

# Quadratic relations of the deformed $W$ -superalgebra $\mathcal{W}_{q,t}(A(M, N))$

Takeo KOJIMA

July 16, 2021

*Department of Mathematics and Physics, Faculty of Engineering, Yamagata University,  
Jonan 4-chome 3-16, Yonezawa 992-8510, JAPAN*

## Abstract

We find the free field construction of the basic  $W$ -current and screening currents for the deformed  $W$ -superalgebra  $\mathcal{W}_{q,t}(A(M, N))$  associated with Lie superalgebra of type  $A(M, N)$ . Using this free field construction, we introduce the higher  $W$ -currents and obtain a closed set of quadratic relations among them. These relations are independent of the choice of Dynkin diagrams for the Lie superalgebra  $A(M, N)$ , though the screening currents are not. This allows us to define  $\mathcal{W}_{q,t}(A(M, N))$  by generators and relations.

## 1 Introduction

The deformed  $W$ -algebra  $\mathcal{W}_{q,t}(\mathfrak{g})$  is a two parameter deformation of the classical  $W$ -algebra  $\mathcal{W}(\mathfrak{g})$ . Shiraishi et al. [1] obtained a free field construction of the deformed Virasoro algebra  $\mathcal{W}_{q,t}(\mathfrak{sl}(2))$ , which is a one-parameter deformation of the Virasoro algebra, to construct a deformation of the correspondence between conformal field theory and the Calogero-Sutherland model. The theory of the deformed  $W$ -algebras  $\mathcal{W}_{q,t}(\mathfrak{g})$  has been developed in papers [2–11]. However, in comparison with the conformal case, the theory of the deformed  $W$ -algebras is still not fully developed and understood. For that matter it is worthwhile to concretely construct  $\mathcal{W}_{q,t}(\mathfrak{g})$  in each case. This paper is a continuation of the paper [11] for  $\mathcal{W}_{q,t}(A(1, 0))$ . The purpose of this paper is to generalize the result of case  $A(1, 0)$  to  $A(M, N)$ .

We follow the method of [10], where a free field construction is found for the deformed  $\mathcal{W}_{q,t}(\mathfrak{sl}(3))$  and  $\mathcal{W}_{q,t}(A(1, 0))$ . Starting from a  $W$  current given as a sum of three vertex operators

$$T_1(z) = \Lambda_1(z) + \Lambda_2(z) + \Lambda_3(z),$$

and two screening currents  $S_j(z)$  given by a vertex operator, the authors of [10] determined them simultaneously by demanding that  $T_1(z)$  and  $S_j(w)$  commute up to a total difference. Higher currents  $T_i(z)$  are defined inductively by the fusion relation

$$\operatorname{Res}_{w=x^i z} T_1(w)T_{i-1}(z) = c_i T_i(x^{i-1}z)$$

with appropriate constants  $x$  and  $c_i$ . In the case of  $\mathcal{W}_{q,t}(\mathfrak{sl}(3))$  it is known that they truncate, i.e.  $T_3(z) = 1$  and  $T_i(z) = 0$  ( $i \geq 4$ ), and that  $T_1(z)$  and  $T_2(z)$  satisfy the quadratic relations [2, 3]

$$\begin{aligned} f_{1,1} \left( \frac{z_2}{z_1} \right) T_1(z_1)T_1(z_2) - f_{1,1} \left( \frac{z_1}{z_2} \right) T_1(z_2)T_1(z_1) &= c \left( \delta \left( \frac{x^{-2}z_2}{z_1} \right) T_2(x^{-1}z_2) - \delta \left( \frac{x^2z_2}{z_1} \right) T_2(xz_2) \right), \\ f_{1,2} \left( \frac{z_2}{z_1} \right) T_1(z_1)T_2(z_2) - f_{2,1} \left( \frac{z_1}{z_2} \right) T_2(z_2)T_1(z_1) &= c \left( \delta \left( \frac{x^{-3}z_2}{z_1} \right) - \delta \left( \frac{x^3z_2}{z_1} \right) \right), \\ f_{2,2} \left( \frac{z_2}{z_1} \right) T_2(z_1)T_2(z_2) - f_{2,2} \left( \frac{z_1}{z_2} \right) T_2(z_2)T_2(z_1) &= c \left( \delta \left( \frac{x^{-2}z_2}{z_1} \right) T_1(x^{-1}z_2) - \delta \left( \frac{x^2z_2}{z_1} \right) T_1(xz_2) \right) \end{aligned}$$

with appropriate constants  $x, c$ , and functions  $f_{i,j}(z)$ . In the case of  $\mathcal{W}_{q,t}(A(1,0))$ , it was shown in [11] that such truncation for  $T_i(z)$  does not take place and that an infinite number of quadratic relations is satisfied by an infinite number of  $T_i(z)$ 's. In the present paper, we extend this result to general  $A(M, N)$ .

Following the method of [10], we construct the basic  $W$ -current  $T_1(z)$  together with the screening currents  $S_j(w)$  for  $\mathcal{W}_{q,t}(A(M, N))$  (See (3) and (4)). We introduce the higher  $W$ -currents  $T_i(z)$  (See (62)) and obtain a closed set of quadratic relations among them (See (64)). We show further that these relations are independent of the choice of Dynkin diagrams for the Lie superalgebra  $A(M, N)$ , though the screening currents are not. This allows us to define  $\mathcal{W}_{q,t}(A(M, N))$  by generators and relations.

Recently, Feigin, Jimbo, Mukhin, and Vilkoviskiy [9] constructed the basic  $W$ -currents  $T_1(z)$  and the screening currents for  $\mathcal{W}_{q,t}(\mathfrak{g})$  in types  $A, B, C, D$  including twisted and supersymmetric cases. Their construction method is completely different from ours. They gave a uniform construction of the basic  $W$ -currents  $T_1(z)$  on a tensor product of Fock spaces of the quantum toroidal  $\mathfrak{gl}_1$  algebra  $\mathcal{E}$  and newly introduced comodule algebra  $\mathcal{K}$  over  $\mathcal{E}$ . Their motivation is to understand a commutative family of integrals of motion associated with affine Dynkin diagrams. They constructed the local integrals of motion associated with Dynkin diagrams of all non-exceptional types except  $D_{l+1}^{(2)}$  by integrals of products of  $T_1(z)$  with the elliptic theta functions. The present paper is not a special case of Ref. [9]. Our motivation is to give the definition of  $\mathcal{W}_{q,t}(\mathfrak{g})$  by generators and relations. We introduce the higher  $W$ -currents  $T_i(z)$  ( $i = 2, 3, 4, \dots$ ) and obtain a closed set of quadratic relations among them, which allows us to define  $\mathcal{W}_{q,t}(A(M, N))$  by generators and relations (See Theorem 4.1). It is still an open problem to find quadratic relations of the deformed  $W$ -algebras  $\mathcal{W}_{q,t}(\mathfrak{g})$  except  $A_N, A(M, N)$ , and the twisted algebra  $A_2^{(2)}$  [4, 8]. In Ref. [9] Feigin, Jimbo, Mukhin, and Vilkoviskiy constructed the local integrals of motion by using  $T_1(z)$ , but did not study the higher  $W$ -currents  $T_i(z)$  ( $i = 2, 3, 4, \dots$ ).

The text is organized as follows. In Section 2, we prepare the notation and formulate the problem. In Section 3, we give a free field construction of the basic  $W$ -current  $T_1(z)$  and the screening currents  $S_j(w)$  for the deformed  $W$ -algebra  $\mathcal{W}_{q,t}(A(M, N))$ . In Section 4, we introduce higher  $W$ -currents  $T_i(z)$  and present a closed set of quadratic relations among them. We show that these quadratic relations are independent of the choice of the Dynkin diagram for the Lie superalgebra  $A(M, N)$ . We also obtain the  $q$ -Poisson algebra in the classical limit. Section 5 is devoted to conclusion and discussion.

## 2 Preliminaries

In this section we prepare the notation and formulate the problem. Throughout this paper we fix a real number  $r > 1$  and a complex number  $x$  with  $0 < |x| < 1$ .

### 2.1 Notation

In this section we use complex numbers  $a$ ,  $w$  ( $w \neq 0$ ),  $q$  ( $q \neq 0, \pm 1$ ), and  $p$  with  $|p| < 1$ . For any integer  $n$ , define  $q$ -integer

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

We use symbols for infinite products

$$(a; p)_\infty = \prod_{k=0}^{\infty} (1 - ap^k), \quad (a_1, a_2, \dots, a_N; p)_\infty = \prod_{i=1}^N (a_i; p)_\infty$$

for complex numbers  $a_1, a_2, \dots, a_N$ . The following standard formulae are useful.

$$\exp\left(-\sum_{m=1}^{\infty} \frac{1}{m} a^m\right) = 1 - a, \quad \exp\left(-\sum_{m=1}^{\infty} \frac{1}{m} \frac{a^m}{1 - p^m}\right) = (a; p)_\infty.$$

We use the elliptic theta function  $\Theta_p(w)$  and the compact notation  $\Theta_p(w_1, w_2, \dots, w_N)$  as

$$\Theta_p(w) = (p, w, pw^{-1}; p)_\infty, \quad \Theta_p(w_1, w_2, \dots, w_N) = \prod_{i=1}^N \Theta_p(w_i)$$

for complex numbers  $w_1, w_2, \dots, w_N \neq 0$ . Define  $\delta(z)$  by the formal series

$$\delta(z) = \sum_{m \in \mathbf{Z}} z^m.$$

### 2.2 Lie superalgebra $A(M, N)$

In this section we introduce the Lie superalgebra  $A(M, N)$ . Let  $\mathbf{Z}_2 = \{\bar{0}, \bar{1}\}$  denote the additive group of two elements. If the vector space  $V$  is a direct sum of two vector subspaces  $V_{\bar{0}}$  and  $V_{\bar{1}}$ , then  $V = V_{\bar{0}} \oplus V_{\bar{1}}$  is called a  $\mathbf{Z}_2$ -graded vector space. An element  $v \in V$  has a unique expression of the form  $v = v_{\bar{0}} + v_{\bar{1}}$  ( $v_i \in V_i$ ). If  $v$  is an element of either  $V_{\bar{0}}$  or  $V_{\bar{1}}$ ,  $v$  is called homogeneous. For homogeneous element  $v \in V_i$ , we set the degree  $|v| = i$ . A  $\mathbf{Z}_2$ -graded vector space  $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$  possessing bilinear multiplication  $[\cdot, \cdot]$  is called the Lie superalgebra if it satisfies the conditions

1.  $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \quad (i, j \in \mathbf{Z}_2)$ ,
2.  $[y, x] = -(-1)^{|x||y|}[x, y]$ ,
3.  $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$ ,

where  $x, y, z \in \mathfrak{g}$  are homogeneous elements. We call this bilinear multiplication  $[\cdot, \cdot]$  the bracket product. Let  $\mathfrak{p}, \mathfrak{i}$  be  $\mathbf{Z}_2$ -graded vector subspace of  $\mathfrak{g}$ .  $\mathfrak{p}$  is called a Lie sub-superalgebra of  $\mathfrak{g}$ , if  $\mathfrak{p}$  satisfies  $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}$ .

$\mathfrak{i}$  is called an ideal of  $\mathfrak{g}$ , if  $\mathfrak{i}$  satisfies  $[\mathfrak{i}, \mathfrak{g}] \subset \mathfrak{i}$ .  $\mathfrak{g}$  is called a simple Lie superalgebra, if it has no ideal except for  $\{0\}$  and  $\mathfrak{g}$  itself.

Let  $(\text{End}(V))_k = \{f \in \text{End}(V) \mid f(V_i) \subset V_{i+k}\}$  ( $k \in \mathbf{Z}_2$ ). If  $f$  is an element of either  $(\text{End}(V))_{\bar{0}}$  or  $(\text{End}(V))_{\bar{1}}$ ,  $f$  is called homogeneous. For homogeneous element  $f \in (\text{End}(V))_k$ , we set the degree  $|f| = k$ . We can make an associative algebra  $\text{End}(V)$  into a Lie superalgebra by letting

$$[f, g] = f \cdot g - (-1)^{|f||g|} g \cdot f, \quad (1)$$

for homogeneous elements  $f, g$  and extending  $[\cdot, \cdot]$  by bilinearity. We fix integers  $M, N$  ( $M + N \geq 1, M, N = 0, 1, 2, \dots$ ). Let the vector subspaces  $V_{\bar{0}} = \mathbf{C}^{M+1}$  and  $V_{\bar{1}} = \mathbf{C}^{N+1}$  respectively.  $(\text{End}(V))_k$  ( $k \in \mathbf{Z}_2$ ) can be realized as follows.

$$\begin{aligned} (\text{End}(V))_{\bar{0}} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \middle| A \in M_{M+1, M+1}(\mathbf{C}), D \in M_{N+1, N+1}(\mathbf{C}) \right\}, \\ (\text{End}(V))_{\bar{1}} &= \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \middle| B \in M_{M+1, N+1}(\mathbf{C}), C \in M_{N+1, M+1}(\mathbf{C}) \right\}. \end{aligned}$$

For  $x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, y = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \text{End}(V)$ , the bracket of these elements can be computed following rule (1) as follows.

$$[x, y] = \begin{pmatrix} AA' - A'A + BC' + B'C & AB' + BD' - (A'B + B'D) \\ CA' + DC' - (C'A + D'C) & DD' - D'D + CB' + C'B \end{pmatrix}.$$

$\text{End}(V_{\bar{0}} \oplus V_{\bar{1}})$ , equipped with the above bracket product, forms a Lie superalgebra called the general linear Lie superalgebra and denoted by

$$\mathfrak{gl}(M+1|N+1).$$

For  $x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(M+1|N+1)$  we define the supertrace  $\text{str}(x)$  as

$$\text{str}(x) = \text{tr}(A) - \text{tr}(D),$$

where  $\text{tr}(y)$  denotes the trace of square matrix  $y$ . We have  $\text{str}([x, y]) = 0$  ( $x, y \in \mathfrak{gl}(M+1|N+1)$ ). Thus the vector subspace

$$\mathfrak{sl}(M+1|N+1) = \{x \in \mathfrak{gl}(M+1|N+1) \mid \text{str}(x) = 0\}$$

is a Lie sub-superalgebra of  $\mathfrak{gl}(M+1|N+1)$ , and it is called the special linear Lie superalgebra.  $\mathfrak{sl}(M+1|N+1)$  is simple for  $M \neq N$ . For  $M = N$ ,  $\mathfrak{sl}(M+1|N+1)$  contains a nontrivial ideal generated by the identity matrix  $I_{M+N+2}$ . We introduce the simple Lie superalgebra  $A(M, N)$  as

$$A(M, N) = \begin{cases} \mathfrak{sl}(M+1|N+1) & (M \neq N \geq 0, M+N \geq 1), \\ \mathfrak{sl}(M+1|N+1)/\mathbf{C}I_{M+N+2} & (M = N \geq 1). \end{cases} \quad (2)$$

### 2.3 Dynkin diagram of $A(M, N)$

In this section we introduce Dynkin diagrams of the Lie superalgebra  $A(M, N)$ . We set  $L = M + N + 1$ .

Let  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{M+1}$  and  $\delta_1, \delta_2, \dots, \delta_{N+1}$  be a basis of  $\mathbf{R}^{L+1}$  with an inner product  $(\ , \ )$  such that

$$\begin{aligned} (\varepsilon_i, \varepsilon_j) &= \delta_{i,j} \quad (1 \leq i, j \leq M+1), \quad (\delta_i, \delta_j) = -\delta_{i,j} \quad (1 \leq i, j \leq N+1), \\ (\varepsilon_i, \delta_j) &= (\delta_j, \varepsilon_i) = 0 \quad (1 \leq i \leq M+1, 1 \leq j \leq N+1). \end{aligned}$$

The standard fundamental system  $\Pi^{st}$  for the Lie superalgebra  $A(M, N)$  is given as

$$\Pi^{st} = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \alpha_{M+1} = \varepsilon_{M+1} - \delta_1, \alpha_{M+1+j} = \delta_j - \delta_{j+1} | 1 \leq i \leq M, 1 \leq j \leq N\}.$$

The standard Dynkin diagram  $\Phi^{st}$  for the Lie superalgebra  $A(M, N)$  is given as

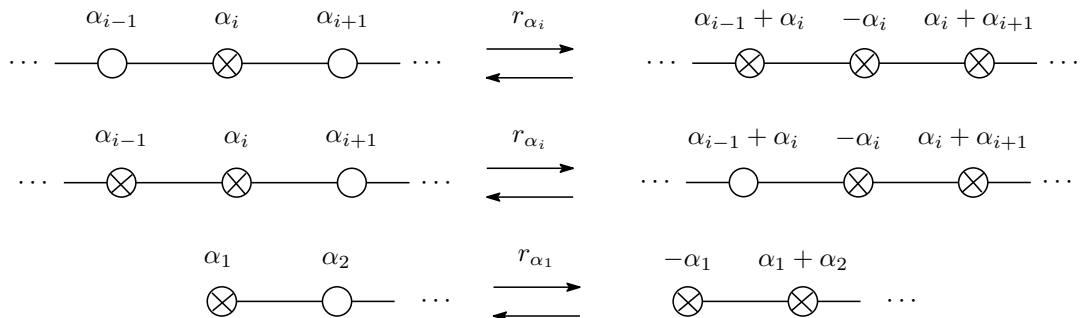
$$\Phi^{st} = \begin{array}{ccccccc} & \alpha_1 & & & \alpha_M & \alpha_{M+1} & \alpha_{M+2} & & & \alpha_{M+N+1} \\ & \circ & \cdots & \cdots & \circ & \otimes & \circ & \cdots & \cdots & \circ \end{array}$$

Here a circle represents an even simple root and a crossed circle represents an odd isotropic simple root. The Lie superalgebra  $A(M, N)$  defined in (2) can be reconstructed from this standard Dynkin diagram  $\Phi^{st}$ . Conversely, the Dynkin diagram  $\Phi^{st}$  can be constructed from the Lie superalgebra  $A(M, N)$ . See details in Refs. [12] and [13].

There is an indeterminacy in how to choose Dynkin diagram for the Lie superalgebra  $A(M, N)$ , which is brought by fundamental reflections  $r_{\alpha_i}$ . For the fundamental system  $\Pi$ , the fundamental reflection  $r_{\alpha_i}$  ( $\alpha_i \in \Pi$ ) satisfies

$$r_{\alpha_i}(\alpha_j) = \begin{cases} -\alpha_i & \text{if } j = i, \\ \alpha_i + \alpha_j & \text{if } j \neq i, (\alpha_i, \alpha_j) \neq 0, \\ \alpha_j & \text{if } j \neq i, (\alpha_i, \alpha_j) = 0. \end{cases}$$

For an odd isotropic root  $\alpha_i$ , we call the fundamental reflection  $r_{\alpha_i}$  odd reflection. For an even root  $\alpha_i$ , we call the fundamental reflection  $r_{\alpha_i}$  real reflection. The Dynkin diagram transformed by  $r_{\alpha_i}$  is represented as  $r_{\alpha_i}(\Phi)$ . Real reflections don't change Dynkin diagram. We illustrate the notion of odd reflections as follows.



$$\cdots \begin{array}{c} \alpha_{L-1} \\ \circ \\ \alpha_L \\ \otimes \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} r_{\alpha_L} \\ \longrightarrow \\ \longleftarrow \end{array} \cdots \begin{array}{c} \alpha_{L-1} + \alpha_L \\ \otimes \\ -\alpha_L \\ \otimes \end{array}$$

**Example**  $A(1, 0)$  and  $A(0, 1)$

$$\begin{array}{c} \delta_1 - \varepsilon_1 \\ \otimes \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \varepsilon_1 - \varepsilon_2 \\ \circ \end{array} \begin{array}{c} r_{\delta_1 - \varepsilon_1} \\ \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \varepsilon_1 - \delta_1 \\ \otimes \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \delta_1 - \varepsilon_2 \\ \otimes \end{array} \begin{array}{c} r_{\delta_1 - \varepsilon_2} \\ \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \varepsilon_1 - \varepsilon_2 \\ \circ \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \varepsilon_2 - \delta_1 \\ \otimes \end{array}$$

Here  $\Pi_1 = \{\delta_1 - \varepsilon_1, \varepsilon_1 - \varepsilon_2\}$  and  $\Pi_2 = \{\varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2\}$  are the other fundamental systems.

**Example**  $A(1, 1)$

$$\begin{array}{c} \varepsilon_1 - \varepsilon_2 \\ \circ \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \varepsilon_2 - \delta_1 \\ \otimes \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \delta_1 - \delta_2 \\ \circ \end{array} \begin{array}{c} r_{\varepsilon_2 - \delta_1} \\ \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \varepsilon_1 - \delta_1 \\ \otimes \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \delta_1 - \varepsilon_2 \\ \otimes \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \varepsilon_2 - \delta_2 \\ \otimes \end{array} \begin{array}{c} r_{\varepsilon_1 - \delta_1} \\ \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \delta_1 - \varepsilon_1 \\ \otimes \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \varepsilon_1 - \varepsilon_2 \\ \circ \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \varepsilon_2 - \delta_2 \\ \otimes \end{array}$$

Here  $\Pi_1 = \{\varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \varepsilon_2 - \delta_2\}$  and  $\Pi_2 = \{\delta_1 - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta_2\}$  are the other fundamental systems.

**Example**  $A(2, 0)$  and  $A(0, 2)$

$$\begin{array}{c} \varepsilon_1 - \varepsilon_2 \\ \circ \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \varepsilon_2 - \varepsilon_3 \\ \circ \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \varepsilon_3 - \delta_1 \\ \otimes \end{array} \begin{array}{c} r_{\varepsilon_3 - \delta_1} \\ \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \varepsilon_1 - \varepsilon_2 \\ \circ \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \varepsilon_2 - \delta_1 \\ \otimes \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \delta_1 - \varepsilon_3 \\ \otimes \end{array} \\ \\ \begin{array}{c} \delta_1 - \varepsilon_1 \\ \otimes \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \varepsilon_1 - \varepsilon_2 \\ \circ \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \varepsilon_2 - \varepsilon_3 \\ \circ \end{array} \begin{array}{c} r_{\varepsilon_1 - \delta_1} \\ \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \delta_1 - \varepsilon_1 \\ \otimes \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \delta_1 - \varepsilon_2 \\ \otimes \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \varepsilon_2 - \varepsilon_3 \\ \circ \end{array} \\ \\ \begin{array}{c} \varepsilon_1 - \varepsilon_2 \\ \circ \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \varepsilon_2 - \delta_1 \\ \otimes \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \delta_1 - \varepsilon_3 \\ \otimes \end{array} \begin{array}{c} r_{\varepsilon_2 - \delta_1} \\ \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \varepsilon_1 - \varepsilon_2 \\ \circ \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \varepsilon_2 - \delta_1 \\ \otimes \end{array} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \begin{array}{c} \delta_1 - \varepsilon_3 \\ \otimes \end{array}$$

Here  $\Pi_1 = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta_1, \delta_1 - \varepsilon_3\}$ ,  $\Pi_2 = \{\varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}$ , and  $\Pi_3 = \{\delta_1 - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}$  are the other fundamental systems.

## 2.4 Ding-Feigin's construction

We introduce the Heisenberg algebra  $\mathcal{H}_{q,t}$  with generators  $a_i(m)$ ,  $Q_i$  ( $m \in \mathbf{Z}$ ,  $1 \leq i \leq L$ ) satisfying

$$\begin{aligned} [a_i(m), a_j(n)] &= \frac{1}{m} A_{i,j}(m) \delta_{m+n,0} \quad (m, n \neq 0, 1 \leq i, j \leq L), \\ [a_i(0), Q_j] &= A_{i,j}(0) \quad (1 \leq i, j \leq L). \end{aligned}$$

The remaining commutators vanish. We impose the following conditions on the parameters  $A_{i,j}(m) \in \mathbf{C}$ :

$$\begin{aligned} A_{i,i}(m) &= 1 \quad (m \neq 0, 1 \leq i \leq L), \quad A_{i,j}(m) = A_{j,i}(-m) \quad (m \in \mathbf{Z}, 1 \leq i \neq j \leq L), \\ \det \left( (A_{i,j}(m))_{i,j=1}^L \right) &\neq 0 \quad (m \in \mathbf{Z}). \end{aligned}$$

We use the normal ordering symbol  $:$  that satisfies

$$\begin{aligned} : a_i(m) a_j(n) : &:= \begin{cases} a_i(m) a_j(n) & (m < 0), \\ a_j(n) a_i(m) & (m \geq 0) \end{cases} \quad (m, n \in \mathbf{Z}, 1 \leq i, j \leq L), \\ : a_i(0) Q_j : &:= Q_j a_i(0) := Q_j a_i(0) \quad (1 \leq i, j \leq L). \end{aligned}$$

Next, we work on Fock space of the Heisenberg algebra. Let  $T_1(z)$  be a sum of vertex operators

$$T_1(z) = g_1 \Lambda_1(z) + g_2 \Lambda_2(z) + \cdots + g_{L+1} \Lambda_{L+1}(z), \quad (3)$$

$$\Lambda_i(z) = e^{\sum_{j=1}^L \lambda_{i,j}(0) a_j(0)} : \exp \left( \sum_{j=1}^L \sum_{m \neq 0} \lambda_{i,j}(m) a_j(m) z^{-m} \right) : \quad (1 \leq i \leq L+1).$$

We call  $T_1(z)$  the basic  $W$ -current. We introduce the screening currents  $S_j(w)$  ( $1 \leq j \leq L$ ) as

$$S_j(w) = w^{\frac{1}{2} A_{j,j}(0)} e^{Q_j} w^{a_j(0)} : \exp \left( \sum_{m \neq 0} s_j(m) a_j(m) w^{-m} \right) : \quad (1 \leq j \leq L). \quad (4)$$

The complex parameters  $A_{i,j}(m)$ ,  $\lambda_{i,j}(m)$ ,  $s_j(m)$  and  $g_i$  are to be determined through the construction given below.

Quite generally, given two vertex operators  $V(z)$ ,  $W(w)$ , their product has the form

$$V(z)W(w) = \varphi_{V,W}(z, w) : V(z)W(w) : \quad (|z| \gg |w|)$$

with some formal power series  $\varphi_{V,W}(z, w) \in \mathbf{C}[[w/z]]$ . The vertex operators  $V(z)$  and  $W(w)$  are said to be mutually local if the following two conditions hold.

- (i)  $\varphi_{V,W}(z, w)$  and  $\varphi_{W,V}(w, z)$  converge to rational functions,
- (ii)  $\varphi_{V,W}(z, w) = \varphi_{W,V}(w, z)$ .

Under this setting, we are going to determine the  $W$ -current  $T_1(z)$  and the screening currents  $S_j(w)$  that satisfy the following mutual locality (5), commutativity (6), and symmetry (7).

**Mutual Locality**  $\Lambda_i(z)$  ( $1 \leq i \leq L+1$ ) and  $S_j(w)$  ( $1 \leq j \leq L$ ) are mutually local, the operator product expansions of their products have at most one pole and one zero, and

$$\varphi_{\Lambda_i, S_j}(z, w) = \varphi_{S_j, \Lambda_i}(w, z) = \frac{w - \frac{z}{p_{i,j}}}{w - \frac{z}{q_{i,j}}} \quad (1 \leq i \leq L+1, 1 \leq j \leq L). \quad (5)$$

We allow the possibility  $p_{i,j} = q_{i,j}$ , in which case  $\Lambda_i(z)S_j(w) = S_j(w)\Lambda_i(z) =: \Lambda_i(z)S_j(w) :$ .

**Commutativity**  $T_1(z)$  commutes with  $S_j(w)$  ( $1 \leq j \leq L$ ) up to a total difference

$$[T_1(z), S_j(w)] = B_j(z) \left( \delta \left( \frac{q_{j,j} w}{z} \right) - \delta \left( \frac{q_{j+1,j} w}{z} \right) \right) \quad (1 \leq j \leq L), \quad (6)$$

with some currents  $B_j(z)$  ( $1 \leq j \leq L$ ).

**Symmetry** For  $\tilde{S}_j(w) = e^{-Q_j} S_j(w)$  ( $1 \leq j \leq L$ ), we impose

$$\begin{aligned} \varphi_{\tilde{S}_k, \tilde{S}_l}(w, z) &= \varphi_{\tilde{S}_l, \tilde{S}_k}(w, z) \quad (1 \leq k, l \leq L), \\ \varphi_{\tilde{S}_k, \tilde{S}_l}(w, z) &= 1 \quad (|k - l| \geq 2, 1 \leq k, l \leq L). \end{aligned} \quad (7)$$

For simplicity, we impose further the following conditions.

$$q_{i,j} \quad (1 \leq i \leq L+1, 1 \leq j \leq L) \text{ are distinct,} \quad (8)$$

$$\left| \frac{q_{j+1,j}}{q_{j,j}} \right| \neq 1 \quad (1 \leq j \leq L), \quad -1 < A_{k,k+1}(0) < 0 \quad (1 \leq k \leq L-1). \quad (9)$$

Consider the following transformations which map operators of form (3), (4) into operators of the same form.

(i) Rearranging indices

$$\Lambda_i(z) \mapsto \Lambda_{i'}(z), \quad S_j(w) \mapsto S_{j'}(w), \quad (10)$$

where  $i \rightarrow i'$  is a permutation of the set  $1, 2, \dots, L+1$  and where  $j \rightarrow j'$  is a permutation of the set  $1, 2, \dots, L$ .

(ii) Scaling variables:  $\Lambda_i(z) \mapsto \Lambda_i(sz)$  ( $s \neq 0$ ), i.e.

$$\lambda_{i,j}(m) \mapsto s^m \lambda_{i,j}(m) \quad q_{i,j} \mapsto s q_{i,j}, \quad p_{i,j} \mapsto s p_{i,j} \quad (m \neq 0, 1 \leq i \leq L+1, 1 \leq j \leq L). \quad (11)$$

(iii) Scaling free fields:

$$\begin{aligned} a_j(m) &\mapsto \alpha_j(m)^{-1} a_j(m), \quad s_j(m) \mapsto \alpha_j(m) s_j(m), \\ \lambda_{i,j}(m) &\mapsto \lambda_{i,j}(m) \alpha_j(m) \\ A_{i,j}(m) &\mapsto \alpha_i(m)^{-1} A_{i,j}(m) \alpha_j(m) \end{aligned} \quad (m \neq 0, 1 \leq i \leq L+1, 1 \leq j \leq L), \quad (12)$$

where  $\alpha_j(m) \neq 0$  ( $1 \leq j \leq L$ ) and  $\alpha_j(-m) = \alpha_j(m)^{-1}$  ( $m > 0, 1 \leq j \leq L$ ).

### 3 Free field construction

In this section we give a free field construction of the basic  $W$ -current and the screening currents for  $\mathcal{W}_{q,t}(A(M, N))$ .



### 3.1 Free field construction

In Ding-Feigin's construction [10], there are  $2^L$  cases to be considered separately according to values of  $A_{j,j}(0)$  ( $1 \leq j \leq L$ ). We fix a pair of integers  $j_1, j_2, \dots, j_K$  ( $1 \leq K \leq L$ ) satisfying  $1 \leq j_1 < j_2 < \dots < j_K \leq L$ . Hereafter, we study the case the following conditions for  $A_{j,j}(0)$  ( $1 \leq j \leq L$ ) are satisfied.

$$A_{j,j}(0) = 1 \quad \text{if } j = j_1, j_2, \dots, j_K, \quad A_{j,j}(0) \neq 1 \quad \text{if } j \neq j_1, j_2, \dots, j_K.$$

First, we prepare the parameters  $A_{i,j}(0)$  to give the free field construction. We have already introduced  $L \times L$  symmetric matrix  $(A_{i,j}(0))_{i,j=1}^L$  as parameters of the Heisenberg algebra. To write  $p_{i,j}$ ,  $q_{i,j}$ ,  $A_{i,j}(m)$ ,  $s_j(m)$ , and  $\lambda_{i,j}(m)$  explicitly, it is convenient to introduce  $(L+1) \times (L+1)$  symmetric matrix  $(A_{i,j}(0))_{i,j=0}^L$  uniquely extended from  $(A_{i,j}(0))_{i,j=1}^L$  as follows.

$$\begin{aligned} A_{0,1}(0) &= \begin{cases} A_{1,2}(0) & \text{if } j_1 \neq 1, \\ -1 - A_{1,2}(0) & \text{if } j_1 = 1, \end{cases} & A_{0,L}(0) &= \begin{cases} A_{L,L-1}(0) & \text{if } j_K \neq L, \\ -1 - A_{L,L-1}(0) & \text{if } j_K = L, \end{cases} \\ A_{0,0}(0) &= \begin{cases} -2A_{0,L}(0) & \text{if } K = \text{even}, \\ 1 & \text{if } K = \text{odd}, \end{cases} & A_{0,i}(0) &= 0 \quad (i \neq 0, 1, L). \end{aligned} \quad (13)$$

The extended matrix  $(A_{i,j}(0))_{i,j=0}^L$  are explicitly written by  $\beta = A_{1,2}(0)$  as follows (See Lemma 3.10).

$$\begin{aligned} A_{i,i}(0) &= \begin{cases} 1 & \text{if } i \in \widehat{J}, \\ -2\beta & \text{if } i \notin \widehat{J}, i \in \widehat{I}(\beta), \\ 2(1+\beta) & \text{if } i \notin \widehat{J}, i \in \widehat{I}(-1-\beta) \end{cases} \quad (1 \leq i \leq L+1), \\ A_{j-1,j}(0) &= A_{j,j-1}(0) = \begin{cases} \beta & \text{if } j \in \widehat{I}(\beta), \\ -1-\beta & \text{if } j \in \widehat{I}(-1-\beta) \end{cases} \quad (1 \leq j \leq L+1), \\ A_{k,l}(0) &= A_{l,k}(0) = 0 \quad (|k-l| \geq 2, 1 \leq k, l \leq L \text{ or } k=0, l \neq 0, 1, L). \end{aligned} \quad (14)$$

Here we set

$$\widehat{J} = \begin{cases} \{j_1, j_2, \dots, j_K\} & \text{if } K = \text{even}, \\ \{j_1, j_2, \dots, j_K, L+1\} & \text{if } K = \text{odd}, \end{cases} \quad \widehat{I}(\delta) = \{1 \leq j \leq L+1 \mid A_{j-1,j}(0) = \delta\}.$$

We understand subscripts of  $A_{i,j}(0)$  with mod.  $L+1$ , i.e.  $A_{0,0}(0) = A_{L+1,L+1}(0)$ . We note  $\widehat{I}(\beta) \cup \widehat{I}(-1-\beta) = \{1, 2, \dots, L+1\}$ .

Next, we introduce the two parameters  $x$  and  $r$  defined as

$$x^{2r} = \frac{q_{2,1}}{q_{1,1}}, \quad r = \begin{cases} \frac{1}{1+\beta} & \text{for } |\widehat{I}(\beta)| > |\widehat{I}(-1-\beta)|, \\ -\frac{1}{\beta} & \text{for } |\widehat{I}(\beta)| \leq |\widehat{I}(-1-\beta)|, \end{cases} \quad (15)$$

where  $|\widehat{I}(\delta)|$  represents the number of elements in  $\widehat{I}(\delta)$ . By this parametrization, we have (19). From (9) and  $q_{i,j} \neq 0$ , we obtain  $|x| \neq 0, 1$  and  $r > 1$ . In this paper, we focus our attention to

$$0 < |x| < 1, \quad r > 1.$$

For the case of  $|x| > 1$ , we obtain the same results under the change  $x \rightarrow x^{-1}$ .

To give the free field construction, we set  $D(k, l; \Phi)$  as

$$D(k, l; \Phi) = \begin{cases} (r-1) \left| \widehat{I} \left( k+1, l+1; -\frac{1}{r}, \widehat{\Phi} \right) \right| + \left| \widehat{I} \left( k+1, l+1; \frac{1-r}{r}, \widehat{\Phi} \right) \right| & (0 \leq k \leq l \leq L), \\ 0 & (0 \leq l < k \leq L), \end{cases}$$

$$\widehat{I}(k, l; \delta, \widehat{\Phi}) = \{1 \leq j \leq L+1 \mid k \leq j \leq l, A_{j-1, j}(0) = \delta\} \quad (1 \leq k \leq l \leq L+1). \quad (16)$$

$D(k, l; \Phi)$  is given by using the matrix  $(A_{i, j}(0))_{i, j=0}^L$ . The matrix  $(A_{i, j}(0))_{i, j=0}^L$  can be constructed from the Dynkin diagrams  $\Phi$  and  $\widehat{\Phi}$ , which we will introduce below.

**Example** We fix integers  $M, N$  ( $M \geq N \geq 0, M+N \geq 1$ ). We set  $K = 1, L = M+N+1$ , and  $j_1 = M+1$ . We have

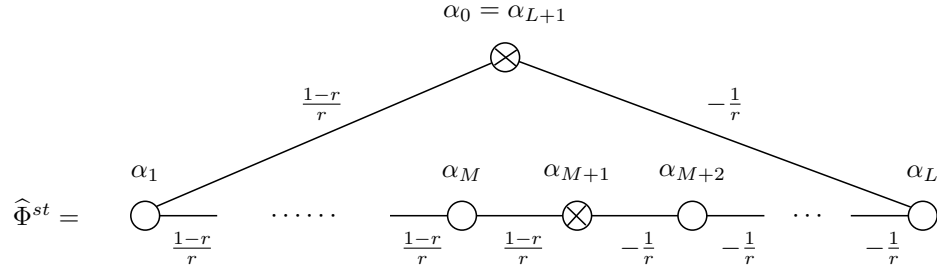
$$A_{i, i}(0) = \begin{cases} \frac{2(r-1)}{r} & \text{if } 1 \leq i \leq M, \\ 1 & \text{if } i = 0, M+1, \\ \frac{2}{r} & \text{if } M+2 \leq i \leq L, \end{cases}$$

$$A_{i, i-1}(0) = A_{i-1, i}(0) = \begin{cases} \frac{1-r}{r} & \text{if } 1 \leq i \leq M+1, \\ -\frac{1}{r} & \text{if } M+2 \leq i \leq L+1, \end{cases}$$

$$A_{k, l}(0) = 0 \quad (|k-l| \geq 2, 1 \leq k, l \leq L \text{ or } k=0, l \neq 0, 1, L).$$

$$\widehat{I} \left( \frac{1-r}{r} \right) = \{1, 2, \dots, M+1\}, \quad \widehat{I} \left( \frac{1}{r} \right) = \{M+2, \dots, L+1\}.$$

We picture  $L \times L$  matrix  $(A_{i, j}(0))_{i, j=1}^L$  as the standard Dynkin diagram  $\Phi^{st}$  of  $A(M, N)$  in Section 2. We picture  $(L+1) \times (L+1)$  matrix  $(A_{i, j}(0))_{i, j=0}^L$  as the Dynkin diagram  $\widehat{\Phi}^{st}$  as follows.



Here a circle represents an even simple root  $(\alpha_i, \alpha_i) = \pm 2$  and a crossed circle represents an odd isotropic simple root  $(\alpha_i, \alpha_i) = 0$ . The inner product  $(\alpha_i, \alpha_j)$  of the roots and the parameters  $A_{i, j}(0)$  correspond as  $(\alpha_i, \alpha_i) = \pm 2 \Leftrightarrow A_{i, i}(0) \neq 1$ ,  $(\alpha_i, \alpha_i) = 0 \Leftrightarrow A_{i, i}(0) = 1$ ,  $(\alpha_i, \alpha_j) = \pm 1 \Leftrightarrow A_{i, j}(0) \neq 0$  ( $i \neq j$ ). As additional information, the values of the parameters  $A_{j-1, j}(0)$  are written beside the line segment connecting  $\alpha_{j-1}$  and  $\alpha_j$ . We have

$$D(0, L; \Phi^{st}) = (N+1)r + M - N.$$

**Example** For  $L = 3$ ,  $K = 2$ ,  $j_1 = 1, j_2 = 3$ , we have

$$(A_{i,j}(0))_{i,j=0}^3 = \begin{pmatrix} \frac{2(r-1)}{r} & \frac{1-r}{r} & 0 & \frac{1-r}{r} \\ \frac{1-r}{r} & 1 & -\frac{1}{r} & 0 \\ 0 & -\frac{1}{r} & \frac{2}{r} & -\frac{1}{r} \\ \frac{1-r}{r} & 0 & -\frac{1}{r} & 1 \end{pmatrix}, \quad \widehat{I}\left(-\frac{1}{r}\right) = \{2, 3\}, \quad \widehat{I}\left(\frac{1-r}{r}\right) = \{1, 4\} = \{0, 1\}.$$

Here we understand subscripts of  $A_{i,j}(0)$  with mod.4, i.e.  $A_{0,3}(0) = A_{4,3}(0)$ . We picture  $3 \times 3$  matrix  $(A_{i,j}(0))_{i,j=1}^3$  as nonstandard Dynkin diagram of  $A(1, 1)$ . We picture  $4 \times 4$  matrix  $(A_{i,j}(0))_{i,j=0}^3$  as the Dynkin diagram  $\widehat{\Phi}$  as follows.



We have

$$D(0, 3; \Phi) = 2r, \quad D(1, 1; \Phi) = r - 1, \quad D(1, 2; \Phi) = 2r - 2.$$

**Theorem 3.1** *Assume that conditions (5), (6), (7), (8) and (9) hold. Then, up to transformations (10), (11) and (12), the parameters  $p_{i,j}, q_{i,j}, A_{i,j}(m), s_i(m), \lambda_{i,j}(m), g_i$ , and the current  $B_j(m)$  are uniquely determined as follows.*

$$\begin{aligned} q_{j,j} &= x^{D(1,j-1;\Phi)}, \quad q_{j+1,j} = x^{2r+D(1,j-1;\Phi)} \quad (1 \leq j \leq L), \\ p_{1,1} &= \begin{cases} x^2 & \text{if } 1 \in \widehat{I}\left(-\frac{1}{r}\right), \\ x^{2r-2} & \text{if } 1 \in \widehat{I}\left(\frac{1-r}{r}\right), \end{cases} \\ p_{j,j} &= x^{D(1,j-2;\Phi)} \times \begin{cases} x^{r+1} & \text{if } j \in \widehat{I}\left(-\frac{1}{r}\right), \\ x^{2r-1} & \text{if } j \in \widehat{I}\left(\frac{1-r}{r}\right) \end{cases} \quad (2 \leq j \leq L), \\ p_{j,j-1} &= x^{D(1,j-2;\Phi)} \times \begin{cases} x^{2r-2} & \text{if } j \in \widehat{I}\left(-\frac{1}{r}\right), \\ x^2 & \text{if } j \in \widehat{I}\left(\frac{1-r}{r}\right) \end{cases} \quad (2 \leq j \leq L+1), \\ p_{k,l} &= q_{k,l} \quad (k \neq l, l+1, 1 \leq k \leq L+1, l \leq l \leq L). \end{aligned} \tag{17}$$

$$s_j(m) = 1 \quad (m > 0, 1 \leq j \leq L),$$

$$s_j(-m) = \begin{cases} -1 & \text{if } j \in \widehat{J}, \\ -\frac{[m]_x [2(r-1)m]_x}{[rm]_x [(r-1)m]_x} & \text{if } j \notin \widehat{J}, j \in \widehat{I}\left(-\frac{1}{r}\right), \\ -\frac{[(r-1)m]_x [2m]_x}{[rm]_x [m]_x} & \text{if } j \notin \widehat{J}, j \in \widehat{I}\left(\frac{1-r}{r}\right) \end{cases} \quad (m > 0, 1 \leq j \leq L). \tag{18}$$

$$\begin{aligned}
A_{i,i}(0) &= \begin{cases} 1 & \text{if } j \in \widehat{\mathcal{J}}, \\ \frac{2}{r} & \text{if } i \notin \widehat{\mathcal{J}}, i \in \widehat{I}(-\frac{1}{r}), \\ \frac{2(r-1)}{r} & \text{if } i \notin \widehat{\mathcal{J}}, i \in \widehat{I}(\frac{1-r}{r}) \end{cases} \quad (1 \leq i \leq L+1), \\
A_{j-1,j}(0) = A_{j,j-1}(0) &= \begin{cases} -\frac{1}{r} & \text{if } j \in \widehat{I}(-\frac{1}{r}), \\ \frac{1-r}{r} & \text{if } j \in \widehat{I}(\frac{1-r}{r}) \end{cases} \quad (1 \leq j \leq L+1), \\
A_{k,l}(0) = A_{l,k}(0) &= 0 \quad (|k-l| \geq 2, 1 \leq k < l \leq L \text{ or } k=0, l \neq 0, 1, L). \tag{19}
\end{aligned}$$

$$\begin{aligned}
A_{j,j}(m) &= 1 \quad (m \neq 0, 1 \leq j \leq L), \\
A_{k,l}(m) &= 0 \quad (m \neq 0, |k-l| \geq 2, 1 \leq k, l \leq L),
\end{aligned}$$

$$\begin{aligned}
A_{j-1,j}(m) &= \frac{[m]_x}{[rm]_x} \times \begin{cases} \frac{1}{s_j(-m)} & (m > 0), \\ \frac{1}{s_{j-1}(m)} & (m < 0) \end{cases} \quad \text{if } j \in \widehat{I}(-\frac{1}{r}), \\
A_{j-1,j}(m) &= \frac{[(r-1)m]_x}{[rm]_x} \times \begin{cases} \frac{1}{s_j(-m)} & (m > 0), \\ \frac{1}{s_{j-1}(m)} & (m < 0) \end{cases} \quad \text{if } j \in \widehat{I}(\frac{1-r}{r}), \\
A_{j-1,j}(-m) &= A_{j,j-1}(m), \quad A_{j,j-1}(-m) = A_{j-1,j}(m) \quad (m > 0, 2 \leq j \leq L). \tag{20}
\end{aligned}$$

$$\lambda_{i,j}(0) = \frac{2r \log x}{D(0, L; \Phi)} \times \begin{cases} D(0, j-1; \Phi) & \text{if } 1 \leq j \leq i-1, \\ -D(j, L; \Phi) & \text{if } i \leq j \leq L \end{cases} \quad (1 \leq i \leq L+1). \tag{21}$$

$$\frac{\lambda_{i,j}(m)}{s_j(m)} = \frac{[rm]_x(x-x^{-1})}{[D(0, L; \Phi)m]_x} \times \begin{cases} -x^{(r+D(1,L;\Phi))m} [D(0, j-1; \Phi)m]_x & \text{if } 1 \leq j \leq i-1, \\ x^{(r-D(0,0;\Phi))m} [D(j, L; \Phi)m]_x & \text{if } i \leq j \leq L \end{cases} \quad (m \neq 0, 1 \leq i \leq L+1). \tag{22}$$

$$g_i = g \times \begin{cases} [r-1]_x & \text{if } i \in \widehat{I}(-\frac{1}{r}), \\ 1 & \text{if } i \in \widehat{I}(\frac{1-r}{r}) \end{cases} \quad (1 \leq i \leq L+1). \tag{23}$$

$$B_j(z) = g_j \left( \frac{q_{j,j}}{p_{j,j}} - 1 \right) : \Lambda_j(z) S_j(q_{j,j}^{-1}z) : \quad (1 \leq j \leq L). \tag{24}$$

Conversely, if the parameters are chosen as above then (5), (6), and (7) are satisfied.

**Proposition 3.2** *The  $\Lambda_i(z)$ 's satisfy the commutation relations*

$$\Lambda_k(z_1) \Lambda_l(z_2) = \frac{\Theta_{x^{2a}} \left( x^2 \frac{z_2}{z_1}, x^{-2r} \frac{z_2}{z_1}, x^{2r-2} \frac{z_2}{z_1} \right)}{\Theta_{x^{2a}} \left( x^{-2} \frac{z_2}{z_1}, x^{2r} \frac{z_2}{z_1}, x^{-2r+2} \frac{z_2}{z_1} \right)} \Lambda_l(z_2) \Lambda_k(z_1) \quad (1 \leq k, l \leq L+1), \tag{25}$$

where  $a = D(0, L; \Phi)$ . We understand (25) in the sense of analytic continuation.

**Proposition 3.3** *The  $S_j(w)$ 's satisfy the commutation relations*

$$\begin{aligned}
S_j(w_1)S_j(w_2) &= S_j(w_2)S_j(w_1) \times \begin{cases} -1 & \text{if } j \in \widehat{J}, \\ -\left(\frac{w_1}{w_2}\right)^{\frac{2}{r}-1} \frac{\Theta_{x^{2r}}\left(x^2 \frac{w_1}{w_2}\right)}{\Theta_{x^{2r}}\left(x^2 \frac{w_2}{w_1}\right)} & \text{if } j \notin \widehat{J}, j \in \widehat{I}\left(-\frac{1}{r}\right), \\ -\left(\frac{w_1}{w_2}\right)^{1-\frac{2}{r}} \frac{\Theta_{x^{2r}}\left(x^2 \frac{w_2}{w_1}\right)}{\Theta_{x^{2r}}\left(x^2 \frac{w_1}{w_2}\right)} & \text{if } j \notin \widehat{J}, j \in \widehat{I}\left(\frac{1-r}{r}\right) \end{cases} \quad (1 \leq j \leq L), \\
S_{j-1}(w_1)S_j(w_2) &= S_j(w_2)S_{j-1}(w_1) \times \begin{cases} \left(\frac{w_1}{w_2}\right)^{-\frac{1}{r}} \frac{\Theta_{x^{2r}}\left(x^{r+1} \frac{w_2}{w_1}\right)}{\Theta_{x^{2r}}\left(x^{r+1} \frac{w_1}{w_2}\right)} & \text{if } j \in \widehat{I}\left(-\frac{1}{r}\right), \\ \left(\frac{w_1}{w_2}\right)^{\frac{1}{r}-1} \frac{\Theta_{x^{2r}}\left(x^{2r-1} \frac{w_2}{w_1}\right)}{\Theta_{x^{2r}}\left(x^{2r-1} \frac{w_1}{w_2}\right)} & \text{if } j \in \widehat{I}\left(\frac{1-r}{r}\right) \end{cases} \quad (2 \leq j \leq L), \\
S_k(w_1)S_l(w_2) &= S_l(w_2)S_k(w_1) \quad (|k-l| \geq 2, 1 \leq k, l \leq L). \quad (26)
\end{aligned}$$

We understand (26) in the sense of the analytic continuation.

In fact, the stronger relation

$$S_j(w_1)S_j(w_2) = (w_1 - w_2) : S_j(w_1)S_j(w_2) : \quad (j \in \widehat{J})$$

holds. This means that the screening currents  $S_j(w)$  ( $j \in \widehat{J}$ ) are ordinary fermions.

### 3.2 Proof of Theorem 3.1

In this section we show Theorem 3.1 and Proposition 3.3.

**Lemma 3.4** *For  $\Lambda_i(z)$  and  $S_j(w)$ , we obtain*

$$\varphi_{\Lambda_i, S_j}(z, w) = e^{\sum_{k=1}^L \lambda_{i,k}(0) A_{k,j}(0)} \exp\left(\sum_{k=1}^L \sum_{m=1}^{\infty} \frac{1}{m} \lambda_{i,k}(m) A_{k,j}(m) s_j(-m) \left(\frac{w}{z}\right)^m\right), \quad (27)$$

$$\varphi_{S_j, \Lambda_i}(w, z) = \exp\left(\sum_{k=1}^L \sum_{m=1}^{\infty} \frac{1}{m} s_j(m) A_{j,k}(m) \lambda_{i,k}(-m) \left(\frac{z}{w}\right)^m\right) \quad (1 \leq i \leq L+1, 1 \leq j \leq L) \quad (28)$$

$$\varphi_{\widetilde{S}_k, \widetilde{S}_l}(w_1, w_2) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} s_k(m) A_{k,l}(m) s_l(-m) \left(\frac{w_2}{w_1}\right)^m\right) \quad (1 \leq k, l \leq L). \quad (29)$$

Assume (5), we obtain

$$\varphi_{\Lambda_k, \Lambda_l}(z_1, z_2) = \exp\left(\sum_{i=1}^L \sum_{m=1}^{\infty} \frac{1}{m} \frac{\lambda_{k,i}(m)}{s_i(m)} (q_{l,i}^{-m} - p_{l,i}^{-m}) \left(\frac{z_2}{z_1}\right)^m\right) \quad (1 \leq k, l \leq L+1). \quad (30)$$

**Lemma 3.5** *Mutual locality (5) holds if and only if (31) and (32) are satisfied*

$$\sum_{k=1}^L \lambda_{i,k}(0) A_{k,j}(0) = \log\left(\frac{q_{i,j}}{p_{i,j}}\right) \quad (1 \leq i \leq L+1, 1 \leq j \leq L), \quad (31)$$

$$\sum_{k=1}^L \lambda_{i,k}(m) A_{k,j}(m) s_j(-m) = q_{i,j}^m - p_{i,j}^m \quad (m \neq 0, 1 \leq i \leq L+1, 1 \leq j \leq L). \quad (32)$$

*Proof of Lemmas 3.4 and 3.5.* Using the standard formula

$$e^A e^B = e^{[A,B]} e^B e^A \quad ([A, B], A = 0 \text{ and } [[A, B], B] = 0),$$

we obtain (27), (28), (29), and

$$\varphi_{\Lambda_k, \Lambda_l}(z_1, z_2) = \exp \left( \sum_{i,j=1}^L \sum_{m=1}^{\infty} \frac{1}{m} \lambda_{k,i}(m) A_{i,j}(m) \lambda_{l,j}(-m) \left( \frac{z_2}{z_1} \right)^m \right) \quad (1 \leq k, l \leq L+1). \quad (33)$$

Considering (27), (28), and the expansions

$$\frac{w - p_{i,j}^{-1} z}{w - q_{i,j}^{-1} z} = \exp \left( \log \left( \frac{q_{i,j}}{p_{i,j}} \right) - \sum_{m=1}^{\infty} \frac{1}{m} (p_{i,j}^m - q_{i,j}^m) \left( \frac{w}{z} \right)^m \right) \quad (|z| \gg |w|), \quad (34)$$

$$\frac{w - p_{i,j}^{-1} z}{w - q_{i,j}^{-1} z} = \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} (p_{i,j}^{-m} - q_{i,j}^{-m}) \left( \frac{z}{w} \right)^m \right) \quad (|w| \gg |z|), \quad (35)$$

we obtain (31) and (32) from (5). Substituting (32) for (33), we have (30).

Conversely, if we assume (31) and (32), we obtain (5) from (27), (28), (34), and (35).  $\square$

From linear equations (31) and (32),  $\lambda_{i,j}(m)$  are expressed in terms of the other parameters.

**Lemma 3.6** *We assume (5) and (8). The commutativity (6) holds if and only if (36), (37), (38), and (39) are satisfied, where*

$$p_{k,l} = q_{k,l} \quad (k \neq l, l+1, 1 \leq k \leq L+1, 1 \leq l \leq L), \quad (36)$$

$$q_{k,k}^{\frac{1}{2} A_{k,k}(0)} : \Lambda_k(z) S_k(q_{k,k}^{-1} z) := q_{k+1,k}^{\frac{1}{2} A_{k,k}(0)} : \Lambda_{k+1}(z) S_k(q_{k+1,k}^{-1} z) : \quad (1 \leq k \leq L), \quad (37)$$

$$\frac{g_{k+1}}{g_k} = - \left( \frac{q_{k+1,k}}{q_{k,k}} \right)^{\frac{1}{2} A_{k,k}(0)} \frac{\frac{q_{k,k}}{p_{k,k}} - 1}{\frac{q_{k+1,k}}{p_{k+1,k}} - 1} \quad (1 \leq k \leq L), \quad (38)$$

$$B_k(z) = g_k \left( \frac{q_{k,k}}{p_{k,k}} - 1 \right) : \Lambda_k(z) S_k(q_{k,k}^{-1} z) : \quad (1 \leq k \leq L). \quad (39)$$

*Proof of Lemma 3.6.* From (5), we obtain

$$[\Lambda_i(z), S_j(w)] = \left( \frac{q_{i,j}}{p_{i,j}} - 1 \right) \delta \left( \frac{q_{i,j} w}{z} \right) : \Lambda_i(z) S_j(q_{i,j}^{-1} z) : \quad (1 \leq i \leq L+1, 1 \leq j \leq L). \quad (40)$$

Considering (8) and (40), we know that (6) holds if and only if (36) and

$$B_j(z) = g_j \left( \frac{q_{j,j}}{p_{j,j}} - 1 \right) : \Lambda_j(z) S_j(q_{j,j}^{-1} z) := -g_{j+1} \left( \frac{q_{j+1,j}}{p_{j+1,j}} - 1 \right) : \Lambda_{j+1}(z) S_j(q_{j+1,j}^{-1} z) : \quad (1 \leq j \leq L) \quad (41)$$

are satisfied. (41) holds if and only if (37), (38), and (39) are satisfied. Hence, we obtain this lemma.  $\square$

We use the abbreviation  $h_{k,l}(w)$  ( $1 \leq k, l \leq L$ ),

$$h_{k,l} \left( \frac{w_2}{w_1} \right) = \varphi_{\tilde{S}_k, \tilde{S}_l}(w_1, w_2). \quad (42)$$

**Lemma 3.7** *We assume (5) and (37). Then,  $h_{k,l}(w)$  in (42) satisfy the  $q$ -difference equations*

$$\begin{aligned} \frac{w - p_{k,k}^{-1}}{w - q_{k,k}^{-1}} h_{k,k} \left( q_{k,k}^{-1} w \right) &= \frac{w - p_{k+1,k}^{-1}}{w - q_{k+1,k}^{-1}} h_{k,k} \left( q_{k+1,k}^{-1} w \right), \\ \left( \frac{q_{k+1,k}}{q_{k,k}} \right)^{A_{k,k}(0)-1} \frac{p_{k+1,k}}{p_{k,k}} \frac{1 - p_{k,k} w}{1 - q_{k,k} w} h_{k,k} \left( q_{k,k} w \right) &= \frac{1 - p_{k+1,k} w}{1 - q_{k+1,k} w} h_{k,k} \left( q_{k+1,k} w \right) \end{aligned} \quad (1 \leq k \leq L), \quad (43)$$

and

$$\begin{aligned}
\frac{h_{k,k+1}(q_{k,k}w)}{h_{k,k+1}(q_{k+1,k}w)} &= \frac{q_{k+1,k+1}}{p_{k+1,k+1}} \left( \frac{q_{k,k}}{q_{k+1,k}} \right)^{A_{k,k+1}(0)} \frac{1-p_{k+1,k+1}w}{1-q_{k+1,k+1}w}, \\
\frac{h_{k,k+1}\left(q_{k+1,k+1}^{-1}w\right)}{h_{k,k+1}\left(q_{k+2,k+1}^{-1}w\right)} &= \frac{1-q_{k+1,k}^{-1}w}{1-p_{k+1,k}^{-1}w}, \\
\frac{h_{k+1,k}(q_{k+2,k+1}w)}{h_{k+1,k}(q_{k+1,k+1}w)} &= \frac{q_{k+1,k}}{p_{k+1,k}} \left( \frac{q_{k+2,k+1}}{q_{k+1,k+1}} \right)^{A_{k,k+1}(0)} \frac{1-p_{k+1,k}w}{1-q_{k+1,k}w}, \\
\frac{h_{k+1,k}\left(q_{k,k}^{-1}w\right)}{h_{k+1,k}\left(q_{k+1,k}^{-1}w\right)} &= \frac{1-p_{k+1,k+1}^{-1}w}{1-q_{k+1,k+1}^{-1}w}
\end{aligned} \tag{44}$$

(1 ≤ k ≤ L - 1).

*Proof of Lemma 3.7.* Multiplying (37) by the screening currents on the left or right and considering the normal orderings, we obtain (43) and (44) as necessary conditions.  $\square$

**Lemma 3.8** *The relations (45) and (46) hold if (5), (7), (9), and (37) are satisfied, where*

$$\begin{aligned}
q_{k+1,k+1} &= q_{k,k}x^{(1+A_{k,k+1}(0))r}, \\
q_{k+1,k} &= q_{k,k}x^{2r}, \\
p_{k+1,k+1} &= q_{k,k}x^{(1-A_{k,k+1}(0))r}, \\
p_{k+1,k} &= q_{k,k}x^{2(1+A_{k,k+1}(0))r}
\end{aligned} \tag{45}$$

(1 ≤ k ≤ L - 1).

$$\begin{aligned}
p_{k,k} &= q_{k,k}x^{A_{k,k}(0)r}, \\
p_{k+1,k} &= q_{k,k}x^{(2-A_{k,k}(0))r} \quad \text{if } k \notin \widehat{J}, \\
p_{k+1,k} &= p_{k,k} \quad \text{if } k \in \widehat{J}
\end{aligned} \tag{46}$$

(1 ≤ k ≤ L).

**Lemma 3.9** *The relation (47) holds, if (5), (7), (9), and (37) are satisfied.*

$$\begin{aligned}
s_k(m)s_k(-m) &= \begin{cases} -1 & \text{if } k \in \widehat{J}, \\ -\frac{[\frac{1}{2}A_{k,k}(0)rm]_x[(2-A_{k,k}(0))rm]_x}{[\frac{1}{2}(2-A_{k,k}(0))rm]_x[rm]_x} & \text{if } k \notin \widehat{J} \end{cases} \quad (m > 0, 1 \leq k \leq L), \\
s_k(m)A_{k,k+1}(m)s_{k+1}(-m) &= -\frac{[A_{k,k+1}(0)rm]_x}{[rm]_x}, \\
s_{k+1}(m)A_{k+1,k}(m)s_k(-m) &= -\frac{[A_{k+1,k}(0)rm]_x}{[rm]_x} \quad (m > 0, 1 \leq k \leq L-1), \\
A_{k,l}(m) &= 0 \quad (m > 0, |k-l| \geq 2, 1 \leq k, l \leq L).
\end{aligned} \tag{47}$$

*Proof of Lemmas 3.8 and 3.9.* From Lemma 3.7, we obtain the  $q$ -difference equations (43) and (44). From (29) and (42), the constant term of  $h_{k,l}(w)$  is 1. Comparing the Taylor expansions for both sides of (43) and (44), we obtain

$$\frac{p_{k+1,k}}{p_{k,k}} \left( \frac{q_{k+1,k}}{q_{k,k}} \right)^{A_{k,k}(0)-1} = 1 \quad (1 \leq k \leq L), \tag{48}$$

$$\frac{q_{k+1,k+1}}{p_{k+1,k+1}} \left( \frac{q_{k,k}}{q_{k+1,k}} \right)^{A_{k,k+1}(0)} = 1, \quad \frac{q_{k+1,k}}{p_{k+1,k}} \left( \frac{q_{k+2,k+1}}{q_{k+1,k+1}} \right)^{A_{k,k+1}(0)} = 1 \quad (1 \leq k \leq L-1). \tag{49}$$

First, we study the  $q$ -difference equations in (44). Upon the specialization (49), we obtain solutions

of (44) as

$$\begin{aligned} h_{k,k+1}(w) &= \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\left( \frac{p_{k+1,k+1}}{q_{k,k}} \right)^m - \left( \frac{q_{k+1,k+1}}{q_{k,k}} \right)^m}{1 - \left( \frac{q_{k+1,k}}{q_{k,k}} \right)^m} w^m \right) \\ &= \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\left( \frac{q_{k+2,k+1}}{p_{k+1,k}} \right)^m - \left( \frac{q_{k+2,k+1}}{q_{k+1,k}} \right)^m}{1 - \left( \frac{q_{k+2,k+1}}{q_{k+1,k+1}} \right)^m} w^m \right), \end{aligned} \quad (50)$$

$$\begin{aligned} h_{k+1,k}(w) &= \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\left( \frac{q_{k+1,k}}{q_{k+1,k+1}} \right)^m - \left( \frac{p_{k+1,k}}{q_{k+1,k+1}} \right)^m}{1 - \left( \frac{q_{k+2,k+1}}{q_{k+1,k+1}} \right)^m} w^m \right) \\ &= \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\left( \frac{q_{k+1,k}}{q_{k+1,k+1}} \right)^m - \left( \frac{q_{k+1,k}}{p_{k+1,k}} \right)^m}{1 - \left( \frac{q_{k+1,k}}{q_{k,k}} \right)^m} w^m \right). \end{aligned} \quad (51)$$

Here we used  $|q_{k+1,k}/q_{k,k}| \neq 1$  ( $1 \leq k \leq L$ ) assumed in (9). From the compatibility of the two formulae for  $h_{k,k+1}(w)$  in (50) ( or  $h_{k+1,k}(w)$  in (51) ), there are two possible restrictions for  $q_{k,k}$ ,  $q_{k+1,k+1}$ ,  $q_{k+1,k}$ , and  $q_{k+2,k+1}$ ,

$$(i) \quad \frac{q_{k+1,k}}{q_{k,k}} = \frac{q_{k+2,k+1}}{q_{k+1,k+1}} \quad \text{or} \quad (ii) \quad \frac{q_{k+1,k}}{q_{k,k}} = \frac{q_{k+1,k+1}}{q_{k+2,k+1}}. \quad (52)$$

First, we consider case (ii)  $q_{k+1,k}/q_{k,k} = q_{k+1,k+1}/q_{k+2,k+1}$  in (52). From the compatibility of the two formulae for  $h_{k,k+1}(w)$  in (50) ( and  $h_{k+1,k}(w)$  in (51) ), we obtain

$$\left( \frac{p_{k+1,k+1}}{q_{k,k}} \right)^m + \left( \frac{q_{k+1,k+1}}{p_{k+1,k}} \right)^m = \left( \frac{q_{k+1,k+1}}{q_{k+1,k}} \right)^m + \left( \frac{q_{k+1,k+1}}{q_{k,k}} \right)^m \quad (m \neq 0). \quad (53)$$

From (53) for  $m = 1, 2$ , we obtain  $p_{k+1,k+1}/p_{k+1,k} = q_{k+1,k+1}/q_{k+1,k}$ . Combining (53) for  $m = 1$  and  $p_{k+1,k+1}/p_{k+1,k} = q_{k+1,k+1}/q_{k+1,k}$ , we obtain  $q_{k,k} = p_{k+1,k}$  or  $q_{k+1,k+1} = p_{k+1,k+1}$ . For the case of  $q_{k,k} = p_{k+1,k}$ , we obtain  $A_{k,k+1}(0) = 1$  from (49). For the case of  $q_{k+1,k+1} = p_{k+1,k+1}$ , we obtain  $A_{k,k+1}(0) = 0$  from (49).  $A_{k,k+1}(0) = 0$  and  $A_{k,k+1}(0) = 1$  contradict with  $-1 < A_{k,k+1}(0) < 0$  assumed in (9). Hence, case (ii)  $q_{k+1,k}/q_{k,k} = q_{k+1,k+1}/q_{k+2,k+1}$  is impossible.

Next, we consider case (i)  $q_{k+1,k}/q_{k,k} = q_{k+2,k+1}/q_{k+1,k+1}$  in (52). From exclusion of case (ii) and the parametrization (15), we can parametrize

$$\frac{q_{2,1}}{q_{1,1}} = \frac{q_{3,2}}{q_{2,2}} = \dots = \frac{q_{L+1,L}}{q_{L,L}} = x^{2r}. \quad (54)$$

From the compatibility of the two formulae for  $h_{k,k+1}(w)$  in (50) ( and  $h_{k+1,k}(w)$  in (51) ), we obtain

$$h_{k,k+1}(w) = \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{[A_{k,k+1}(0)rm]_x}{[rm]_x} x^{-(A_{k,k+1}(0)+1)rm} \left( \frac{q_{k+1,k+1}}{q_{k,k}} \right)^m w^m \right), \quad (55)$$

$$h_{k+1,k}(w) = \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{[A_{k,k+1}(0)rm]_x}{[rm]_x} x^{(A_{k,k+1}(0)+1)rm} \left( \frac{q_{k,k}}{q_{k+1,k+1}} \right)^m w^m \right). \quad (56)$$

We used (49) and (54). From  $h_{k,k+1}(w) = h_{k+1,k}(w)$  assumed in (7), we obtain

$$\frac{q_{k+1,k+1}}{q_{k,k}} = x^{(A_{k,k+1}(0)+1)r}. \quad (57)$$



Considering (49), (54), and (57), we obtain (45). From (55), (56), and (57), we obtain

$$h_{k,k+1}(w) = h_{k+1,k}(w) = \exp\left(-\sum_{m=1}^{\infty} \frac{1}{m} \frac{[A_{k,k+1}(0)rm]_x}{[rm]_x} w^m\right). \quad (58)$$

Considering (29), (42), and (58), we obtain the second half of (47).

Next, we study the  $q$ -difference equations in (43). Upon the specialization (48), the compatibility condition of the equations in (43) is

$$(p_{k,k} - p_{k+1,k})(p_{k,k}p_{k+1,k} - q_{k,k}q_{k+1,k}) = 0 \quad (1 \leq k \leq L). \quad (59)$$

First, we study the case of  $A_{k,k}(0) = 1$ . We obtain  $p_{k,k} = p_{k+1,k}$  in the second half of (46) from (48). Solving (43) upon  $p_{k,k} = p_{k+1,k}$ , we obtain  $h_{k,k}(w) = 1 - w$ . Considering (29) and (42), we obtain  $s_k(m)s_k(-m) = -1$  ( $m > 0$ ) in the first half of (47).

Next, we study the case of  $A_{k,k}(0) \neq 1$ . We obtain  $p_{k,k} \neq p_{k+1,k}$  from (9) and (48). Then, we obtain  $p_{k,k}p_{k+1,k} = q_{k,k}q_{k+1,k}$  from (59). Combining  $p_{k,k}p_{k+1,k} = q_{k,k}q_{k+1,k}$  and (48), we obtain (46). Solving (43), we obtain

$$h_{k,k}(w) = \exp\left(-\sum_{m=1}^{\infty} \frac{1}{m} \frac{[\frac{1}{2}A_{k,k}(0)rm]_x [(2 - A_{k,k}(0))rm]_x}{[\frac{1}{2}(2 - A_{k,k}(0))rm]_x [rm]_x} w^m\right).$$

We used  $|q_{k+1,k}/q_{k,k}| \neq 1$  ( $1 \leq k \leq L$ ) in (9) and  $q_{k+1,k}/q_{k,k} = x^{2r}$  ( $1 \leq k \leq L$ ) in (54). Considering (29) and (42), we obtain the first half of (47).  $\square$

**Lemma 3.10** *The relation (14) holds for  $(A_{i,j}(0))_{i,j=0}^L$  if (5), (6), (7), (8), and (9) are satisfied.*

*Proof of Lemma 3.10.* We obtain  $A_{k,l}(0)$  ( $|k-l| \geq 2, 1 \leq k, l \leq L$ ) from (7). From Lemma 3.6, we have (37). From Lemma 3.8, we have (45) and (46). From the compatibility of (45) and (46), we obtain the following relations for  $(A_{i,j}(0))_{i,j=1}^L$ .

$$\begin{aligned} A_{k+1,k}(0) &= \begin{cases} A_{k-1,k}(0) & \text{if } k \notin \widehat{J}, \\ -1 - A_{k-1,k}(0) & \text{if } k \in \widehat{J} \end{cases} \quad (2 \leq k \leq L-1), \\ A_{k,k}(0) &= \begin{cases} -2A_{k-1,k}(0) & \text{if } k \notin \widehat{J}, \\ 1 & \text{if } k \in \widehat{J} \end{cases} \quad (2 \leq k \leq L), \quad A_{1,1}(0) = \begin{cases} -2A_{1,2}(0) & \text{if } 1 \notin \widehat{J}, \\ 1 & \text{if } 1 \in \widehat{J}, \end{cases} \\ A_{k,l}(0) &= 0 \quad (|k-l| \geq 2, 1 \leq k, l \leq L). \end{aligned} \quad (60)$$

Solving these equations, we obtain (14) for  $1 \leq i \leq L, 2 \leq j \leq L$ , and  $1 \leq k, l \leq L$ . The extension to  $(A_{i,j}(0))_{i,j=0}^L$  is direct consequence of the definition (13).  $\square$

**Proposition 3.11** *The relations (5), (6), and (7) hold if the parameters  $p_{i,j}, q_{i,j}, A_{i,j}(m), s_i(m), g_i$ , and  $\lambda_{i,j}(m)$  are determined by (19), (31), (32), (36), (38), (45), (46), and (47).*

*Proof of Proposition 3.11.* First, we will show the formulae (17), (18), (20), (21), (22), and (23) in

Theorem 3.1. Let  $q_{1,1} = s$ . From (19), (36), (45), and (46), we have

$$\begin{aligned}
q_{j,j} &= sx^{D(1,j-1;\Phi)}, \quad q_{j+1,j} = sx^{2r+D(1,j-1;\Phi)} \quad (1 \leq j \leq L), \\
p_{1,1} &= s \times \begin{cases} x^2 & \text{if } 1 \in \widehat{I}(-\frac{1}{r}), \\ x^{2r-2} & \text{if } 1 \in \widehat{I}(\frac{1-r}{r}) \end{cases}, \quad p_{j,j} = sx^{D(1,j-2;\Phi)} \times \begin{cases} x^{r+1} & \text{if } j \in \widehat{I}(-\frac{1}{r}), \\ x^{2r-1} & \text{if } j \in \widehat{I}(\frac{1-r}{r}) \end{cases} \quad (2 \leq j \leq L), \\
p_{j,j-1} &= sx^{D(1,j-2;\Phi)} \times \begin{cases} x^{2r-2} & \text{if } j \in \widehat{I}(-\frac{1}{r}), \\ x^2 & \text{if } j \in \widehat{I}(\frac{1-r}{r}) \end{cases} \quad (2 \leq j \leq L+1), \\
p_{k,l} &= q_{k,l} \quad (k \neq l, l+1, 1 \leq k \leq L+1, l \leq l \leq L).
\end{aligned}$$

Upon the specialization  $s = 1$ , we have (17). From (17), (19), and (38), we have (23). From (47) we have

$$\begin{aligned}
s_k(-m) &= -\frac{1}{\alpha_k(m)} \times \begin{cases} 1 & \text{if } k \in \widehat{J}, \\ \frac{[\frac{1}{2}A_{k,k}(0)rm]_x[(2-A_{k,k}(0))rm]_x}{[\frac{1}{2}(2-A_{k,k}(0))rm]_x[rm]_x} & \text{if } k \notin \widehat{J} \end{cases} \quad (m > 0), \\
A_{k\pm 1,k}(m) &= \frac{\alpha_k(m)}{\alpha_{k\pm 1}(m)} \frac{[A_{k\pm 1,k}(0)rm]_x}{[rm]_x} \times \begin{cases} 1 & \text{if } k \in \widehat{J}, \\ \frac{[rm]_x[\frac{1}{2}(2-A_{k,k}(0))rm]_x}{[\frac{1}{2}A_{k,k}(0)rm]_x[(2-A_{k,k}(0))rm]_x} & \text{if } k \notin \widehat{J} \end{cases} \quad (m > 0), \\
A_{k,k\pm 1}(-m) &= A_{k\pm 1,k}(m) \quad (m > 0).
\end{aligned}$$

Here the signs of the formulae are in the same order. Here we set  $s_k(m) = \alpha_k(m)$  ( $m > 0, 1 \leq k \leq L$ ). Setting  $\alpha_k(m) = 1$  ( $m > 0, 1 \leq k \leq L$ ) provides (18) and (20). Solving (31) and (32), we obtain  $\lambda_{i,j}(m)$  in (21) and (22). Solving (31) and (32) for arbitrary  $\alpha_k(m)$ , we obtain  $\lambda_{i,j}(m)\alpha_j(m)$ . Now we obtained the formulae (17), (18), (20), (21), (22), and (23). As a by-product of calculation, we proved that there is no indeterminacy in the free field realization except for (10), (11), and (12), which is part of Theorem 3.1.

Next, we will derive (5), (6), and (7). From (18) and (20) we obtain the symmetry (7) by direct calculation. Because  $\lambda_{i,j}(m)$  are determined by (31) and (32), the mutual locality (5) holds from Lemma 3.5. From (18) and (22), we have (37) by direct calculation. From (5), (36), (37), and (38), we obtain  $[T_1(z), S_j(w)] = \left(\frac{q_{j,j}}{p_{j,j}} - 1\right) : \Lambda_j(z)S_j(q_{j,j}^{-1}z) : \left(\delta\left(\frac{q_{j,j}w}{z}\right) - \delta\left(\frac{q_{j+1,j}w}{z}\right)\right)$  ( $1 \leq j \leq L$ ). Hence, we have commutativity (6) upon the condition (24). We derived (5), (6), and (7).  $\square$

*Proof of Theorem 3.1.* We assume the relations (5), (6), (7), (8), and (9). From Lemmas 3.5, 3.6, 3.8, 3.9, and 3.10, we obtain the relations (19), (31), (32), (36), (38), (45), (46), and (47). In the proof of Proposition 3.11, we have obtained  $p_{i,j}$ ,  $q_{i,j}$ ,  $s_j(m)$ ,  $A_{i,j}(m)$ ,  $g_i$ ,  $\lambda_{i,j}(m)$ , and  $B_j(z)$  in (17), (18), (20), (21), (22), (23), and (24) from the relations (19), (31), (32), (36), (38), (45), (46), and (47). Moreover, in the proof of Proposition 3.11, we have proved that there is no indeterminacy in the free field realization except for (10), (11), and (12).

Conversely, in the proof of Proposition 3.11, we have proved that the relations (5), (6), and (7) hold, if the relations (17), (18), (19), (20), (21), (22), (23), and (24) are satisfied.  $\square$

*Proof of Proposition 3.3.* Using  $h_{k,l}(w)$  in (42) we obtain

$$S_k(w_1)S_l(w_2) = \left(\frac{w_1}{w_2}\right)^{A_{k,l}(0)} \frac{h_{k,l}\left(\frac{w_2}{w_1}\right)}{h_{l,k}\left(\frac{w_1}{w_2}\right)} S_l(w_2)S_k(w_1) \quad (1 \leq k, l \leq L).$$

Using (18), (19) and (20) we obtain (26).  $\square$

By direct calculation, we have the following lemma.

**Lemma 3.12** *The determinants of  $(A_{i,j}(m))_{i,j=1}^L$  in (19) and (20) are given by*

$$\det \left( (A_{i,j}(m))_{i,j=1}^L \right) = (-1)^L \frac{[D(0, L; \Phi)m]_x [(r-1)m]_x^{|\widehat{T}(\frac{1-r}{r})|-1} [m]_x^{|\widehat{T}(-\frac{1}{r})|-1}}{[rm]_x^L \prod_{j=1}^L s_j(m)s_j(-m)} \quad (m \neq 0),$$

$$\det \left( (A_{i,j}(0))_{i,j=1}^L \right) = r^{-L} (r-1)^{|\widehat{T}(\frac{1-r}{r})|-1} D(0, L; \Phi).$$

Hence, the condition  $\det \left( (A_{i,j}(m))_{i,j=1}^L \right) \neq 0$  ( $m \in \mathbf{Z}$ ) is satisfied.

## 4 Quadratic relations

In this section we introduce the higher  $W$ -currents  $T_i(z)$  and obtain a set of quadratic relations of  $T_i(z)$  for the deformed  $W$ -superalgebra  $\mathcal{W}_{q,t}(A(M, N))$ . We show that these relations are independent of the choice of Dynkin diagrams.

### 4.1 Quadratic relations

We define the functions  $\Delta_i(z)$  ( $i = 0, 1, 2, \dots$ ) as

$$\Delta_i(z) = \frac{(1 - x^{2r-i}z)(1 - x^{-2r+i}z)}{(1 - x^i z)(1 - x^{-i}z)}.$$

We have

$$\Delta_i(z) - \Delta_i(z^{-1}) = \frac{[r]_x [r-i]_x}{[i]_x} (x - x^{-1}) (\delta(x^{-i}z) - \delta(x^i z)) \quad (i = 1, 2, 3, \dots).$$

We define the structure functions  $f_{i,j}(z; a)$  ( $i, j = 0, 1, 2, \dots$ ) as

$$f_{i,j}(z; a) = \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{[(r-1)m]_x [rm]_x [\text{Min}(i, j)m]_x [(a - \text{Max}(i, j))m]_x}{[m]_x [am]_x} (x - x^{-1})^2 z^m \right). \quad (61)$$

In the case of  $a = D(0, L; \Phi)$ , the ratio of the structure function

$$\frac{f_{1,1}(z^{-1}; a)}{f_{1,1}(z; a)} = \frac{\Theta_{x^{2a}}(x^2 z, x^{-2r} z, x^{2r-2} z)}{\Theta_{x^{2a}}(x^{-2} z, x^{2r} z, x^{-2r+2} z)}$$

coincides with those of (25).

We introduce the higher  $W$ -currents  $T_i(z)$  and give the quadratic relations. From now on, we set  $g = 1$  in (23), but this is not an essential limitation. Hereafter, we use the abbreviations

$$c(r, x) = [r]_x[r-1]_x(x-x^{-1}), \quad d_j(r, x) = \begin{cases} \prod_{l=1}^j \frac{[r-l]_x}{[l]_x} & (j \geq 1), \\ 1 & (j = 0). \end{cases}$$

We introduce the  $W$ -currents  $T_i(z)$  ( $i = 0, 1, 2, \dots$ ) as

$$\begin{aligned} T_0(z) &= 1, \quad T_1(z) = \sum_{k \in \hat{I}(\frac{1-r}{r})} \Lambda_k(z) + d_1(r, x) \sum_{k \in \hat{I}(-\frac{1}{r})} \Lambda_k(z), \\ T_i(z) &= \sum_{\substack{(m_1, m_2, \dots, m_{L+1}) \in \hat{N}(\Phi) \\ m_1 + m_2 + \dots + m_{L+1} = i}} \prod_{k \in \hat{I}(-\frac{1}{r})} d_{m_k}(r, x) \Lambda_{m_1, m_2, \dots, m_{L+1}}^{(i)}(z) \quad (i = 2, 3, 4, \dots). \end{aligned} \quad (62)$$

Here we set

$$\begin{aligned} \Lambda_{m_1, m_2, \dots, m_{L+1}}^{(i)}(z) &=: \prod_{k \in \hat{I}(\frac{1-r}{r})} \Lambda_k^{(m_k)}(x^{-i+1+2(m_1+\dots+m_{k-1})}z) \\ &\quad \times \prod_{k \in \hat{I}(-\frac{1}{r})} \Lambda_k^{(m_k)}(x^{-i+1+2(m_1+\dots+m_{k-1})}z) : \text{ for } (m_1, m_2, \dots, m_{L+1}) \in \hat{N}(\Phi), \end{aligned}$$

where

$$\begin{aligned} \Lambda_k^{(0)}(z) &= 1, \quad \Lambda_k^{(m)}(z) =: \Lambda_k(z) \Lambda_k(x^2 z) \cdots \Lambda_k(x^{2m-2} z) :, \\ \hat{N}(\Phi) &= \left\{ (m_1, m_2, \dots, m_{L+1}) \in \mathbf{N}^{L+1} \mid 0 \leq m_k \leq 1 \text{ if } k \in \hat{I}\left(\frac{1-r}{r}\right), m_k \geq 0 \text{ if } k \in \hat{I}\left(-\frac{1}{r}\right) \right\}. \end{aligned} \quad (63)$$

We have  $T_i(z) \neq 1$  ( $i = 1, 2, 3, \dots$ ) and  $T_i(z) \neq T_j(z)$  ( $i \neq j$ ).

The following is **the main theorem** of this paper.

**Theorem 4.1** *For the deformed  $W$ -superalgebra  $\mathcal{W}_{q,t}(A(M, N))$  the  $W$ -currents  $T_i(z)$  satisfy the set of quadratic relations*

$$\begin{aligned} & f_{i,j} \left( \frac{z_2}{z_1}; a \right) T_i(z_1) T_j(z_2) - f_{j,i} \left( \frac{z_1}{z_2}; a \right) T_j(z_2) T_i(z_1) \\ &= c(r, x) \sum_{k=1}^i \prod_{l=1}^{k-1} \Delta_1(x^{2l+1}) \left( \delta \left( \frac{x^{-j+i-2k} z_2}{z_1} \right) f_{i-k, j+k}(x^{j-i}; a) T_{i-k}(x^k z_1) T_{j+k}(x^{-k} z_2) \right. \\ &\quad \left. - \delta \left( \frac{x^{j-i+2k} z_2}{z_1} \right) f_{i-k, j+k}(x^{-j+i}; a) T_{i-k}(x^{-k} z_1) T_{j+k}(x^k z_2) \right) \quad (j \geq i \geq 1). \end{aligned} \quad (64)$$

Here we use  $f_{i,j}(z; a)$  in (61) with the specialization  $a = D(0, L; \Phi)$ . The quadratic relations (64) are independent of the choice of Dynkin diagrams for the Lie superalgebra  $A(M, N)$ .

In view of Theorem 4.1, we arrive at the following definition.

**Definition 4.2** *Set  $\bar{T}_i(z) = \sum_{m \in \mathbf{Z}} \bar{T}_i[m] z^{-m}$  ( $i = 1, 2, 3, \dots$ ). The deformed  $W$ -superalgebra  $\mathcal{W}_{q,t}(A(M, N))$  is the associative algebra over  $\mathbf{C}$  with the generators  $\bar{T}_i[m]$  ( $m \in \mathbf{Z}, i = 1, 2, 3, \dots$ ) and defining relations (64).*

As a merit of this definition it becomes clear that  $\mathcal{W}_{q,t}(A(M, N))$  is independent of the choice of Dynkin diagrams. In Section 5 we discuss about justification of this definition. We compare Definition 4.2 with the other definitions of the deformed  $W$ -superalgebra of type  $A(M, N)$ , that are based on the Miura transformation or the screening operators.

## 4.2 Proof of Theorem 4.1

**Proposition 4.3** *The  $\Lambda_i(z)$ 's satisfy*

$$\begin{aligned}
f_{1,1} \left( \frac{z_2}{z_1}; a \right) \Lambda_k(z_1) \Lambda_l(z_2) &= \Delta_1 \left( \frac{x^{-1} z_2}{z_1} \right) : \Lambda_k(z_1) \Lambda_l(z_2) :, \\
f_{1,1} \left( \frac{z_2}{z_1}; a \right) \Lambda_l(z_1) \Lambda_k(z_2) &= \Delta_1 \left( \frac{x z_2}{z_1} \right) : \Lambda_l(z_1) \Lambda_k(z_2) : \quad (1 \leq k < l \leq L+1), \\
f_{1,1} \left( \frac{z_2}{z_1}; a \right) \Lambda_i(z_1) \Lambda_i(z_2) &=: \Lambda_i(z_1) \Lambda_i(z_2) : \quad \text{if } i \in \hat{I} \left( \frac{1-r}{r} \right), \\
f_{1,1} \left( \frac{z_2}{z_1}; a \right) \Lambda_i(z_1) \Lambda_i(z_2) &= \Delta_2 \left( \frac{z_2}{z_1} \right) : \Lambda_i(z_1) \Lambda_i(z_2) : \quad \text{if } i \in \hat{I} \left( -\frac{1}{r} \right), \tag{65}
\end{aligned}$$

where we set  $a = D(0, L; \Phi)$ .

*Proof of Proposition 4.3.* Substituting (17) and (22) into  $\varphi_{\Lambda_k, \Lambda_l}(z_1, z_2)$  in (30), we obtain (65).  $\square$

*Proof of Proposition 3.2.* Using  $\varphi_{\Lambda_k, \Lambda_l}(z_1, z_2)$  in (30), we obtain

$$\Lambda_k(z_1) \Lambda_l(z_2) = \frac{\varphi_{\Lambda_k, \Lambda_l}(z_1, z_2)}{\varphi_{\Lambda_l, \Lambda_k}(z_2, z_1)} \Lambda_l(z_2) \Lambda_k(z_1) \quad (1 \leq k, l \leq L+1).$$

Using the explicit formulae of  $\varphi_{\Lambda_k, \Lambda_l}(z_1, z_2)$ , we obtain (25).  $\square$

**Lemma 4.4** *The  $D(0, L; \Phi)$  given in (16) is independent of the choice of the Dynkin diagrams for the Lie superalgebra  $A(M, N)$ .*

$$D(0, L; \Phi) = D(0, L; r_{\alpha_i}(\Phi)) \quad (\alpha_i \in \Pi). \tag{66}$$

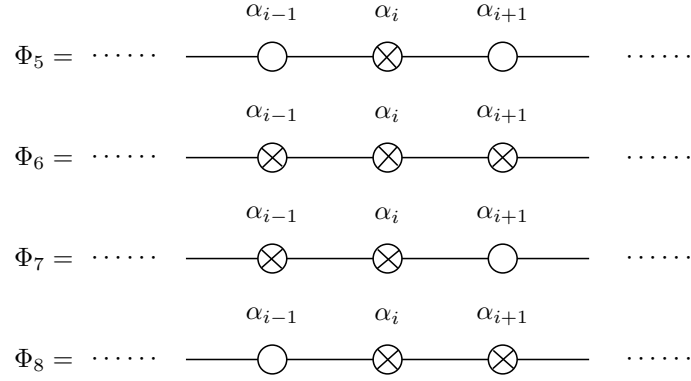
Here  $\Pi$  is a fundamental system.

*Proof of Lemma 4.4.* We show (66) by checking all cases. We set the Dynkin diagrams  $\Phi_j$  ( $1 \leq j \leq 8$ ) as follows. Let  $K = K(\Phi_j)$  the number of odd isotropic roots  $(\alpha_i, \alpha_i) = 0$  in the Dynkin diagram  $\Phi_j$ .

We set

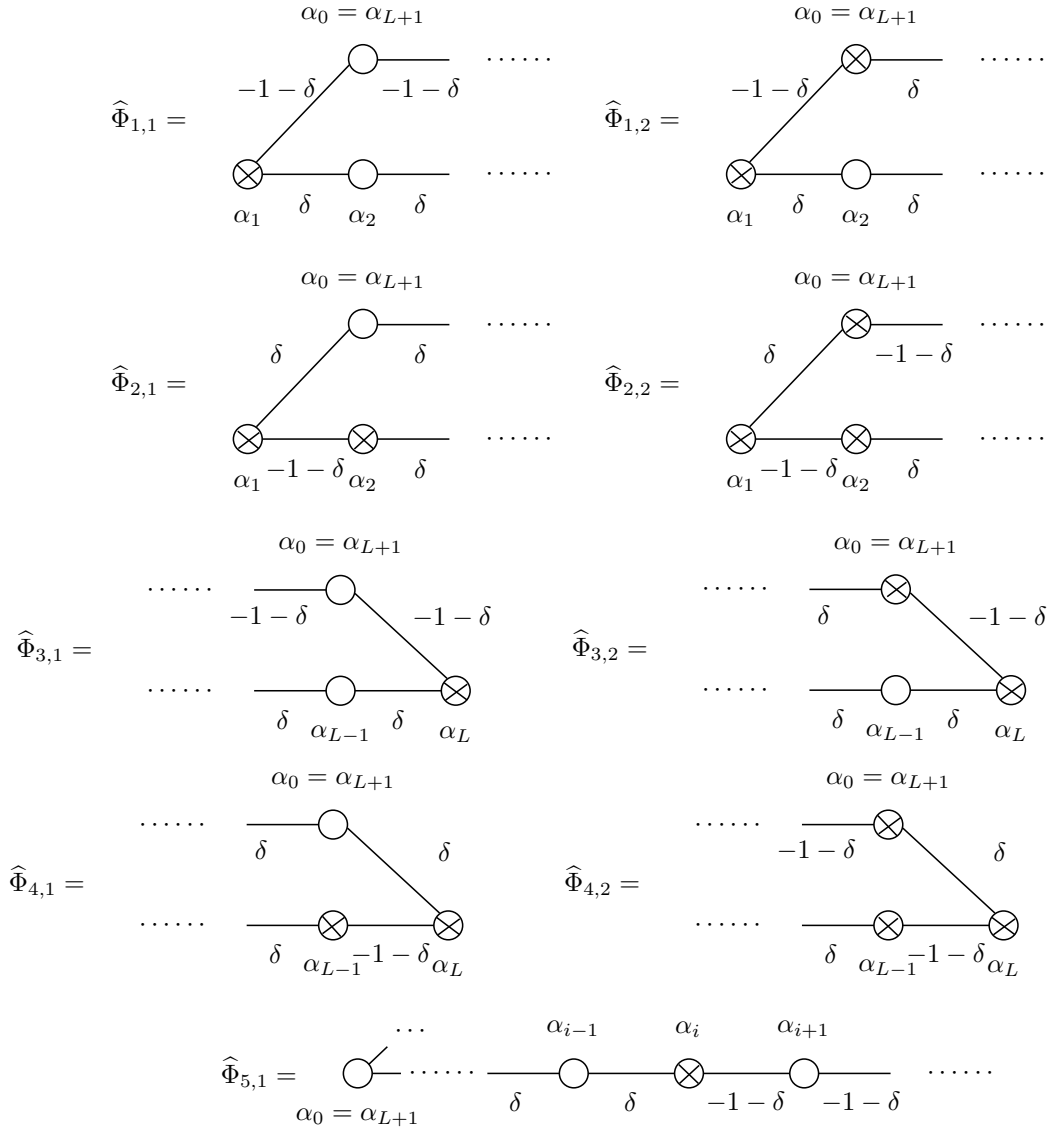
$$\begin{array}{ll}
\Phi_1 = \begin{array}{c} \alpha_1 \quad \alpha_2 \\ \otimes \text{---} \circ \text{---} \dots \end{array} & \Phi_2 = \begin{array}{c} \alpha_1 \quad \alpha_2 \\ \otimes \text{---} \otimes \text{---} \dots \end{array} \\
\Phi_3 = \begin{array}{c} \alpha_{L-1} \quad \alpha_L \\ \dots \text{---} \circ \text{---} \otimes \end{array} & \Phi_4 = \begin{array}{c} \alpha_{L-1} \quad \alpha_L \\ \dots \text{---} \otimes \text{---} \otimes \end{array}
\end{array}$$

For  $2 \leq i \leq L-1$ , we set



We have  $r_{\alpha_1}(\Phi_1) = \Phi_2$ ,  $r_{\alpha_L}(\Phi_3) = \Phi_4$ ,  $r_{\alpha_i}(\Phi_5) = \Phi_6$ , and  $r_{\alpha_i}(\Phi_7) = \Phi_8$ .

The affinized Dynkin diagrams  $\widehat{\Phi}_j$  from  $\Phi_j$  ( $1 \leq j \leq 8$ ) are given as  $\widehat{\Phi}_j = \begin{cases} \widehat{\Phi}_{j,1} & \text{if } K(\Phi_j) = \text{even,} \\ \widehat{\Phi}_{j,2} & \text{if } K(\Phi_j) = \text{odd.} \end{cases}$



$$\begin{aligned}
\widehat{\Phi}_{5,2} &= \begin{array}{c} \dots \\ \alpha_0 = \alpha_{L+1} \end{array} \begin{array}{c} \text{---} \otimes \text{---} \dots \dots \text{---} \circ \text{---} \text{---} \otimes \text{---} \text{---} \circ \text{---} \dots \dots \end{array} \\
&\quad \delta \qquad \delta \qquad -1-\delta \qquad -1-\delta \\
&\quad \alpha_{i-1} \qquad \alpha_i \qquad \alpha_{i+1} \\
\widehat{\Phi}_{6,1} &= \begin{array}{c} \dots \\ \alpha_0 = \alpha_{L+1} \end{array} \begin{array}{c} \text{---} \circ \text{---} \dots \dots \text{---} \otimes \text{---} \text{---} \otimes \text{---} \text{---} \otimes \text{---} \dots \dots \end{array} \\
&\quad \delta \qquad -1-\delta \qquad \delta \qquad -1-\delta \\
&\quad \alpha_{i-1} \qquad \alpha_i \qquad \alpha_{i+1} \\
\widehat{\Phi}_{6,2} &= \begin{array}{c} \dots \\ \alpha_0 = \alpha_{L+1} \end{array} \begin{array}{c} \text{---} \otimes \text{---} \dots \dots \text{---} \otimes \text{---} \text{---} \otimes \text{---} \text{---} \otimes \text{---} \dots \dots \end{array} \\
&\quad \delta \qquad -1-\delta \qquad \delta \qquad -1-\delta \\
&\quad \alpha_{i-1} \qquad \alpha_i \qquad \alpha_{i+1} \\
\widehat{\Phi}_{7,1} &= \begin{array}{c} \dots \\ \alpha_0 = \alpha_{L+1} \end{array} \begin{array}{c} \text{---} \circ \text{---} \dots \dots \text{---} \otimes \text{---} \text{---} \otimes \text{---} \text{---} \circ \text{---} \dots \dots \end{array} \\
&\quad \delta \qquad -1-\delta \qquad \delta \qquad \delta \\
&\quad \alpha_{i-1} \qquad \alpha_i \qquad \alpha_{i+1} \\
\widehat{\Phi}_{7,2} &= \begin{array}{c} \dots \\ \alpha_0 = \alpha_{L+1} \end{array} \begin{array}{c} \text{---} \otimes \text{---} \dots \dots \text{---} \otimes \text{---} \text{---} \otimes \text{---} \text{---} \circ \text{---} \dots \dots \end{array} \\
&\quad \delta \qquad -1-\delta \qquad \delta \qquad \delta \\
&\quad \alpha_{i-1} \qquad \alpha_i \qquad \alpha_{i+1} \\
\widehat{\Phi}_{8,1} &= \begin{array}{c} \dots \\ \alpha_0 = \alpha_{L+1} \end{array} \begin{array}{c} \text{---} \circ \text{---} \dots \dots \text{---} \circ \text{---} \text{---} \otimes \text{---} \text{---} \otimes \text{---} \dots \dots \end{array} \\
&\quad \delta \qquad \delta \qquad -1-\delta \qquad \delta \\
&\quad \alpha_{i-1} \qquad \alpha_i \qquad \alpha_{i+1} \\
\widehat{\Phi}_{8,2} &= \begin{array}{c} \dots \\ \alpha_0 = \alpha_{L+1} \end{array} \begin{array}{c} \text{---} \otimes \text{---} \dots \dots \text{---} \circ \text{---} \text{---} \otimes \text{---} \text{---} \otimes \text{---} \dots \dots \end{array} \\
&\quad \delta \qquad \delta \qquad -1-\delta \qquad \delta \\
&\quad \alpha_{i-1} \qquad \alpha_i \qquad \alpha_{i+1}
\end{aligned}$$

The values of  $A_{j-1,j}(0)$  are written beside the line segment connecting  $\alpha_{j-1}$  and  $\alpha_j$ . We have

$$|\widehat{I}(1, L+1; \delta, \widehat{\Phi}_{2j-1,1})| = |\widehat{I}(1, L+1; \delta, \widehat{\Phi}_{2j,2})|, \quad |\widehat{I}(1, L+1; \delta, \widehat{\Phi}_{2j-1,2})| = |\widehat{I}(1, L+1; \delta, \widehat{\Phi}_{2j,1})| \quad (1 \leq j \leq 2),$$

$$|\widehat{I}(1, L+1; \delta, \widehat{\Phi}_{2j-1,1})| = |\widehat{I}(1, L+1; \delta, \widehat{\Phi}_{2j,1})|, \quad |\widehat{I}(1, L+1; \delta, \widehat{\Phi}_{2j-1,2})| = |\widehat{I}(1, L+1; \delta, \widehat{\Phi}_{2j,2})| \quad (3 \leq j \leq 4),$$

where  $\delta = \frac{1-r}{r}$  or  $-\frac{1}{r}$ . Hence we have

$$D(0, L; \Phi_{2j-1}) = D(0, L; \Phi_{2j}) \quad (1 \leq j \leq 4).$$

In other words, we have

$$D(0, L; \Phi_j) = D(0, L; r_{\alpha_1}(\Phi_j)) \quad (1 \leq j \leq 2),$$

$$D(0, L; \Phi_j) = D(0, L; r_{\alpha_L}(\Phi_j)) \quad (3 \leq j \leq 4),$$

$$D(0, L; \Phi_j) = D(0, L; r_{\alpha_i}(\Phi_j)) \quad (5 \leq j \leq 8, 2 \leq i \leq L-1).$$

Now we have proved (66).  $\square$

**Lemma 4.5**  $\Delta_i(z)$  and  $f_{i,j}(z; a)$  satisfy the following fusion relations.

$$f_{i,j}(z; a) = f_{j,i}(z; a) = \prod_{k=1}^i f_{1,j}(z^{-i-1+2k}z; a) \quad (1 \leq i \leq j), \quad (67)$$

$$f_{1,i}(z; a) = \left( \prod_{k=1}^{i-1} \Delta_1(x^{-i+2k}z) \right)^{-1} \prod_{k=1}^i f_{1,1}(x^{-i-1+2k}z; a) \quad (i \geq 2), \quad (68)$$

$$f_{1,i}(z; a)f_{j,i}(x^{\pm(j+1)}z; a) = \begin{cases} f_{j+1,i}(x^{\pm j}z; a)\Delta_1(x^{\pm i}z) & (1 \leq i \leq j), \\ f_{j+1,i}(x^{\pm j}z; a) & (1 \leq j < i), \end{cases} \quad (69)$$

$$f_{1,i}(z; a)f_{1,j}(x^{\pm(i+j)}z; a) = f_{1,i+j}(x^{\pm j}z; a)\Delta_1(x^{\pm i}z) \quad (i, j \geq 1), \quad (70)$$

$$f_{1,i}(z; a)f_{1,j}(x^{\pm(i-j-2k)}z; a) = f_{1,i-k}(x^{\mp k}z; a)f_{1,j+k}(x^{\pm(i-j-k)}z; a) \quad (i, j, i-k, j+k \geq 1), \quad (71)$$

$$\Delta_{i+1}(z) = \left( \prod_{k=1}^{i-1} \Delta_1(x^{-i+2k}z) \right)^{-1} \prod_{k=1}^i \Delta_2(x^{-i-1+2k}z) \quad (i \geq 2). \quad (72)$$

*Proof of Lemma 4.5.* We obtain (67) and (72) by straightforward calculation from the definitions. We show (68) here. From definitions, we have

$$\begin{aligned} & \left( \prod_{k=1}^{i-1} \Delta_1(x^{-i+2k}z) \right)^{-1} \prod_{k=1}^i f_{1,1}(x^{-i-1+2k}z; a) \\ &= \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{[rm]_x [(r-1)m]_x}{[am]_x} (x - x^{-1})^2 \right. \\ & \quad \left. \times \left( [(a-1)m]_x \sum_{k=1}^i x^{(-i+2k-1)m} - [am]_x \sum_{k=1}^{i-1} x^{(-i+2k)m} \right) z^m \right). \end{aligned}$$

Using the relation

$$[(a-1)m]_x \sum_{k=1}^i x^{(-i+2k-1)m} - [am]_x \sum_{k=1}^{i-1} x^{(-i+2k)m} = [(a-i)m]_x,$$

we have  $f_{1,i}(z; a)$ . Using (67) and (68), we obtain the relations (69), (70), and (71).  $\square$

The following relations (73), (74), and (75) give special cases of (64).

**Lemma 4.6** The  $T_i(z)$ 's satisfy the fusion relation

$$\begin{aligned} & \lim_{z_1 \rightarrow x^{\pm(i+j)}z_2} \left( 1 - x^{\pm(i+j)} \frac{z_2}{z_1} \right) f_{i,j} \left( \frac{z_2}{z_1}; a \right) T_i(z_1) T_j(z_2) \\ &= \mp c(r, x) \prod_{k=1}^{\text{Min}(i,j)-1} \Delta_1(x^{2k+1}) T_{i+j}(x^{\pm i}z_2) \quad (i, j \geq 1). \end{aligned} \quad (73)$$

Here we set  $a = D(0, L; \Phi)$ .

*Proof of Lemma 4.6.* Summing up the relations (A 1)–(A 4) in Appendix A gives (73).  $\square$

**Lemma 4.7** The  $T_i(z)$ 's satisfy the exchange relation as meromorphic functions

$$f_{i,j} \left( \frac{z_2}{z_1}; a \right) T_i(z_1) T_j(z_2) = f_{j,i} \left( \frac{z_1}{z_2}; a \right) T_j(z_2) T_i(z_1) \quad (j \geq i \geq 1). \quad (74)$$

Here we set  $a = D(0, L; \Phi)$ .



*Proof of Lemma 4.7.* Using the commutation relation (65) repeatedly, (74) is obtained except for poles in both sides.  $\square$

**Lemma 4.8** *The  $T_i(z)$ 's satisfy the quadratic relations*

$$\begin{aligned} & f_{1,i} \left( \frac{z_2}{z_1}; a \right) T_1(z_1) T_i(z_2) - f_{i,1} \left( \frac{z_1}{z_2}; a \right) T_i(z_2) T_1(z_1) \\ &= c(r, x) \left( \delta \left( \frac{x^{-i-1} z_2}{z_1} \right) T_{i+1}(x^{-1} z_2) - \delta \left( \frac{x^{i+1} z_2}{z_1} \right) T_{i+1}(x z_2) \right) \quad (i \geq 1). \end{aligned} \quad (75)$$

Here we set  $a = D(0, L; \Phi)$ .

*Proof of Lemma 4.8.* Summing up the relations (B 2)–(B 6) in Appendix B gives (75).  $\square$

*Proof of Theorem 4.1.* We prove Theorem 4.1 by induction. Lemma 4.8 is the basis of induction for the proof. In what follows we set  $a = D(0, L; \Phi)$ .

We define  $\text{LHS}_{i,j}$  and  $\text{RHS}_{i,j}(k)$  with  $(1 \leq k \leq i \leq j)$  as

$$\begin{aligned} \text{LHS}_{i,j} &= f_{i,j} \left( \frac{z_2}{z_1}; a \right) T_i(z_1) T_j(z_2) - f_{j,i} \left( \frac{z_1}{z_2}; a \right) T_j(z_2) T_i(z_1), \\ \text{RHS}_{i,j}(k) &= c(r, x) \prod_{l=1}^{k-1} \Delta_1(x^{2l+1}) \left( \delta \left( \frac{x^{-j+i-2k} z_2}{z_1} \right) f_{i-k,j+k}(x^{j-i}; a) T_{i-k}(x^k z_1) T_{j+k}(x^{-k} z_2) \right. \\ &\quad \left. - \delta \left( \frac{x^{j-i+2k} z_2}{z_1} \right) f_{i-k,j+k}(x^{-j+i}; a) T_{i-k}(x^{-k} z_1) T_{j+k}(x^k z_2) \right) \quad (1 \leq k \leq i-1), \\ \text{RHS}_{i,j}(i) &= c(r, x) \prod_{l=1}^{i-1} \Delta_1(x^{2l+1}) \left( \delta \left( \frac{x^{-j-i} z_2}{z_1} \right) T_{j+i}(x^{-i} z_2) - \delta \left( \frac{x^{j+i} z_2}{z_1} \right) T_{j+i}(x^i z_2) \right). \end{aligned}$$

We prove the following relation by induction on  $i$  ( $1 \leq i \leq j$ ).

$$\text{LHS}_{i,j} = \sum_{k=1}^i \text{RHS}_{i,j}(k). \quad (76)$$

The starting point of  $i = 1 \leq j$  was previously proven in Lemma 4.8. We assume that the relation (76) holds for some  $i$  ( $1 \leq i < j$ ), and we show  $\text{LHS}_{i+1,j} = \sum_{k=1}^{i+1} \text{RHS}_{i+1,j}(k)$  from this assumption. Multiplying  $\text{LHS}_{i,j}$  by  $f_{1,i}(z_1/z_3; a) f_{1,j}(z_2/z_3; a) T_1(z_3)$  on the left and using the quadratic relation (76) with  $i = 1$  along with fusion relation (69) gives

$$\begin{aligned} & f_{1,j} \left( \frac{z_2}{z_3}; a \right) f_{i,j} \left( \frac{z_2}{z_1}; a \right) f_{1,i} \left( \frac{z_1}{z_3}; a \right) T_1(z_3) T_i(z_1) T_j(z_2) \\ & - f_{j,1} \left( \frac{z_3}{z_2}; a \right) f_{j,i} \left( \frac{z_1}{z_2}; a \right) T_j(z_2) f_{1,i} \left( \frac{z_1}{z_3}; a \right) T_1(z_3) T_i(z_1) \\ & - c(r, x) \delta \left( \frac{x^{-j-1} z_2}{z_3} \right) \Delta_1 \left( \frac{x^{-i} z_1}{z_3} \right) f_{j+1,i} \left( \frac{x^{-j} z_1}{z_3}; a \right) T_{j+1}(x^j z_3) T_i(z_1) \\ & + c(r, x) \delta \left( \frac{x^{j+1} z_2}{z_3} \right) \Delta_1 \left( \frac{x^i z_1}{z_3} \right) f_{j+1,i} \left( \frac{x^j z_1}{z_3}; a \right) T_{j+1}(x^{-j} z_3) T_i(z_1). \end{aligned} \quad (77)$$

Taking the limit  $z_3 \rightarrow x^{-i-1}z_1$  of (77) multiplied by  $c(r, x)^{-1} (1 - x^{-i-1}z_1/z_3)$  and using fusion relation (73) along with the relation  $\lim_{z_3 \rightarrow x^{-i-1}z_1} (1 - x^{-i-1}z_1/z_3) \Delta_1(x^{-i}z_1/z_3) = c(r, x)$  gives

$$\begin{aligned} & f_{1,j} \left( \frac{x^{i+1}z_2}{z_1}; a \right) f_{i,j} \left( \frac{z_2}{z_1}; a \right) T_{i+1}(x^{-1}z_1)T_j(z_2) - \\ & f_{j,1} \left( \frac{x^{-i-1}z_1}{z_2}; a \right) f_{j,i} \left( \frac{z_1}{z_2}; a \right) T_j(z_2)T_{i+1}(x^{-1}z_1) \\ & - c(r, x)\delta \left( \frac{x^{i-j}z_2}{z_1} \right) f_{j+1,i}(x^{i-j+1}; a)T_{j+1}(x^{j-i-1}z_1)T_i(z_1) \\ & + c(r, x)\delta \left( \frac{x^{i+j+2}z_2}{z_1} \right) \prod_{l=1}^i \Delta_1(x^{2l+1})T_{i+j+1}(x^{i+1}z_2). \end{aligned}$$

Using fusion relation (69) and

$$f_{j+1,i}(x^{i-j+1}; a)T_{j+1}(x^{j-i-1}z_1)T_i(z_1) = f_{i,j+1}(x^{j-i-1}; a)T_i(z_1)T_{j+1}(x^{j-i-1}z_1)$$

in (76) with  $j \rightarrow j+1$  gives

$$\begin{aligned} & f_{i+1,j} \left( \frac{xz_2}{z_1}; a \right) T_{i+1}(x^{-1}z_1)T_j(z_2) - f_{j,i+1} \left( \frac{x^{-1}z_1}{z_2}; a \right) T_j(z_2)T_{i+1}(x^{-1}z_1) \\ & - c(r, x)\delta \left( \frac{x^{i-j}z_2}{z_1} \right) f_{i,j+1}(x^{-i+j+1}; a)T_i(z_1)T_{j+1}(x^{-1}z_2) \\ & + c(r, x)\delta \left( \frac{x^{i+j+2}z_2}{z_1} \right) \prod_{l=1}^i \Delta_1(x^{2l+1})T_{i+j+1}(x^{i+1}z_2). \end{aligned} \quad (78)$$

Multiplying  $\text{RHS}_{i,j}(i)$  by  $f_{1,i}(z_1/z_3; a) f_{1,j}(z_2/z_3; a) T_1(z_3)$  from the left and using fusion relation (70) gives

$$\begin{aligned} & c(r, x) \prod_{l=1}^{i-1} \Delta_1(x^{2l+1}) \left( \delta \left( \frac{x^{-i-j}z_2}{z_1} \right) f_{1,i+1} \left( \frac{x^j z_1}{z_3}; a \right) \Delta_1 \left( \frac{x^i z_1}{z_3} \right) T_1(z_3)T_{i+j}(x^j z_1) \right. \\ & \left. - \delta \left( \frac{x^{i+j}z_2}{z_1} \right) f_{1,i+1} \left( \frac{x^{-j} z_1}{z_3}; a \right) \Delta_1 \left( \frac{x^{-i} z_1}{z_3} \right) T_1(z_3)T_{i+j}(x^{-j} z_1) \right). \end{aligned} \quad (79)$$

Taking the limit  $z_3 \rightarrow x^{-i-1}z_1$  of (79) multiplied by  $c(r, x)^{-1} (1 - x^{-i-1}z_1/z_3)$  and using fusion relation (73) along with the relation  $\lim_{z_3 \rightarrow x^{-i-1}z_1} (1 - x^{-i-1}z_1/z_3) \Delta_1(x^{-i}z_1/z_3) = c(r, x)$  gives

$$\begin{aligned} & c(r, x)\delta \left( \frac{x^{-i-j}z_2}{z_1} \right) \prod_{l=1}^i \Delta_1(x^{2l+1})T_{i+j+1}(x^{-i-1}z_2) \\ & - c(r, x)\delta \left( \frac{x^{i+j}z_2}{z_1} \right) \prod_{l=1}^{i-1} \Delta_1(x^{2l+1})f_{1,i+j}(x^{i-j+1}; a)T_1(x^{-i-1}z_1)T_{i+j}(x^i z_2). \end{aligned} \quad (80)$$

Multiplying  $\text{RHS}_{i,j}(k)$  ( $1 \leq k \leq i-1$ ) by  $f_{1,i}(z_1/z_3; a) f_{1,j}(z_2/z_3; a) T_1(z_3)$  from the left and using fusion relation (71) along with

$$f_{i-k,j+k}(x^{j-i}; a)T_{i-k}(x^k z_1)T_{j+k}(x^{j-i+k} z_1) = f_{j+k,i-k}(x^{i-j}; a)T_{j+k}(x^{j-i+k} z_1)T_{i-k}(x^k z_1)$$

in (76) with  $j \rightarrow j + k$  gives

$$\begin{aligned}
& c(r, x) \prod_{l=1}^{k-1} \Delta_1(x^{2l+1}) \\
& \times \left( \delta \left( \frac{x^{-j+i-2k} z_2}{z_1} \right) f_{1, i-k} \left( \frac{x^k z_1}{z_3}; a \right) f_{j+k, i-k}(x^{i-j}; a) f_{1, j+k} \left( \frac{x^{-i+j+k} z_1}{z_3}; a \right) \right. \\
& \quad \times T_1(z_3) T_{j+k}(x^{j-i+k} z_1) T_{i-k}(x^k z_1) \\
& \quad - \delta \left( \frac{x^{j-i+2k} z_2}{z_1} \right) f_{1, i-k} \left( \frac{x^{-k} z_1}{z_3}; a \right) f_{i-k, j+k}(x^{i-j}; a) f_{1, j+k} \left( \frac{x^{i-j-k} z_1}{z_3}; a \right) \\
& \quad \left. \times T_1(z_3) T_{i-k}(x^{-k} z_1) T_{j+k}(x^k z_2) \right). \tag{81}
\end{aligned}$$

Taking the limit  $z_3 \rightarrow x^{-i-1} z_1$  of (81) multiplied by  $c(r, x)^{-1} (1 - x^{-i-1} z_1/z_3)$  and using fusion relations (69) and (73) along with

$$\begin{aligned}
& f_{i-k+1, j+k}(x^{i-j+1}; a) T_{i-k+1}(x^{-k-1} z_1) T_{j+k}(x^{-j+i-k} z_1) \\
& = f_{j+k, i-k+1}(x^{j-i-1}; a) T_{j+k}(x^{-j+i-k} z_1) T_{i-k+1}(x^{-k-1} z_1)
\end{aligned}$$

in (76) with  $j \rightarrow j + k$  gives

$$\begin{aligned}
& c(r, x) \prod_{l=1}^k \Delta_1(x^{2l+1}) \delta \left( \frac{x^{-j+i-2k} z_2}{z_1} \right) f_{j+k-1, i-k}(x^{i-j+1}; a) T_{i-k}(x^k z_1) T_{j+k+1}(x^{-k-1} z_2) \\
& - c(r, x) \prod_{l=1}^{k-1} \Delta_1(x^{2l+1}) \delta \left( \frac{x^{j-i+2k} z_2}{z_1} \right) f_{i-k+1, j+k}(x^{i-j+1}; a) T_{i-k+1}(x^{-k-1} z_1) T_{j+k}(x^k z_2). \tag{82}
\end{aligned}$$

Summing (78), (80), and (82) for  $1 \leq k \leq i-1$  and shifting the variable  $z_1 \mapsto x z_1$  gives  $\text{LHS}_{i+1, j} = \sum_{k=1}^{i+1} \text{RHS}_{i+1, j}(k)$ . By induction on  $i$ , we have shown quadratic relation (64).

The quadratic relations (64) are independent of the choice of Dynkin diagrams for the Lie superalgebra  $A(M, N)$ , because  $a = D(0, L; \Phi)$  is independent of the choice of Dynkin diagrams. See Lemma 4.4.  $\square$

### 4.3 Classical limit

The deformed  $W$ -algebra  $\mathcal{W}_{q,t}(\mathfrak{g})$  yields a  $q$ -Poisson  $W$ -algebra in the classical limit. As an application of the quadratic relations (64), we obtain a  $q$ -Poisson  $W$ -algebra [6, 14, 15]. We study  $\mathcal{W}_{q,t}(A(M, N))$  ( $M \geq N \geq 0, M + N \geq 1$ ). We set parameters  $q = x^{2r}$  and  $\beta = (r-1)/r$ . We define the  $q$ -Poisson bracket  $\{\cdot, \cdot\}$  by taking the classical limit  $\beta \rightarrow 0$  with  $q$  fixed as

$$\{T_i^{\text{PB}}[m], T_j^{\text{PB}}[n]\} = - \lim_{\beta \rightarrow 0} \frac{1}{\beta \log q} [T_i[m], T_j[n]].$$

Here, we set  $T_i^{\text{PB}}[m]$  as  $T_i(z) = \sum_{m \in \mathbf{Z}} T_i[m] z^{-m} \rightarrow T_i^{\text{PB}}(z) = \sum_{m \in \mathbf{Z}} T_i^{\text{PB}}[m] z^{-m}$  ( $\beta \rightarrow 0, q$  fixed). The  $\beta$ -expansions of the structure functions are given as

$$\begin{aligned}
f_{i,j}(z; a) &= 1 + \beta \log q \sum_{m=1}^{\infty} \frac{[\frac{1}{2} \text{Min}(i, j) m]_q [(\frac{1}{2} (\text{Max}(i, j) - M - 1) m)]_q}{[\frac{1}{2} (M + 1) m]_q} (q - q^{-1}) + O(\beta^2) \quad (i, j \geq 1), \\
c(r, x) &= -\beta \log q + O(\beta^2),
\end{aligned}$$

where  $a = D(0, M + N + 1; \Phi) = (N + 1)r + M - N$ .

**Proposition 4.9** For the  $q$ -Poisson  $W$ -superalgebra for  $A(M, N)$  ( $M \geq N \geq 0, M + N \geq 1$ ) the generating functions  $T_i^{PB}(z)$  satisfy

$$\begin{aligned} \{T_i^{PB}(z_1), T_j^{PB}(z_2)\} &= (q - q^{-1})C_{i,j} \left( \frac{z_2}{z_1} \right) T_i^{PB}(z_1)T_j^{PB}(z_2) \\ &+ \sum_{k=1}^i \delta \left( \frac{q^{-\frac{j+i}{2}-k} z_2}{z_1} \right) T_{i-k}^{PB}(q^{\frac{k}{2}} z_1) T_{j+k}^{PB}(q^{-\frac{k}{2}} z_2) \\ &- \sum_{k=1}^i \delta \left( \frac{q^{\frac{j-i}{2}+k} z_2}{z_1} \right) T_{i-k}^{PB}(q^{-\frac{k}{2}} z_1) T_{j+k}^{PB}(q^{\frac{k}{2}} z_2) \quad (1 \leq i \leq j). \end{aligned} \quad (83)$$

Here we set the structure functions  $C_{i,j}(z)$  ( $i, j \geq 1$ ) as

$$C_{i,j}(z) = \sum_{m \in \mathbf{Z}} \frac{[\frac{1}{2} \text{Min}(i, j)m]_q [\frac{1}{2} (\text{Max}(i, j) - M - 1)m]_q}{[\frac{1}{2} (M + 1)m]_q} z^m \quad (i, j \geq 1).$$

The structure functions satisfy  $C_{i, M+1}(z) = C_{M+1, i}(z) = 0$  ( $1 \leq i \leq M + 1$ ).

## 5 Conclusion and Discussion

In this paper, we found the free field construction of the basic  $W$ -current  $T_1(z)$  (See (21) and (22)) and the screening currents  $S_j(w)$  (See (18)) for the deformed  $W$ -superalgebra  $\mathcal{W}_{q,t}(A(M, N))$ . Using the free field construction, we introduced the higher  $W$ -currents  $T_i(z)$  (See (62)) and obtained a closed set of quadratic relations among them (See (64)). These relations are independent of the choice of Dynkin diagrams for the Lie superalgebra  $A(M, N)$ , though the screening currents are not. This allows us to define  $\mathcal{W}_{q,t}(A(M, N))$  by generators and relations.

Recently, Feigin, Jimbo, Mukhin, and Vilkoviskiy [9] introduced the free field construction of the basic  $W$ -current and the screening currents for  $\mathcal{W}_{q,t}(\mathfrak{g})$  in types  $A, B, C, D$  including twisted and supersymmetric cases in terms of the quantum toroidal algebras. Their motivation is to understand a commutative family of integrals of motion associated with affine Dynkin diagrams [16, 17]. In the case of type  $A$ , their basic  $W$ -current  $T_1(z)$  satisfies

$$T_1(z_1)T_1(z_2) = \frac{\Theta_\mu \left( q_1 \frac{z_2}{z_1}, q_2 \frac{z_2}{z_1}, q_3 \frac{z_2}{z_1} \right)}{\Theta_\mu \left( q_1^{-1} \frac{z_2}{z_1}, q_2^{-1} \frac{z_2}{z_1}, q_3^{-1} \frac{z_2}{z_1} \right)} T_1(z_2)T_1(z_1) \quad (q_1 q_2 q_3 = 1) \quad (84)$$

in the sense of analytic continuation. Upon the specialization  $q_1 = x^2, q_2 = x^{-2r}, q_3 = x^{2r-2}, \mu = x^{2a}$  ( $a = D(0, L; \Phi)$ ), their commutation relation (84) coincides with those of this paper (See (25)). In the case of  $\mathfrak{sl}(N)$ , their basic  $W$ -current  $T_1(z)$  coincides with those of [16, 17], which gives a one-parameter deformation of  $\mathcal{W}_{q,t}(\mathfrak{sl}(N))$  in Ref. [2, 3]. In the case of  $A(M, N)$ , their basic  $W$ -current  $T_1(z)$  gives a deformation of those of  $\mathcal{W}_{q,t}(A(M, N))$  in this paper.

Here we discuss about justification of the definition of the deformed  $W$ -superalgebra of type  $A(M, N)$ . We compare Definition 4.2 with other definitions based on the Miura transformation or the screening operators. In what follows we set  $q = x^r, t = x^{1-r}$  and  $\beta = \frac{r-1}{r}$ . In Definition 4.2 we define  $\mathcal{W}_{q,t}(A(M, N))$

by generators and relations. In Refs. [2, 3] the deformed  $W$ -algebra  $\mathcal{W}_{q,t}(A_N)$  was proposed by considering a deformation of the Miura transformation that is the affine counterpart of the Harish-Chandra homomorphism. First, we study definition of the deformed  $W$ -algebras based on the Miura transformation. Recently, in the conformal case Gaiotto and Rapčák [18] proposed a new class of  $W$ -algebras from D3-branes attached to a 5-brane junction, which contains the  $W$ -algebra  $\mathcal{W}_\beta(A(M, N))$  as a special case. Their algebras are denoted as  $Y_{L,M,N}$  and are referred to as  $Y$ -algebra. Recently, Harada, Matsuo, Noshita and Watanabe [19] obtained a deformation of  $Y$ -algebra  $Y_{L,M,N}$  by considering a deformation of Miura transformation. Their deformed  $Y$ -algebra  $q$ - $Y_{L,M,N}$  coincided with the deformed  $W$ -superalgebra of type  $A$  introduced in [9]. They conjectured quadratic relations of  $q$ - $Y_{L,M,N}$  that were similar as our quadratic relations (64). In this revised manuscript we cite Ref. [19] that appeared in arXiv after submitting the first version of the present paper to *J. Phys. A*. A deformation of Miura transformation in Ref. [19] is represented in our notation as follows.

**Lemma 5.1** [19] *The formula (62) is rewritten as follows.*

$$: \prod_{k \in \tilde{I}(\frac{1-r}{r})} R_k^{(1)}(z) \prod_{k \in \tilde{I}(-\frac{1}{r})} R_k^{(2)}(z) := \sum_{i=0}^{\infty} (-1)^i T_i(x^{i-1}z) x^{2iD_z}. \quad (85)$$

Here we set  $D_z = z \frac{d}{dz}$  and

$$R_k^{(1)}(z) = 1 - \Lambda_k(z) x^{2D_z}, \quad R_k^{(2)}(z) = \sum_{i=0}^{\infty} (-1)^i \prod_{l=1}^i d_l(x, r) \Lambda_k(x^{2(l-1)}z) x^{2iD_z}.$$

In the conformal limit  $x \rightarrow 1$ , (85) leads to Miura transformation of the  $W$ -algebra  $\mathcal{W}_\beta(A(M, N))$ .

*Proof of Lemma 5.1.* Using (65) and  $p^{D_z} f(z) p^{-D_z} = f(pz)$  we obtain (85).  $\square$

Taking into account of this Miura transformation we define the associative algebra  $\mathcal{M}_{q,t}(\Phi)$  as a subalgebra of  $\mathcal{H}_{q,t}$  generated by the Fourier coefficients  $T_i[m]$  ( $m \in \mathbf{Z}, i = 1, 2, 3, \dots$ ) of the free field construction of the  $W$ -currents  $T_i(z) = \sum_{m \in \mathbf{Z}} T_i[m] z^{-m}$ . The algebra  $\mathcal{M}_{q,t}(\Phi)$  is  $A(M, N)$  counterpart of  $\mathcal{W}_{q,t}(A_N)$  defined in Refs. [2, 3]. We can set a homomorphism of an associative algebra  $\varphi : \mathcal{W}_{q,t}(A(M, N)) \rightarrow \mathcal{M}_{q,t}(\Phi)$  by letting  $\varphi(\bar{T}_i[m]) = T_i[m]$ , because the free field construction of the  $W$ -currents  $T_i(z)$  satisfies (64).  $\varphi$  is surjective. We conjecture that  $\varphi$  is injective. In other words, we conjecture that there doesn't exist independent relation other than (64) in  $\mathcal{M}_{q,t}(\Phi)$ . Here are two pieces of evidence to support this claim. In the classical limit, the second Hamiltonian structure  $\{\cdot, \cdot\}$  of the  $q$ -Poisson algebra [14, 15] is obtained from the quadratic relations. See (83). In the conformal limit all defining relations of the  $W$ -algebra  $\mathcal{W}_\beta(A_N)$  ( $N = 1, 2$ ) are obtained from the quadratic relations of  $\mathcal{W}_{q,t}(A_N)$  upon appropriate condition. See Ref. [2].

**Conjecture 5.2**  $\mathcal{W}_{q,t}(A(M, N))$  and  $\mathcal{M}_{q,t}(\Phi)$  are isomorphic as an associative algebra.

$$\mathcal{W}_{q,t}(A(M, N)) \cong \mathcal{M}_{q,t}(\Phi).$$

As a merit of Definition 4.2 it becomes clear that  $\mathcal{W}_{q,t}(A(M, N))$  is independent of the choice of Dynkin diagrams.

Next, we study definitions of the deformed  $W$ -algebras based on the screening operators. In Ref. [6] the deformed  $W$ -algebra of type  $\mathfrak{g} = A_N^{(1)}, B_N^{(1)}, C_N^{(1)}, D_N^{(1)}$  and  $A_{2N}^{(2)}$  were proposed as the intersection of kernels of the screening operators, that gives a deformation of the  $W$ -algebra  $\mathcal{W}_\beta(\mathfrak{g})$  in Ref. [20]. In what follows we work on the vacuum module  $\pi_0$  of the Heisenberg algebra  $\mathcal{H}_{q,t}$ . We set the screening operators  $S_j$  ( $1 \leq j \leq L$ ) acting on  $\pi_0$  as

$$S_j = \oint S_j(z) dz,$$

where  $S_j(w)$  are the screening currents associated with  $A(M, N)$ . The screening operators  $S_j$  depend on the choice of Dynkin diagrams  $\Phi$  of  $A(M, N)$ .

**Lemma 5.3**

$$[T_i(z), S_j] = 0 \quad (i \geq 0, 1 \leq j \leq L). \quad (86)$$

*Proof of Lemma 5.3.* From commutativity (6) we obtain  $[T_1(z), S_j] = 0$ . From quadratic relations (75)  $T_i[m]$  ( $i \geq 2$ ) are generated by  $T_1[n]$ . Hence we obtain (86).  $\square$

Let  $\mathbf{H}_{q,t}$  be the vector space spanned by formal power series of the form

$$: \partial_z^{n_1} \Lambda_{i_1}(x^{rj_1+k_1})^{\varepsilon_1} \dots \partial_z^{n_l} \Lambda_{i_l}(x^{rj_l+k_l})^{\varepsilon_l} :,$$

where  $\varepsilon_i = \pm 1$ . We define  $\mathbf{W}_{q,t}(\Phi)$  as the vector subspace of  $\mathbf{H}_{q,t}$  consisting of all currents that commute with the screening operators  $S_j$  ( $1 \leq j \leq L$ )

$$\mathbf{W}_{q,t}(\Phi) = \bigcap_{j=1}^L \text{Ker } S_j.$$

We define the algebra  $\mathcal{W}_{q,t}(\Phi)$  as the associative algebra generated by the Fourier coefficients of currents in  $\mathbf{W}_{q,t}(\Phi)$ . As a consequence of (86) we obtain the following inclusion relation.

**Corollary 5.4**

$$\mathcal{M}_{q,t}(\Phi) \subseteq \mathcal{W}_{q,t}(\Phi).$$

In the same way as for  $\mathfrak{g} = A_N^{(1)}$  we conjecture the following equality.

**Conjecture 5.5**

$$\mathcal{M}_{q,t}(\Phi) = \mathcal{W}_{q,t}(\Phi).$$

Finally, we comment on the definition by the quantum Drinfeld-Sokolov reduction. The definitions of the deformed  $W$ -algebra  $\mathcal{W}_{q,t}(\mathfrak{g})$  to nontwisted affine Lie algebra  $\mathfrak{g}$  were formulated in terms of the quantum Drinfeld-Sokolov reduction in Ref. [7]. It is still an open problem to formulate definitions of the

deformed  $W$ -algebras  $\mathcal{W}_{q,t}(\mathfrak{g})$  in terms of quantum Drinfeld-Sokolov reduction to twisted or supersymmetric cases. To summarize discussions, we conjecture

$$\mathcal{W}_{q,t}(A(M, N)) \simeq \mathcal{M}_{q,t}(\Phi) = \mathcal{W}_{q,t}(\Phi).$$

In this paper we choose the definition by generators and relations. As a merit this definition it becomes clear that  $\mathcal{W}_{q,t}(A(M, N))$  is independent of the choice of Dynkin diagrams.

The author would like to mention an open problem. It is still an open problem to find quadratic relations of the deformed  $W$ -algebra  $\mathcal{W}_{q,t}(\mathfrak{g})$ , except  $A_N^{(1)}$ ,  $A(M, N)^{(1)}$ , and the twisted algebra  $A_2^{(2)}$ . It seems to be possible to extend Ding-Feigin's construction to other Lie algebras and obtain their quadratic relations.

## ACKNOWLEDGMENTS

The author would like to thank Professor Michio Jimbo very much for carefully reading the manuscript and for giving lots of useful advice. The author would like to thank Professors Jun'ichi Shiraishi, Hitoshi Konno, Hiroaki Kanno, Koichi Harada, Yusuke Ohkubo, and Hiraku Nakajima for their interest to this work. This work is supported by JSPS KAKENHI Grant Number JP19K03509.

## A Fusion relations

In this appendix we summarize the fusion relations of  $\Lambda_i(z)$ . We use the abbreviation

$$F_{i,j}^{(\pm)}(z; a) = (1 - x^{\pm(i+j)}z)f_{i,j}(z; a) \quad (a = D(0, L; \Phi)).$$

For  $(m_1, m_2, \dots, m_{L+1}), (n_1, n_2, \dots, n_{L+1}) \in \hat{N}(\Phi)$  defined in (63), we set  $i = m_1 + m_2 + \dots + m_{L+1}$  and  $j = n_1 + n_2 + \dots + n_{L+1}$ .

- If  $\text{Max}\{1 \leq k \leq L+1 | n_k \neq 0\} < \text{Min}\{1 \leq k \leq L+1 | m_k \neq 0\}$  holds, we have

$$\begin{aligned} & \lim_{z_1 \rightarrow x^{i+j}z_2} F_{i,j}^{(+)}\left(\frac{z_2}{z_1}; a\right) \Lambda_{m_1, m_2, \dots, m_{L+1}}^{(i)}(z_1) \Lambda_{n_1, n_2, \dots, n_{L+1}}^{(j)}(z_2) \\ &= -c(r, x) \prod_{l=1}^{\text{Min}(i,j)-1} \Delta_1(x^{2l+1}) \Lambda_{m_1+n_1, m_2+n_2, \dots, m_{L+1}+n_{L+1}}^{(i+j)}(x^i z_2). \end{aligned} \quad (\text{A } 1)$$

- If  $\text{Max}\{1 \leq k \leq L+1 | m_k \neq 0\} < \text{Min}\{1 \leq k \leq L+1 | n_k \neq 0\}$  holds, we have

$$\begin{aligned} & \lim_{z_1 \rightarrow x^{-(i+j)}z_2} F_{i,j}^{(-)}\left(\frac{z_2}{z_1}; a\right) \Lambda_{m_1, m_2, \dots, m_{L+1}}^{(i)}(z_1) \Lambda_{n_1, n_2, \dots, n_{L+1}}^{(j)}(z_2) \\ &= c(r, x) \prod_{l=1}^{\text{Min}(i,j)-1} \Delta_1(x^{2l+1}) \Lambda_{m_1+n_1, m_2+n_2, \dots, m_{L+1}+n_{L+1}}^{(i+j)}(x^{-i} z_2). \end{aligned} \quad (\text{A } 2)$$

- If  $l$  satisfies  $l \in \hat{I}(-\frac{1}{r})$  and  $l = \text{Max}\{1 \leq k \leq L+1 | n_k \neq 0\} = \text{Min}\{1 \leq k \leq L+1 | m_k \neq 0\}$ , we have

$$\begin{aligned} & \lim_{z_1 \rightarrow x^{i+j}z_2} F_{i,j}^{(+)}\left(\frac{z_2}{z_1}; a\right) \Lambda_{m_1, m_2, \dots, m_{L+1}}^{(i)}(z_1) \Lambda_{n_1, n_2, \dots, n_{L+1}}^{(j)}(z_2) \\ &= -\frac{c(r, x)d_{m_l+n_l}(r, x)}{d_{m_l}(r, x)d_{n_l}(r, x)} \prod_{l=1}^{\text{Min}(i,j)-1} \Delta_1(x^{2l+1}) \Lambda_{m_1+n_1, m_2+n_2, \dots, m_{L+1}+n_{L+1}}^{(i+j)}(x^i z_2). \end{aligned} \quad (\text{A } 3)$$

- If  $l$  satisfies  $l \in \widehat{I}(-\frac{1}{r})$  and  $l = \text{Max}\{1 \leq k \leq L+1 | m_k \neq 0\} = \text{Min}\{1 \leq k \leq L+1 | n_k \neq 0\}$ , we have

$$\begin{aligned} & \lim_{z_1 \rightarrow x^{-(i+j)} z_2} F_{i,j}^{(-)} \left( \frac{z_2}{z_1}; a \right) \Lambda_{m_1, m_2, \dots, m_{L+1}}^{(i)}(z_1) \Lambda_{n_1, n_2, \dots, n_{L+1}}^{(j)}(z_2) \\ &= \frac{c(r, x) d_{m_l + n_l}(r, x)}{d_{m_l}(r, x) d_{n_l}(r, x)} \prod_{l=1}^{\text{Min}(i,j)-1} \Delta_1(x^{2l+1}) \Lambda_{m_1+n_1, m_2+n_2, \dots, m_{L+1}+n_{L+1}}^{(i+j)}(x^{-i} z_2). \end{aligned} \quad (\text{A } 4)$$

The remaining fusions vanish.

## B Exchange relations

In this appendix we give the exchange relations of  $\Lambda_i(z)$  and  $\Lambda_{m_1, m_2, \dots, m_{L+1}}^{(i)}(z)$ , which are obtained from Proposition 4.3. For  $(m_1, m_2, \dots, m_{L+1}) \in \widehat{N}(\Phi)$  in (63), we set  $i = m_1 + m_2 + \dots + m_{L+1}$ . We assume  $i \geq 1$ . We calculate

$$f_{1,i} \left( \frac{z_2}{z_1}; a \right) \Lambda_l(z_1) \Lambda_{m_1, m_2, \dots, m_{L+1}}^{(i)}(z_2) - f_{i,1} \left( \frac{z_1}{z_2}; a \right) \Lambda_{m_1, m_2, \dots, m_{L+1}}^{(i)}(z_2) \Lambda_l(z_1), \quad (\text{B } 1)$$

where  $a = D(0, L; \Phi)$ .

- If  $l$  satisfies  $m_l \neq 0$  and  $l \in \widehat{I}(\frac{1-r}{r})$ , (B 1) is deformed as

$$f_{1,i} \left( \frac{z_2}{z_1}; a \right) \Lambda_l(z_1) \Lambda_{m_1, m_2, \dots, m_{L+1}}^{(i)}(z_2) - f_{i,1} \left( \frac{z_1}{z_2}; a \right) \Lambda_{m_1, m_2, \dots, m_{L+1}}^{(i)}(z_2) \Lambda_l(z_1) = 0. \quad (\text{B } 2)$$

- If  $l$  satisfies  $m_l \neq 0$  and  $l \in \widehat{I}(-\frac{1}{r})$ , (B 1) is deformed as

$$\begin{aligned} & \frac{c(r, x) d_{m_l+1}(r, x)}{d_1(r, x) d_{m_l}(r, x)} : \Lambda_l(z_1) \Lambda_{m_1, m_2, \dots, m_{L+1}}^{(i)}(z_2) : \\ & \times \left( \delta \left( x^{-i-1+2(m_1+m_2+\dots+m_{l-1})} \frac{z_2}{z_1} \right) - \delta \left( x^{i+1-2(m_{l+1}+m_{l+2}+\dots+m_{L+1})} \frac{z_2}{z_1} \right) \right). \end{aligned} \quad (\text{B } 3)$$

- If  $l$  satisfies  $l < \text{Min}\{1 \leq k \leq L+1 | m_k \neq 0\}$ , (B 1) is deformed as

$$c(r, x) : \Lambda_l(z_1) \Lambda_{m_1, m_2, \dots, m_{L+1}}^{(i)}(z_2) : \left( \delta \left( \frac{x^{-i-1} z_2}{z_1} \right) - \delta \left( \frac{x^{-i+1} z_2}{z_1} \right) \right). \quad (\text{B } 4)$$

- If  $l$  satisfies  $l > \text{Max}\{1 \leq k \leq L+1 | m_k \neq 0\}$ , (B 1) is deformed as

$$c(r, x) : \Lambda_l(z_1) \Lambda_{m_1, m_2, \dots, m_{L+1}}^{(i)}(z_2) : \left( \delta \left( \frac{x^{i-1} z_2}{z_1} \right) - \delta \left( \frac{x^{i+1} z_2}{z_1} \right) \right). \quad (\text{B } 5)$$

- If  $l$  satisfies  $m_l = 0$  and  $\text{Min}\{1 \leq k \leq L+1 | m_k \neq 0\} < l < \text{Max}\{1 \leq k \leq L+1 | m_k \neq 0\}$ , (B 1) is deformed as

$$\begin{aligned} & c(r, x) : \Lambda_l(z_1) \Lambda_{m_1, m_2, \dots, m_{L+1}}^{(i)}(z_2) : \\ & \times \left( \delta \left( x^{-i+1+2(m_1+m_2+\dots+m_{l-2})} \frac{z_2}{z_1} \right) - \delta \left( x^{i+1-2(m_l+m_{l+1}+\dots+m_{L+1})} \frac{z_2}{z_1} \right) \right). \end{aligned} \quad (\text{B } 6)$$

## References

- [1] J. Shiraishi, H. Kubo, H. Awata, and S. Odake, A quantum deformation of the Virasoro algebra and the Macdonald symmetric functions, *Lett. Math. Phys.* **38**, 33-51 (1996)



- [2] H. Awata, H. Kubo, S. Odake, and J. Shiraishi, Quantum  $\mathcal{W}_N$  algebras and Macdonald polynomials, *Commun. Math. Phys.* **179**, 401-416 (1996)
- [3] B. Feigin and E. Frenkel, Quantum  $\mathcal{W}$ -algebras and elliptic algebras, *Commun. Math. Phys.* **178**, 653-678 (1996)
- [4] V. Brazhnikov and S. Lukyanov, Angular quantization and form-factors in massive integrable models, *Nucl. Phys.* **B512**, 616-636 (1998).
- [5] Y. Hara, M. Jimbo, H. Konno, S. Odake, and J. Shiraishi, Free field approach to the dilute  $A_L$  models, *J. Math. Phys.* **40**, 3791-3826 (1999)
- [6] E. Frenkel and N. Reshetikhin, Deformations of  $\mathcal{W}$  algebras associated to simple Lie algebras, *Commun. Math. Phys.* **197**, 1-31 (1998)
- [7] A. Sevostyanov, Drinfeld-Sokolov reduction for quantum groups and deformations of  $W$ -algebras, *Selecta Math.* **8**, 637-703 (2002)
- [8] S. Odake, Comments on the deformed  $W_N$  algebra, *Int. J. Mod. Phys.* **B16**, 2055-2064 (2002)
- [9] B. Feigin, M. Jimbo, E. Mukhin, and I. Vilkoviskiy, Deformation of  $W$  algebras via quantum toroidal algebras, arXiv: 2003.04234 (2020)
- [10] J. Ding and B. Feigin, Quantized  $W$ -algebra of  $\mathfrak{sl}(2, 1)$ : A construction from the quantization of screening operators, *Contemp. Math.* **248**, 83-108 (1998)
- [11] T. Kojima, Quadratic relations of the deformed  $W$ -superalgebra  $\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$ , *J. Math. Phys.* **62**, 051702 (2021)
- [12] V. G. Kac, Lie superalgebras, *Advances in Math.* **26**, 8-96 (1977)
- [13] M. Wakimoto, Infinite-dimensional Lie algebras, Iwanami Series in modern mathematics **195** (Translations of mathematical monographs), Amer. Math. Soc. 2001
- [14] E. Frenkel and N. Reshetikhin, Quantum affine algebras and deformation of the Virasoro algebra and  $\mathcal{W}$ -algebra, *Commun. Math. Phys.* **178**, 237-264 (1996)
- [15] E. Frenkel, N. Reshetikhin, and M. Semenov-Tian-Shansky, Drinfeld-Sokolov reduction for difference operators and deformation of  $\mathcal{W}$ -algebras I. The case of Virasoro algebra, *Commun. Math. Phys.* **192**, 605-629 (1998)
- [16] B. Feigin, T. Kojima, J. Shiraishi, and H. Watanabe, The integrals of motion for the deformed  $W$ -algebra  $\mathcal{W}_{q,t}(\widehat{\mathfrak{sl}}_N)$ , *Proceedings of Symposium on Representation Theory 2006*, 102-114 (2006), ISBN4-9902328-2-8, arXiv: 0705.0627v1
- [17] T. Kojima and J. Shiraishi, The integrals of motion for the deformed  $W$ -algebra  $\mathcal{W}_{q,t}(\widehat{\mathfrak{gl}}_N)$ . II. Proof of the commutation relations, *Commun. Math. Phys.* **283**, 795-851 (2008)

- [18] D. Gaiotto and M. Rapčák, Vertex algebras at the corner, *JHEP* **01** 160, (2019)
- [19] K. Harada, Y. Matsuo, G. Noshita and A. Watanabe,  $q$ -Deformation of corner vertex operator algebras by Miura transformation, arXiv:2101.03953 (2021)
- [20] E. Frenkel and N. Reshetikhin, Integrals of motion and quantum groups, Lect. Notes. in Math. **1620**, 349-418 (1995)