, \dots, N, 0, \bar{N}, \dots, \bar{1}.$$

For $i, j = 1, 2, 3, \dots$ we set

$$\begin{aligned} f_{i,j}(z) &= \exp \left(- \sum_{m=1}^{\infty} \frac{1}{m} [(r-1)m]_x [rm]_x (x-x^{-1})^2 \times \right. \\ &\quad \left. \times \frac{[\text{Min}(i,j)m]_x ([(N+1 - \text{Max}(i,j))m]_x - [(N - \text{Max}(i,j))m]_x)}{[m]_x ([(N+1)m]_x - [Nm]_x)} z^m \right), \\ d(z) &= \frac{(1-x^{2r-1}z)(1-x^{-2r+1}z)}{(1-xz)(1-x^{-1}z)}, \quad c(x,r) = [r]_x [r-1]_x (x-x^{-1}). \end{aligned}$$

We define the basic W -current $\mathbf{T}_1(z)$ for $\mathcal{W}_{x,r}(A_{2N}^{(2)})$ by

$$\mathbf{T}_1(z) = \sum_{k=1}^N \Lambda_k(z) + \frac{k_2^B}{b_2} \Lambda_0(z) + \sum_{k=1}^N \Lambda_{\bar{k}}(z).$$

Here indices are ordered as

$$1 \prec 2 \prec \cdots \prec N \prec 0 \prec \bar{N} \prec \cdots \prec \bar{2} \prec \bar{1}.$$

Lemma 1. [8]¹ *In the case of $\mathcal{W}_{x,r}(A_{2N}^{(2)})$, the dressed currents $\mathbf{\Lambda}_i(z)$ satisfy*

$$\begin{aligned} f_{1,1} \left(\frac{z_2}{z_1} \right) \mathbf{\Lambda}_i(z_1) \mathbf{\Lambda}_j(z_2) &= d \left(\frac{x^{-1} z_2}{z_1} \right) : \mathbf{\Lambda}_i(z_1) \mathbf{\Lambda}_j(z_2) :, \quad i \prec j, j \neq \bar{i}, \\ f_{1,1} \left(\frac{z_2}{z_1} \right) \mathbf{\Lambda}_j(z_1) \mathbf{\Lambda}_i(z_2) &= d \left(\frac{x z_2}{z_1} \right) : \mathbf{\Lambda}_j(z_1) \mathbf{\Lambda}_i(z_2) :, \quad i \prec j, j \neq \bar{i}, \\ f_{1,1} \left(\frac{z_2}{z_1} \right) \mathbf{\Lambda}_0(z_1) \mathbf{\Lambda}_0(z_2) &= d \left(\frac{z_2}{z_1} \right) : \mathbf{\Lambda}_0(z_1) \mathbf{\Lambda}_0(z_2) :, \\ f_{1,1} \left(\frac{z_2}{z_1} \right) \mathbf{\Lambda}_i(z_1) \mathbf{\Lambda}_i(z_2) &=: \mathbf{\Lambda}_i(z_1) \mathbf{\Lambda}_i(z_2) :, \quad i \neq 0, \\ f_{1,1} \left(\frac{z_2}{z_1} \right) \mathbf{\Lambda}_k(z_1) \mathbf{\Lambda}_{\bar{k}}(z_2) &= d \left(\frac{x^{-1} z_2}{z_1} \right) d \left(\frac{x^{-2N-2+2k} z_2}{z_1} \right) : \mathbf{\Lambda}_k(z_1) \mathbf{\Lambda}_{\bar{k}}(z_2) :, \quad 1 \leq k \leq N, \\ f_{1,1} \left(\frac{z_2}{z_1} \right) \mathbf{\Lambda}_{\bar{k}}(z_1) \mathbf{\Lambda}_k(z_2) &= d \left(\frac{x z_2}{z_1} \right) d \left(\frac{x^{2N+2-2k} z_2}{z_1} \right) : \mathbf{\Lambda}_{\bar{k}}(z_1) \mathbf{\Lambda}_k(z_2) :, \quad 1 \leq k \leq N. \end{aligned}$$

3.2 Quadratic relations

In this section, we introduce the higher W -currents $\mathbf{T}_j(z)$ and obtain quadratic relations of them. We define the higher W -currents $\mathbf{T}_i(z)$ ($i \in \mathbf{N}$) for $\mathcal{W}_{x,r}(A_{2N}^{(2)})$ by fusion relation

$$\begin{aligned} &\lim_{z_1 \rightarrow x^{\pm(i+j)} z_2} \left(1 - \frac{x^{\pm(i+j)} z_2}{z_1} \right) f_{i,j} \left(\frac{z_2}{z_1} \right) \mathbf{T}_i(z_1) \mathbf{T}_j(z_2) \\ &= \mp c(x, r) \prod_{l=1}^{\text{Min}(i,j)-1} d(x^{2l+1}) \mathbf{T}_{i+j}(x^{\pm i} z_2), \quad i, j \geq 1, \end{aligned}$$

and $\mathbf{T}_0(z) = 1$. We obtain $\mathbf{T}_i(z) = 0$ for $i \geq 2N + 2$.

Proposition 1. [12] *In the case of $\mathcal{W}_{x,r}(A_{2N}^{(2)})$, the W -currents $\mathbf{T}_i(z)$ satisfy the duality*

$$\mathbf{T}_{2N+1-i}(z) = \frac{[r - \frac{1}{2}]_x}{[\frac{1}{2}]_x} \prod_{k=1}^{N-i} d(x^{2k}) \mathbf{T}_i(z), \quad 0 \leq i \leq N. \quad (1)$$

¹ Frenkel-Reshetikhin [5] constructed the bosonic operators $\mathbf{\Lambda}_i^{\text{FR}}(z)$ in kernel of screening operators, that satisfy the same normal ordering relations as those of Lemma 1 in Ref.[8].

Theorem 1. [12] *In the case of $\mathcal{W}_{x,r}(A_{2N}^{(2)})$, the W -currents $\mathbf{T}_i(z)$ satisfy the set of quadratic relations*

$$\begin{aligned}
& f_{i,j} \left(\frac{z_2}{z_1} \right) \mathbf{T}_i(z_1) \mathbf{T}_j(z_2) - f_{j,i} \left(\frac{z_1}{z_2} \right) \mathbf{T}_j(z_2) \mathbf{T}_i(z_1) \\
&= c(x,r) \sum_{k=1}^i \prod_{l=1}^{k-1} d(x^{2l+1}) \times \\
& \quad \times \left(\delta \left(\frac{x^{-j+i-2k} z_2}{z_1} \right) f_{i-k,j+k}(x^{j-i}) \mathbf{T}_{i-k}(x^k z_1) \mathbf{T}_{j+k}(x^{-k} z_2) \right. \\
& \quad \left. - \delta \left(\frac{x^{j-i+2k} z_2}{z_1} \right) f_{i-k,j+k}(x^{-j+i}) \mathbf{T}_{i-k}(x^{-k} z_1) \mathbf{T}_{j+k}(x^k z_2) \right) \\
& \quad + c(x,r) \prod_{l=1}^{i-1} d(x^{2l+1}) \prod_{l=N+1-j}^{N+i-j} d(x^{2N}) \\
& \quad \times \left(\delta \left(\frac{x^{-2N+j-i-1} z_2}{z_1} \right) \mathbf{T}_{j-i}(x^{-i} z_2) - \delta \left(\frac{x^{2N-j+i+1} z_2}{z_1} \right) \mathbf{T}_{j-i}(x^i z_2) \right), \\
& \quad 1 \leq i \leq j \leq N. \tag{2}
\end{aligned}$$

Definition 1. Let W be the free complex associative algebra generated by elements $\overline{\mathbf{T}}_i[m], m \in \mathbf{Z}, 0 \leq i \leq 2N+1$, I_K the left ideal generated by elements $\overline{\mathbf{T}}_i[m], m \geq K \in \mathbf{N}, 0 \leq i \leq 2N+1$, and

$$\widehat{W} = \varprojlim W/I_K.$$

The deformed W -algebra $\mathcal{W}_{x,r}(A_{2N}^{(2)})$ is the quotient of \widehat{W} by the two-sided ideal generated by the coefficients of the generating series which are the differences of the right hand sides and of the left hand sides of the relations (1) and (2), where the generating series $\mathbf{T}_i(z)$ are replaced with $\overline{\mathbf{T}}_i(z) = \sum_{m \in \mathbf{Z}} \overline{\mathbf{T}}_i[m] z^{-m}, 0 \leq i \leq 2N+1$.

4 Quadratic relations of q - Y_{L_1, L_2, L_3}

We fix natural numbers L_1, L_2, L_3 such that $L_1 + L_2 + L_3 \geq 1$, $L_1, L_2, L_3 \in \mathbf{N}$. We fix real numbers $\lambda_1, \lambda_2, \lambda_3$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 0$. We fix $0 < |x| < 1$. Throughout this section we set

$$q_1 = x^{2\lambda_1}, \quad q_2 = x^{2\lambda_2}, \quad q_3 = x^{2\lambda_3}, \quad L = L_1 + L_2 + L_3.$$

4.1 Basic W -current

Consider a tensor product of L Fock modules of \mathcal{E} :

$$\mathcal{F}_{c_1}(u_1) \otimes \cdots \otimes \mathcal{F}_{c_L}(u_L).$$

Here we choose colors $c_1, c_2, \dots, c_L \in \{1, 2, 3\}$ such that

$$L_c = n(I(c)), \quad I(c) = \{1 \leq i \leq L | c_i = c\} \quad (c = 1, 2, 3).$$

The total level of L Fock modules is $C = \prod_{j=1}^L s_{c_j}$. The current $e(z)$ acts as a sum of vertex operators in L bosons of the form

$$\Delta^{(L-1)}e(z) = \sum_{i=1}^L b_{c_i} \Lambda_i(z).$$

Here, for $i = 1, 2, \dots, L$ we set

$$\Lambda_i(z) = 1 \otimes \cdots \otimes 1 \otimes V_{c_i}(a_i z; u_i) \otimes \psi^+(s_{c_{i+1}}^{-1} a_{i+1} z) \otimes \cdots \otimes \psi^+(s_{c_L}^{-1} a_L z),$$

where a_i are given by $a_i = \prod_{j=i+1}^L s_{c_j}^{-1}$. Define the dressed currents $\mathbf{\Lambda}_i(z)$ depending on free parameter μ by

$$\mathbf{\Lambda}_i(z) = \Lambda_i(z) \Delta^{(L-1)} K_\mu^+(z)^{-1}, \quad K_\mu^+(z) = \prod_{s=0}^{\infty} K^+(\mu^{-s} z), \quad \mu = x^{2\alpha}.$$

For $i, j = 1, 2, 3, \dots$, we set

$$g_{i,j}(z) = \exp \left(\sum_{m=1}^{\infty} \frac{1}{m} \frac{[\lambda_1 m]_x [\lambda_3 m]_x [\text{Min}(i, j) \lambda_2 m]_x [(\alpha - \text{Max}(i, j) \lambda_2) m]_x}{[\lambda_2 m]_x [\alpha m]_x} (x - x^{-1})^2 z^m \right),$$

$$d_\lambda(z) = \frac{(1 - x^{\lambda_1 - \lambda_3} z)(1 - x^{\lambda_3 - \lambda_1} z)}{(1 - x^{\lambda_2} z)(1 - x^{-\lambda_2} z)}, \quad c_\lambda = \frac{[\lambda_1]_x [\lambda_3]_x}{[\lambda_2]_x} (x - x^{-1}),$$

$$\gamma_c(z) = \frac{(1 - x^{2\lambda_c} z)(1 - x^{-2\lambda_c} z)}{(1 - x^{2\lambda_2} z)(1 - x^{-2\lambda_2} z)}, \quad c = 1, 2, 3.$$

We define the basic W -current $\mathbf{T}_1(z)$ for q - Y_{L_1, L_2, L_3} by

$$\mathbf{T}_1(z) = \sum_{i=1}^L b_{c_i} \mathbf{\Lambda}_i(z).$$

Lemma 2. [8] *In the case of q - Y_{L_1, L_2, L_3} , the dressed currents $\mathbf{\Lambda}_i(z)$ satisfy*

$$g_{1,1} \left(\frac{z_2}{z_1} \right) \mathbf{\Lambda}_i(z_1) \mathbf{\Lambda}_j(z_2) = d_\lambda \left(\frac{x^{\lambda_2} z_2}{z_1} \right) : \mathbf{\Lambda}_i(z_1) \mathbf{\Lambda}_j(z_2) :, \quad 1 \leq i < j \leq L,$$

$$g_{1,1} \left(\frac{z_2}{z_1} \right) \mathbf{\Lambda}_j(z_1) \mathbf{\Lambda}_i(z_2) = d_\lambda \left(\frac{x^{-\lambda_2} z_2}{z_1} \right) : \mathbf{\Lambda}_j(z_1) \mathbf{\Lambda}_i(z_2) :, \quad 1 \leq i < j \leq L,$$

$$g_{1,1} \left(\frac{z_2}{z_1} \right) \mathbf{\Lambda}_i(z_1) \mathbf{\Lambda}_i(z_2) = \gamma_{c_i} \left(\frac{z_2}{z_1} \right) : \mathbf{\Lambda}_i(z_1) \mathbf{\Lambda}_i(z_2) :, \quad 1 \leq i \leq L.$$

4.2 Quadratic relations

In this section, we introduce the higher W -currents $\mathbf{T}_i(z)$ and obtain quadratic relations of them. We define the higher W -currents $\mathbf{T}_i(z)$, ($i \in \mathbf{N}$) for q - Y_{L_1, L_2, L_3} by fusion relation

$$\lim_{z_1 \rightarrow x^{\pm(i+j)\lambda_2} z_2} \left(1 - \frac{x^{\pm(i+j)\lambda_2} z_2}{z_1} \right) g_{i,j} \left(\frac{z_2}{z_1} \right) \mathbf{T}_i(z_1) \mathbf{T}_j(z_2)$$

$$= \mp c_\lambda \prod_{l=1}^{\text{Min}(i,j)-1} d_\lambda(x^{(2l+1)\lambda_2}) \mathbf{T}_{i+j}(x^{\pm i\lambda_2} z_2), \quad i, j \geq 1,$$

and $\mathbf{T}_0(z) = 1$.

Proposition 2. *In the case of q - Y_{L_1, L_2, L_3} , the W -currents $\mathbf{T}_i(z)$ don't vanish.*

$$\mathbf{T}_i(z) \neq 0, \quad i \in \mathbf{N}.$$

Upon the specialization $(L_1, L_2, L_3) = (0, N+1, 0)$, the W -currents $\mathbf{T}_i(z)$ of q - Y_{L_1, L_2, L_3} coincide with those of $\mathcal{W}_{x,r}(A_N^{(1)})$ in Refs.[1, 2]. Upon the specialization $(L_1, L_2, L_3) = (0, N+1, M+1)$, the W -currents $\mathbf{T}_i(z)$ of q - Y_{L_1, L_2, L_3} coincide with those of $\mathcal{W}_{x,r}(A(M, N)^{(1)})$ in Refs.[10, 11].

$$\mathbf{T}_i(z) = 0 \quad (i \geq N+1) \quad \text{for } \mathcal{W}_{x,r}(A_N^{(1)}) : \text{non-super},$$

$$\mathbf{T}_i(z) \neq 0 \quad (i \in \mathbf{N}) \quad \text{for } \mathcal{W}_{x,r}(A(M, N)^{(1)}) : \text{super}.$$

Theorem 2. ² *In the case of q - Y_{L_1, L_2, L_3} , the W -currents $\mathbf{T}_i(z)$ satisfy the set of quadratic relations*

$$g_{i,j} \left(\frac{z_2}{z_1} \right) \mathbf{T}_i(z_1) \mathbf{T}_j(z_2) - g_{j,i} \left(\frac{z_1}{z_2} \right) \mathbf{T}_j(z_2) \mathbf{T}_i(z_1)$$

$$= c_\lambda \sum_{k=1}^i \prod_{l=1}^{k-1} d_\lambda(x^{(2l+1)\lambda_2}) \times$$

² Harada, Matsuo, Noshita and Watanabe conjectured similar quadratic relations in Ref.[13]. The quadratic relations (3) of q - Y_{L_1, L_2, L_3} in Theorem 2 can be proved similarly to those of $\mathcal{W}_{x,r}(A(M, N)^{(1)})$ in Refs.[10, 11].

$$\begin{aligned}
 & \times \left(\delta \left(\frac{x^{(j-i+2k)\lambda_2} z_2}{z_1} \right) g_{i-k, j+k}(x^{(i-j)\lambda_2}) \mathbf{T}_{i-k}(x^{-k\lambda_2} z_1) \mathbf{T}_{j+k}(x^{k\lambda_2} z_2) \right. \\
 & \left. - \delta \left(\frac{x^{(-j+i-2k)\lambda_2} z_2}{z_1} \right) g_{i-k, j+k}(x^{(j-i)\lambda_2}) \mathbf{T}_{i-k}(x^{k\lambda_2} z_1) \mathbf{T}_{j+k}(x^{-k\lambda_2} z_2) \right), \\
 & 1 \leq i \leq j. \tag{3}
 \end{aligned}$$

Upon the specialization $(L_1, L_2, L_3) = (0, N+1, M+1)$ and $(\lambda_1, \lambda_2, \lambda_3) = (r, -1, 1-r)$, the set of quadratic relations (3) of q - Y_{L_1, L_2, L_3} coincides with those of $\mathcal{W}_{x,r}(A(M, N))^{(1)}$ in Refs.[10, 11].

Definition 2. Let W be the free complex associative algebra generated by elements $\overline{\mathbf{T}}_i[m], m \in \mathbf{Z}, i \in \mathbf{N}$, I_K the left ideal generated by elements $\overline{\mathbf{T}}_i[m], m \geq K \in \mathbf{N}, i \in \mathbf{N}$, and

$$\widehat{W} = \varprojlim W/I_K.$$

The q - Y_{L_1, L_2, L_3} is the quotient of \widehat{W} by the two-sided ideal generated by the coefficients of the generating series which are the differences of the right hand sides and of the left hand sides of the relations (3), where the generating series $\mathbf{T}_i(z)$ are replaced with $\overline{\mathbf{T}}_i(z) = \sum_{m \in \mathbf{Z}} \overline{\mathbf{T}}_i[m] z^{-m}, i \in \mathbf{N}$.

Acknowledgements The author is thankful for the kind hospitality by the organizing committee of the 15-th International Workshop "Lie theory and its application in physics" (LT15). The author would like to thank Professor Michio Jimbo for giving advice. This work is supported by the Grant-in-Aid for Scientific Research C(19K0350900) from Japan Society for Promotion of Science.

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