Quadratic relations of the deformed W-superalgebra $\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$

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Abstract

We revisit the free field construction of the deformed W-superalgebras $\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$ by J. Ding and B. Feigin, *Contemp. Math.* **248**, 83-108 (1998), where the basic W-current and screening currents have been found. In this paper we introduce higher W-currents and obtain a closed set of quadratic relations among them. These relations are independent of the choice of Dynkin diagrams for the superalgebra $\mathfrak{sl}(2|1)$, though the screening currents are not. This allows us to define $\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$ by generators and relations.

I Introduction

The deformed W-algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ is a two parameter deformation of the classical W-algebra $\mathcal{W}(\mathfrak{g})$, including the W-algebra and the q-Poisson W-algebra as special cases. Shiraishi et al.[1] obtained a free field realization of the deformed Virasoro algebra $\mathcal{W}_{q,t}(\mathfrak{sl}(2))$, which is a one-parameter deformation of the Virasoro algebra, to construct a deformation of the correspondence between conformal field theory and the Calogero-Sutherland model. Subsequently this work has been extended in various directions [2–9] concerning the deformed W-algebra $\mathcal{W}_{q,t}(\mathfrak{g})$. In comparison with the conformal case, the theory of deformed W-algebras is still not fully developed and understood. For that matter it is worthwhile to concretely construct $\mathcal{W}_{q,t}(\mathfrak{g})$ in each case.

In a way different from the above papers, Ding and Feigin[10] constructed free field realizations of the deformed W-algebras $\mathcal{W}_{q,t}(\mathfrak{sl}(3))$ and $\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$. They started from a W-current $T_1(z)$ given as a sum of three vertex operators $\Lambda_i(z)$ (see (1)), and two screening currents $S_j(w)$ each given by a vertex operator (see (2)). The idea of Ref.[10] is to determine them simultaneously by demanding that $T_1(z)$ and $S_j(w)$ commute up to a total difference. Using this method Ding and Feigin obtained three kinds of free field realizations. The first realization coincided with that of the deformed W-algebra $\mathcal{W}_{q,t}(\mathfrak{sl}(3))$ obtained earlier in Refs.[2,3] The second and the third realizations gave objects which were new at that time. In the conformal limit the second and the third screening charges in Ref.[10] satisfy the Serre relations of $U_q(\mathfrak{sl}(2|1))$ associated with the Dynkin diagrams ○—⊗ and ⊗—⊗, respectively. For that reason they called the algebras generated by the second and the third realizations the deformed W-superalgebra $\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$. Although the term “realization” is used, we stress that an abstract definition of $\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$ was missing at this stage.

In this paper we continue the study started in Ding and Feigin[10]. We begin by a review of their work. Since the derivation of the results given in Ref.[10] is somewhat sketchy, we give here the proofs in full detail. We then introduce higher W-currents $T_i(z)$ $(i = 1, 2, 3, \cdots)$, and present a closed set of quadratic relations for them. In the case of $\mathcal{W}_{q,t}(\mathfrak{sl}(3))$ it is known[2,3] that these currents truncate, i.e., $T_3(z) = 1$ and $T_i(z) = 0$ $(i \geq 4)$, so that the quadratic relations close between $T_1(z)$ and $T_2(z)$ (see (69)). In contrast, such truncation does not take place for the deformed W-superalgebra $\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$. We show that an infinite number of quadratic relations are satisfied by an infinite number of $T_i(z)$'s (see (70) and (71)). Moreover, these quadratic relations do not depend on the choice of the Dynkin diagram for the superalgebra $\mathfrak{sl}(2|1)$, even though the screening currents do. This leads us to define the algebra $\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$ abstractly by generators $T_i(z)$ $(i = 1, 2, 3, \cdots)$ and defining relations. This is the main result of this paper.

The text is organized as follows. In Section II, we introduce our notation and review Ding-Feigin's construction of the deformed W-algebra $\mathcal{W}_{q,t}(\mathfrak{sl}(3))$ and $\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$. In Section III, we introduce higher W-currents $T_i(z)$ and present a closed set of quadratic relations among them. We also obtain the q-Poisson algebra in the classical limit. Section IV is devoted to conclusion and discussion.

II Preliminaries

In this section we prepare the notation and review Ding-Feigin construction of the deformed W-algebra[10]. Throughout this paper, we fix a real number $r > 1$ and a complex number x with $0 < |x| < 1$.

A Notation

In this section we use complex numbers a, w $(w \neq 0)$, q $(q \neq 0, \pm 1)$, and p with $|p| < 1$. For any integer n, define q-integer

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

We use symbols for infinite products

$$(a;p)_\infty = \prod_{k=0}^{\infty}(1 - ap^k), \quad (a_1, a_2, \ldots, a_N; p)_\infty = \prod_{i=1}^{N}(a_i;p)_\infty.$$

for complex numbers $a_1, a_2, \ldots, a_N$. The following standard formulae are useful.

$$\exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}a^m\right) = 1 - a, \quad \exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}\frac{a^m}{1-p^m}\right) = (a;p)_\infty.$$

We use the elliptic theta function

$$\Theta_p(w) = (p, w, pw^{-1}; p)_\infty, \quad \Theta_p(w_1, w_2, \ldots, w_N) = \prod_{i=1}^{N}\Theta_p(w_i)$$

for complex numbers $w_1, w_2, \dots, w_N \neq 0$. Define $\delta(z)$ by the formal series

$$\delta(z) = \sum_{m \in \mathbf{Z}} z^m.$$

B Ding-Feigin realization

In the following we review the results of Ding and Feigin[10]. We state and rederive their results, giving the full detail and correcting some minor mistakes in their paper.

We introduce the Heisenberg algebra with generators $a_i(m)$, Q_i $(m \in \mathbf{Z}, 1 \le i \le 2)$ satisfying

$$[a_i(m), a_j(n)] = \frac{1}{m} A_{i,j}(m) \delta_{m+n,0} \quad (m, n \neq 0, 1 \le i, j \le 2),$$
$$[a_i(0), Q_j] = A_{i,j}(0) \quad (1 \le i, j \le 2).$$

The remaining commutators vanish. We impose the following conditions on the parameters $A_{i,j}(m) \in \mathbf{C}$:

$$A_{1,1}(m) = A_{2,2}(m) = 1 \ (m \neq 0), \quad A_{1,2}(m) = A_{2,1}(-m) \ (m \in \mathbf{Z}),$$
$$A_{1,2}(m) A_{2,1}(m) \neq A_{1,1}(m) A_{2,2}(m) \quad (m \in \mathbf{Z}).$$

We use the normal ordering symbol $: \quad :$ that satisfies

$$: a_i(m) a_j(n) : = \begin{cases} a_i(m) a_j(n) & (m < 0), \\ a_j(n) a_i(m) & (m \ge 0) \end{cases} \qquad (m, n \in \mathbf{Z}, 1 \le i, j \le 2),$$
$$: a_i(0) Q_j :=: Q_j a_i(0) := Q_j a_i(0) \quad (1 \le i, j \le 2).$$

Next, we work on Fock space of the Heisenberg algebra. Let $T_1(z)$ be a sum of three vertex operators

$$T_1(z) = g_1 \Lambda_1(z) + g_2 \Lambda_2(z) + g_3 \Lambda_3(z), \tag{1}$$
$$\Lambda_i(z) = e^{\sum_{j=1}^{2} \lambda_{i,j}(0) a_j(0)} : \exp\left(\sum_{j=1}^{2} \sum_{m \neq 0} \lambda_{i,j}(m) a_j(m) z^{-m} \right) : \quad (1 \le i \le 3).$$

We call $T_1(z)$ the basic W-current. We introduce the screening currents $S_j(w)$ $(1 \le j \le 2)$ as

$$S_j(w) = w^{\frac{1}{2} A_{j,j}(0)} e^{Q_j} w^{a_j(0)} : \exp\left(\sum_{m \neq 0} s_j(m) a_j(m) w^{-m} \right) : \quad (1 \le j \le 2). \tag{2}$$

The complex parameters $A_{ij}(m)$, $\lambda_{ij}(m), s_j(m)$ and g_i are to be determined through the construction given below.

Quite generally, given two vertex operators $V(z)$, $W(w)$, their product has the form

$$V(z) W(w) = \varphi_{V,W}(z, w) : V(z) W(w) : \quad (|z| \gg |w|),$$

with some formal power series $\varphi_{V,W}(z, w) \in \mathbf{C}[[w/z]]$. The vertex operators $V(z)$ and $W(w)$ are said to be mutually local if the following two conditions hold.

(i) $\varphi_{V,W}(z, w)$ and $\varphi_{W,V}(w, z)$ converge to rational functions,

(ii) $\varphi_{V,W}(z, w) = \varphi_{W,V}(w, z)$.

Under this setting, we are going to determine the W-currents $T_1(z)$ and the screening currents $S_j(w)$ that satisfy the following mutual locality (3), commutativity (4), and symmetry (5).

Mutual Locality $\Lambda_i(z)$ $(1\leq i\leq 3)$ and $S_j(w)$ $(1\leq j\leq 2)$ are mutually local, the operator product expansions of their products have at most one pole and one zero, and

$$\varphi_{\Lambda_i,S_j}(z,w)=\varphi_{S_j,\Lambda_i}(w,z)=\frac{w-\frac{z}{p_{i,j}}}{w-\frac{z}{q_{i,j}}}\quad(1\leq i\leq 3, 1\leq j\leq 2). \tag{3}$$

We allow the possibility $p_{i,j}=q_{i,j}$, in which case $\Lambda_i(z)S_j(w)=S_j(w)\Lambda_i(z)=:\Lambda_i(z)S_j(w):$.

Commutativity $T_1(z)$ commutes with $S_j(w)$ $(1\leq j\leq 2)$ up to a total difference

$$[T_1(z),S_j(w)]=B_j(z)\left(\delta\left(\frac{q_{j,j}w}{z}\right)-\delta\left(\frac{q_{j+1,j}w}{z}\right)\right)\quad(1\leq j\leq 2), \tag{4}$$

with some currents $B_j(z)$ $(1\leq j\leq 2)$.

Symmetry For $\widetilde{S}_j(w)=e^{-Q_j}S_j(w)$ $(1\leq j\leq 2)$, we impose

$$\varphi_{\widetilde{S}_1,\widetilde{S}_2}(w,z)=\varphi_{\widetilde{S}_2,\widetilde{S}_1}(w,z). \tag{5}$$

For simplicity, we impose further the following conditions.

$$q_{i,j}\quad(1\leq i\leq 3, 1\leq j\leq 2)\quad\text{are distinct.} \tag{6}$$

$$\left|\frac{q_{j+1,j}}{q_{j,j}}\right|\neq 1\quad(1\leq j\leq 2),\qquad -1<A_{1,2}(0)=A_{2,1}(0)<0. \tag{7}$$

We introduce the two parameters x and r defined as

$$x^{2r}=\frac{q_{2,1}}{q_{1,1}},\quad r=\begin{cases}\dfrac{1}{A_{1,2}(0)+1} & (A_{1,1}(0)\neq 1),\\ -\dfrac{1}{A_{1,2}(0)} & (A_{1,1}(0)=1).\end{cases} \tag{8}$$

From (7) and $q_{i,j}\neq 0$, we obtain $|x|\neq 0,1$ and $r>1$. In this paper, we focus our attention to

$$0<|x|<1,\quad r>1.$$

For the case of $|x|>1$, we obtain the same results under the change $x\mapsto x^{-1}$.

Consider the following transformations which map operators of form (1), (2) into operators of the same form.

(i) Rearranging indices

$$\Lambda_i(z)\mapsto\Lambda_{i'}(z),\quad S_j(w)\mapsto S_{j'}(w), \tag{9}$$

where $i\to i'$ is a permutation of the set $1,2,3$ and $j\to j'$ is a permutation of the set $1,2$.

(ii) Scaling variables: $\Lambda_i(z)\mapsto\Lambda_i(sz)$ $(s\neq 0)$, i.e.

$$\lambda_{i,j}(m)\mapsto s^m\lambda_{i,j}(m)\quad q_{i,j}\mapsto sq_{i,j},\quad p_{i,j}\mapsto sp_{i,j}\quad(m\neq 0, 1\leq i\leq 3, 1\leq j\leq 2). \tag{10}$$

(iii) Scaling free fields :

$$\begin{aligned} a_j(m) \mapsto \alpha_j(m)^{-1} a_j(m), \quad s_j(m) \mapsto \alpha_j(m) s_j(m), \\ \lambda_{i,j}(m) \mapsto \lambda_{i,j}(m)\alpha_j(m) \end{aligned} \qquad (m \neq 0, 1 \leq i \leq 3, 1 \leq j \leq 2),$$
$$A_{i,j}(m) \mapsto \alpha_i(m)^{-1} A_{i,j}(m) \alpha_j(m) \qquad (m \neq 0, 1 \leq i, j \leq 2), \tag{11}$$

where $\alpha_j(m) \neq 0$ $(1 \leq j \leq 2)$ and $\alpha_j(-m) = \alpha_j(m)^{-1}$ $(m > 0, 1 \leq j \leq 2)$.

In Ding-Feigin's construction[10], there are four cases to be considered separately.

$$\begin{aligned} &\text{Case 1}: \ A_{1,1}(0) \neq 1, \ A_{2,2}(0) \neq 1, \quad \text{Case 2}: \ A_{1,1}(0) \neq 1, \ A_{2,2}(0) = 1, \\ &\text{Case 3}: \ A_{1,1}(0) = 1, \ A_{2,2}(0) = 1, \quad \text{Case 4}: \ A_{1,1}(0) = 1, \ A_{2,2}(0) \neq 1. \end{aligned} \tag{12}$$

From (9), either Case 2 or Case 4 can be omitted. In what follows, we omit Case 4 and fix the index of $\Lambda_i(z)$ and $S_j(w)$.

Theorem II.1 *Assume that conditions (3), (4), (5), (6) and (7) hold. Then, up to transformations* (9), (10) *and* (11), *the parameters* $p_{i,j}, q_{i,j}$, $A_{i,j}(m)$, $s_i(m)$, $\lambda_{i,j}(m)$, g_i, *and the current* $B_j(m)$ *are uniquely determined as follows.*

- *In all Cases 1, 2, and 3,*

$$p_{1,2} = q_{1,2}, \quad p_{3,1} = q_{3,1}, \quad s_j(m) = 1 \quad (m > 0, 1 \leq j \leq 2), \tag{13}$$
$$A_{1,1}(m) = A_{2,2}(m) = 1, \quad A_{2,1}(m) = A_{1,2}(-m) \quad (m \neq 0),$$
$$B_j(z) = g_j \left(\frac{q_{j,j}}{p_{j,j}} - 1 \right) : \Lambda_j(z) S_j(q_{j,j}^{-1} z) : \quad (1 \leq j \leq 2). \tag{14}$$

- *In Case 1,*

$$A_{1,1}(0) = A_{2,2}(0) = \frac{2(r-1)}{r}, \quad A_{1,2}(0) = A_{2,1}(0) = -\frac{r-1}{r}, \quad g_1 = g_2 = g_3, \tag{15}$$
$$q_{i,i} = x^{i-1}, \ q_{i+1,i} = x^{2r+i-1}, \ p_{i,i} = x^{2r+i-3}, \ p_{i+1,i} = x^{i+1} \quad (1 \leq i \leq 2), \tag{16}$$
$$A_{1,2}(m) = -\frac{[m]_x}{[2m]_x} \quad (m \neq 0), \quad s_j(m) = -\frac{[(r-1)m]_x [2m]_x}{[rm]_x [m]_x} \quad (m < 0, 1 \leq j \leq 2), \tag{17}$$

$$\lambda_{i,j}(0) = \frac{2r}{3} \log x \times \begin{cases} j & (1 \leq j < i \leq 3), \\ j-3 & (1 \leq i \leq j \leq 2), \end{cases} \tag{18}$$
$$\frac{\lambda_{i,j}(m)}{s_j(m)} = -\frac{[rm]_x}{[3m]_x}(x - x^{-1}) \times \begin{cases} x^{(r+2)m}[jm]_x & (1 \leq j < i \leq 3), \\ x^{(r-1)m}[(j-3)m]_x & (1 \leq i \leq j \leq 2) \end{cases} \quad (m \neq 0). \tag{19}$$

- *In Case 2,*

$$A_{1,1}(0) = \frac{2(r-1)}{r}, \ A_{2,2}(0) = 1, \ \ A_{1,2}(0) = A_{2,1}(0) = -\frac{r-1}{r}, \quad g_1 = g_2, \quad g_3 = [r-1]_x g_1, \tag{20}$$
$$q_{i,i} = x^{i-1}, \ q_{i+1,i} = x^{2r+i-1}, \ p_{i,i} = x^{2r+i-3}, \ p_{i+1,i} = x^{-2r+5+(2r-3)i} \quad (1 \leq i \leq 2), \tag{21}$$
$$A_{1,2}(m) = \begin{cases} -\dfrac{[(r-1)m]_x}{[rm]_x} & (m > 0), \\ -\dfrac{[m]_x}{[2m]_x} & (m < 0), \end{cases} \quad s_1(m) = -\frac{[(r-1)m]_x [2m]_x}{[rm]_x [m]_x}, \ s_2(m) = -1 \ (m < 0), \tag{22}$$

$$\lambda_{i,j}(0)=\frac{2r}{r+1}\log x\times\begin{cases}-r & (i,j)=(1,1),\\ 1-r & (i,j)=(1,2),(2,2),\\ 1 & (i,j)=(2,1),(3,1),\\ 2 & (i,j)=(3,2),\end{cases} \tag{23}$$

$$\frac{\lambda_{i,j}(m)}{s_j(m)}=\frac{[rm]_x}{[(r+1)m]_x}(x-x^{-1})\times\begin{cases}x^{(r-1)m}[rm]_x & (i,j)=(1,1),\\ x^{(r-1)m}[(r-1)m]_x & (i,j)=(1,2),(2,2),\\ -x^{2rm}[m]_x & (i,j)=(2,1),(3,1),\\ -x^{2rm}[2m]_x & (i,j)=(3,2)\end{cases}\quad (m\neq 0). \tag{24}$$

- *In Case 3,*

$$A_{1,1}(0)=A_{2,2}(0)=1,\quad A_{1,2}(0)=A_{2,1}(0)=-\frac{1}{r},\quad g_2=[r-1]_x g_1,\quad g_3=g_1, \tag{25}$$

$$q_{i,i}=x^{(r-1)(i-1)},\ q_{i+1,i}=x^{r+1+(r-1)i},\ p_{i,i}=p_{i+1,i}=x^{3r-5+(-r+3)i}\quad (1\le i\le 2), \tag{26}$$

$$A_{1,2}(m)=-\frac{[m]_x}{[rm]_x}\quad (m\neq 0),\quad s_1(m)=s_2(m)=-1\quad (m<0), \tag{27}$$

$$\lambda_{i,j}(0)=\frac{2r}{r+1}\log x\times\begin{cases}-r & (i,j)=(1,1),\\ -1 & (i,j)=(1,2),(2,2),\\ 1 & (i,j)=(2,1),(3,1),\\ r & (i,j)=(3,2),\end{cases} \tag{28}$$

$$\frac{\lambda_{i,j}(m)}{s_j(m)}=\frac{[rm]_x}{[(r+1)m]_x}(x-x^{-1})\times\begin{cases}x^{(r-1)m}[rm]_x & (i,j)=(1,1),\\ x^{(r-1)m}[m]_x & (i,j)=(1,2),(2,2),\\ -x^{2rm}[m]_x & (i,j)=(2,1),(3,1),\\ -x^{2rm}[rm]_x & (i,j)=(3,2)\end{cases}\quad (m\neq 0). \tag{29}$$

Conversely, if the parameters are chosen as above then (3), (4), and (5) are satisfied.

Proposition II.2 *The $\Lambda_i(z)$'s satisfy the commutation relations*

$$\Lambda_k(z_1)\Lambda_l(z_2)=-\frac{z_2}{z_1}\frac{\Theta_{x^{2s}}\left(x^2\frac{z_2}{z_1},\quad x^{2s-2r}\frac{z_2}{z_1},\quad x^{2s+2r-2}\frac{z_2}{z_1}\right)}{\Theta_{x^{2s}}\left(x^2\frac{z_1}{z_2},\quad x^{2s-2r}\frac{z_1}{z_2},\quad x^{2s+2r-2}\frac{z_1}{z_2}\right)}\Lambda_l(z_2)\Lambda_k(z_1)\quad (1\le k,l\le 3), \tag{30}$$

where

$$s=\begin{cases}3 & \textit{for Case 1},\\ r+1 & \textit{for Cases 2 and 3}.\end{cases}$$

We understand (30) in the sense of analytic continuation.

Proposition II.3 *For Case 1, $S_j(w)$ satisfy*

$$S_1(w_1)S_2(w_2)=\left(\frac{w_1}{w_2}\right)^{1+\frac{1}{r}}\frac{\Theta_{x^{2r}}\left(x^{-1}\frac{w_2}{w_1}\right)}{\Theta_{x^{2r}}\left(x^{-1}\frac{w_1}{w_2}\right)}S_2(w_2)S_1(w_1), \tag{31}$$

$$S_j(w_1)S_j(w_2)=-\left(\frac{w_1}{w_2}\right)^{1-\frac{2}{r}}\frac{\Theta_{x^{2r}}\left(x^2\frac{w_2}{w_1}\right)}{\Theta_{x^{2r}}\left(x^2\frac{w_1}{w_2}\right)}S_j(w_2)S_j(w_1)\quad (1\le j\le 2). \tag{32}$$

For Case 2, $S_j(w)$ satisfy (31), (32) for $j=1$, and

$$S_2(w_1)S_2(w_2)=-S_2(w_2)S_2(w_1). \tag{33}$$

For Case 3, $S_j(w)$ satisfy

$$S_1(w_1)S_2(w_2)=\left(\frac{w_1}{w_2}\right)^{-\frac{1}{r}}\frac{\Theta_{x^{2r}}\left(x^{r+1}\frac{w_2}{w_1}\right)}{\Theta_{x^{2r}}\left(x^{r+1}\frac{w_1}{w_2}\right)}S_2(w_2)S_1(w_1), \tag{34}$$

$$S_j(w_1)S_j(w_2)=-S_j(w_2)S_j(w_1) \quad (1\le j\le 2). \tag{35}$$

We understand (31), (32), (33), (34), and (35) on the analytic continuation.

In fact, the stronger relation

$$S_j(w_1)S_j(w_2)=(w_1-w_2):S_j(w_1)S_j(w_2):$$

holds instead of (33) and (35). This means that the screening currents $S_2(w)$ in Case 2 and $S_1(w), S_2(w)$ in Case 3 are ordinary fermions.

We give a few words about Ref.[10]. Apart from some typos, the essential content is correct. We believe that their assumption $A_{1,2}(m)=A_{2,1}(m)$ is a misprint. The free field realizations of $\Lambda_i(z)$ $(1\le i\le 3)$ were not completely constructed. In Case 1, all $\Lambda_i(z)$ $(1\le i\le 3)$ were not constructed, in Case 2, $\Lambda_2(z)$ was not constructed, and in Cass 3, $\Lambda_i(z)$ $(i=1,2)$ were not constructed. The exchange relations (30) were only partially investigated. In Case 1, all exchange relations were not investigated, in Case 2, only $k=l=1$ and $k=l=3$, and in Case 3, only $k=l=3$ was investigated.

C Proof of Theorem II.1

In this section we show Theorem II.1 and Proposition II.3.

Lemma II.4 *For $\Lambda_i(z)$ and $S_j(w)$, we obtain*

$$\varphi_{\Lambda_i,S_j}(z,w)=e^{\sum_{k=1}^2\lambda_{i,k}(0)A_{k,j}(0)}\exp\left(\sum_{k=1}^2\sum_{m=1}^\infty\frac{1}{m}\lambda_{i,k}(m)A_{k,j}(m)s_j(-m)\left(\frac{w}{z}\right)^m\right), \tag{36}$$

$$\varphi_{S_j,\Lambda_i}(w,z)=\exp\left(\sum_{k=1}^2\sum_{m=1}^\infty\frac{1}{m}s_j(m)A_{j,k}(m)\lambda_{i,k}(-m)\left(\frac{z}{w}\right)^m\right) \quad (1\le i\le 3, 1\le j\le 2), \tag{37}$$

$$\varphi_{\widetilde{S}_k,\widetilde{S}_l}(w_1,w_2)=\exp\left(\sum_{m=1}^\infty\frac{1}{m}s_k(m)A_{k,l}(m)s_l(-m)\left(\frac{w_2}{w_1}\right)^m\right) \quad (1\le k,l\le 2), \tag{38}$$

$$\varphi_{\Lambda_k,\Lambda_l}(z_1,z_2)=\exp\left(\sum_{i,j=1}^2\sum_{m=1}^\infty\frac{1}{m}\lambda_{k,i}(m)A_{i,j}(m)\lambda_{l,j}(-m)\left(\frac{z_2}{z_1}\right)^m\right) \quad (1\le k,l\le 3). \tag{39}$$

Proof. Using the standard formula

$$e^Ae^B=e^{[A,B]}e^Be^A \quad ([[A,B],A]=0 \text{ and } [[A,B],B]=0),$$

we obtain (36), (37), (38), and (39). $\square$

Lemma II.5 *Mutual locality (3) holds if and only if (40) and (41) are satisfied*

$$\sum_{k=1}^{2}\lambda_{i,k}(0)A_{k,j}(0)=\log\left(\frac{q_{i,j}}{p_{i,j}}\right)\quad(1\le i\le 3,1\le j\le 2),\tag{40}$$

$$\sum_{k=1}^{2}\lambda_{i,k}(m)A_{k,j}(m)s_j(-m)=q_{i,j}^m-p_{i,j}^m\quad(m\neq 0,1\le i\le 3,1\le j\le 2).\tag{41}$$

Proof. Considering (36), (37), and the expansions

$$\frac{w-p_{i,j}^{-1}z}{w-q_{i,j}^{-1}z}=\exp\left(\log\left(\frac{q_{i,j}}{p_{i,j}}\right)-\sum_{m=1}^{\infty}\frac{1}{m}(p_{i,j}^m-q_{i,j}^m)\left(\frac{w}{z}\right)^m\right)\quad(|z|\gg|w|),\tag{42}$$

$$\frac{w-p_{i,j}^{-1}z}{w-q_{i,j}^{-1}z}=\exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}(p_{i,j}^{-m}-q_{i,j}^{-m})\left(\frac{z}{w}\right)^m\right)\quad(|w|\gg|z|),\tag{43}$$

we obtain (40) and (41) from (3).

Conversely, if we assume (40) and (41), we obtain (3) from (36), (37), (42), and (43). □

From linear equations (40) and (41), $\lambda_{i,j}(m)$ are expressed in terms of the other parameters.

Lemma II.6 *We assume (3) and (6). The commutativity (4) holds if and only if* $p_{1,2}=q_{1,2}$, $p_{3,1}=q_{3,1}$, *(44), (45), and (46) are satisfied, where*

$$q_{j,j}^{\frac{1}{2}A_{j,j}(0)}:\Lambda_j(z)S_j\left(q_{j,j}^{-1}z\right):=q_{j+1,j}^{\frac{1}{2}A_{j,j}(0)}:\Lambda_{j+1}(z)S_j\left(q_{j+1,j}^{-1}z\right):\quad(1\le j\le 2),\tag{44}$$

$$\frac{g_{j+1}}{g_j}=-\left(\frac{q_{j+1,j}}{q_{j,j}}\right)^{\frac{1}{2}A_{j,j}(0)}\frac{\frac{q_{j,j}}{p_{j,j}}-1}{\frac{q_{j+1,j}}{p_{j+1,j}}-1}\quad(1\le j\le 2),\tag{45}$$

$$B_j(z)=g_j\left(\frac{q_{j,j}}{p_{j,j}}-1\right):\Lambda_j(z)S_j(q_{j,j}^{-1}z):\quad(1\le j\le 2).\tag{46}$$

Proof. From (3), we obtain

$$[\Lambda_i(z),S_j(w)]=\left(\frac{q_{i,j}}{p_{i,j}}-1\right)\delta\left(\frac{q_{i,j}w}{z}\right):\Lambda_i(z)S_j(q_{i,j}^{-1}z):\quad(1\le i\le 3,1\le j\le 2).\tag{47}$$

Considering (6) and (47), we know that (4) holds if and only if $p_{1,2}=q_{1,2}$, $p_{3,1}=q_{3,1}$, and

$$B_j(z)=g_j\left(\frac{q_{j,j}}{p_{j,j}}-1\right):\Lambda_j(z)S_j(q_{j,j}^{-1}z):=-g_{j+1}\left(\frac{q_{j+1,j}}{p_{j+1,j}}-1\right):\Lambda_{j+1}(z)S_j(q_{j+1,j}^{-1}z):\ (1\le j\le 2)\tag{48}$$

are satisfied. (48) holds if and only if (44), (45), and (46) are satisfied. Hence, we obtain this lemma. □

We use the abbreviation $h_{k,l}(w)$ $(1\le k,l\le 2)$,

$$h_{k,l}\left(\frac{w_2}{w_1}\right)=\varphi_{\widetilde{S}_k,\widetilde{S}_l}(w_1,w_2).\tag{49}$$

Lemma II.7 *We assume (3) and (44). Then,* $h_{k,l}(w)$ *in (49) satisfy the q-difference equations*

$$\begin{aligned}&\frac{h_{1,2}(q_{1,1}w)}{h_{1,2}(q_{2,1}w)}=\frac{q_{2,2}}{p_{2,2}}\left(\frac{q_{1,1}}{q_{2,1}}\right)^{A_{1,2}(0)}\frac{1-p_{2,2}w}{1-q_{2,2}w},\quad \frac{h_{1,2}\left(q_{2,2}^{-1}w\right)}{h_{1,2}\left(q_{3,2}^{-1}w\right)}=\frac{1-q_{2,1}^{-1}w}{1-p_{2,1}^{-1}w},\\ &\frac{h_{2,1}(q_{3,2}w)}{h_{2,1}(q_{2,2}w)}=\frac{q_{2,1}}{p_{2,1}}\left(\frac{q_{3,2}}{q_{2,2}}\right)^{A_{1,2}(0)}\frac{1-p_{2,1}w}{1-q_{2,1}w},\quad \frac{h_{2,1}\left(q_{1,1}^{-1}w\right)}{h_{2,1}\left(q_{2,1}^{-1}w\right)}=\frac{1-p_{2,2}^{-1}w}{1-q_{2,2}^{-1}w},\end{aligned}\tag{50}$$

$$\begin{array}{c}\frac{1-(p_{j,j}w)^{-1}}{1-(q_{j,j}w)^{-1}}h_{j,j}\left(q_{j,j}^{-1}w\right)=\frac{1-(p_{j+1,j}w)^{-1}}{1-(q_{j+1,j}w)^{-1}}h_{j,j}\left(q_{j+1,j}^{-1}w\right),\\ \left(\frac{q_{j+1,j}}{q_{j,j}}\right)^{A_{j,j}(0)-1}\frac{p_{j+1,j}}{p_{j,j}}\frac{1-p_{j,j}w}{1-q_{j,j}w}h_{j,j}\left(q_{j,j}w\right)=\frac{1-p_{j+1,j}w}{1-q_{j+1,j}w}h_{j,j}\left(q_{j+1,j}w\right)\end{array}\quad (1\le j\le 2). \tag{51}$$

Proof. Multiplying (44) by the screening currents on the left or right and considering the normal orderings, we obtain (50) and (51) as necessary conditions. □

Lemma II.8 *The relation (52) holds if (3), (5), (7), and (44) are satisfied, where*

$$\begin{aligned}&q_{1,1}=s,\quad q_{2,2}=sx^{(1+A_{1,2}(0))r},\quad q_{2,1}=sx^{2r},\quad q_{3,2}=sx^{(3+A_{1,2}(0))r},\\&p_{2,1}=sx^{2(1+A_{1,2}(0))r},\quad p_{2,2}=sx^{(1-A_{1,2}(0))r},\\&s_1(m)A_{1,2}(m)s_2(-m)=s_2(m)A_{2,1}(m)s_1(-m)=-\frac{[A_{1,2}(0)rm]_x}{[rm]_x}\quad (m>0).\end{aligned} \tag{52}$$

Proof. From lemma II.7, we obtain (50). From (38) and (49), the constant term of $h_{k,l}(w)$ is 1. Comparing the Taylor expansions for both sides of (50), we obtain

$$\frac{q_{2,2}}{p_{2,2}}\left(\frac{q_{1,1}}{q_{2,1}}\right)^{A_{1,2}(0)}=1,\quad \frac{q_{2,1}}{p_{2,1}}\left(\frac{q_{3,2}}{q_{2,2}}\right)^{A_{1,2}(0)}=1. \tag{53}$$

Upon the specialization (53), we obtain solutions of (50) as

$$h_{1,2}(w)=\exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}\frac{\left(\frac{p_{2,2}}{q_{1,1}}\right)^m-\left(\frac{q_{2,2}}{q_{1,1}}\right)^m}{1-\left(\frac{q_{2,1}}{q_{1,1}}\right)^m}w^m\right)=\exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}\frac{\left(\frac{q_{3,2}}{p_{2,1}}\right)^m-\left(\frac{q_{3,2}}{q_{2,1}}\right)^m}{1-\left(\frac{q_{3,2}}{q_{2,2}}\right)^m}w^m\right), \tag{54}$$

$$h_{2,1}(w)=\exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}\frac{\left(\frac{q_{2,1}}{q_{2,2}}\right)^m-\left(\frac{p_{2,1}}{q_{2,2}}\right)^m}{1-\left(\frac{q_{3,2}}{q_{2,2}}\right)^m}w^m\right)=\exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}\frac{\left(\frac{q_{2,1}}{q_{2,2}}\right)^m-\left(\frac{q_{2,1}}{p_{2,2}}\right)^m}{1-\left(\frac{q_{2,1}}{q_{1,1}}\right)^m}w^m\right). \tag{55}$$

Here we used $|q_{j+1,j}/q_{j,j}|\neq 1$ $(1\le j\le 2)$ assumed in (7). From the compatibility of the two formulae for $h_{1,2}(w)$ in (54) [or $h_{2,1}(w)$ in (55)], there are two possible restrictions for $q_{1,1}$, $q_{2,2}$, $q_{2,1}$, and $q_{3,2}$,

$$\text{(i)}\quad \frac{q_{2,1}}{q_{1,1}}=\frac{q_{3,2}}{q_{2,2}}\quad \text{or}\quad \text{(ii)}\quad \frac{q_{2,1}}{q_{1,1}}=\frac{q_{2,2}}{q_{3,2}}. \tag{56}$$

First, we consider case (i) $q_{2,1}/q_{1,1}=q_{3,2}/q_{2,2}$ in (56). From the compatibility of the two formulae for $h_{1,2}(w)$ in (54) [and $h_{2,1}(w)$ in (55)], we obtain

$$h_{1,2}(w)=\exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}\frac{[A_{1,2}(0)rm]_x}{[rm]_x}x^{-(A_{1,2}(0)+1)rm}\left(\frac{q_{2,2}}{q_{1,1}}\right)^m w^m\right), \tag{57}$$

$$h_{2,1}(w)=\exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}\frac{[A_{1,2}(0)rm]_x}{[rm]_x}x^{(A_{1,2}(0)+1)rm}\left(\frac{q_{1,1}}{q_{2,2}}\right)^m w^m\right). \tag{58}$$

We used (53) and $q_{2,1}/q_{1,1}=q_{3,2}/q_{2,2}=x^{2r}$. From $h_{1,2}(w)=h_{2,1}(w)$ assumed in (5), we obtain

$$\frac{q_{2,2}}{q_{1,1}}=x^{(A_{1,2}(0)+1)r}. \tag{59}$$

Considering (53) and (59), we obtain the first half of (52). From (57), (58), and (59), we obtain

$$h_{1,2}(w)=h_{2,1}(w)=\exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}\frac{[A_{1,2}(0)rm]_x}{[rm]_x}w^m\right). \tag{60}$$

Considering (38) and (49), we obtain the second half of (52).

Next, we consider case (ii) $q_{2,1}/q_{1,1} = q_{2,2}/q_{3,2}$ in (56). From the compatibility of the two formulae for $h_{1,2}(w)$ in (54) [and $h_{2,1}(w)$ in (55)], we obtain

$$\left(\frac{p_{2,2}}{q_{1,1}}\right)^m + \left(\frac{q_{2,2}}{p_{2,1}}\right)^m = \left(\frac{q_{2,2}}{q_{2,1}}\right)^m + \left(\frac{q_{2,2}}{q_{1,1}}\right)^m \quad (m \neq 0). \tag{61}$$

From (61) for $m = 1, 2$, we obtain $p_{2,2}/p_{2,1} = q_{2,2}/q_{2,1}$. Combining (61) for $m = 1$ and $p_{2,2}/p_{2,1} = q_{2,2}/q_{2,1}$, we obtain $p_{2,1} = q_{2,1}$ or $p_{2,1} = q_{1,1}$. For the case of $p_{2,1} = q_{2,1}$, we obtain $A_{1,2}(0) = 0$ from (53). For the case of $p_{2,1} = q_{1,1}$, we obtain $A_{1,2}(0) = 1$ from (53). $A_{1,2}(0) = 0$ and $A_{1,2}(0) = 1$ contradict with $-1 < A_{1,2}(0) < 0$ assumed in (7). Hence, the case (ii) $q_{2,1}/q_{1,1} = q_{2,2}/q_{3,2}$ is impossible. $\square$

Lemma II.9 *The relations (62) and (63) hold if (3), (7), and (44) are satisfied, where*

$$\begin{aligned} &\frac{p_{j,j}}{q_{j,j}} = x^{A_{j,j}(0)r}, \quad \frac{p_{j+1,j}}{q_{j,j}} = x^{(2-A_{j,j}(0))r}, \\ &s_j(m)s_j(-m) = -\frac{\left[\frac{1}{2}A_{j,j}(0)rm\right]_x [(2-A_{j,j}(0))rm]_x}{\left[\frac{1}{2}(2-A_{j,j}(0))rm\right]_x [rm]_x} \end{aligned} \quad (m > 0, A_{j,j}(0) \neq 1), \tag{62}$$

$$p_{j,j} = p_{j+1,j}, \quad s_j(m)s_j(-m) = -1 \quad (m > 0, A_{j,j}(0) = 1). \tag{63}$$

Proof. From Lemma II.7, we obtain (51). From (38) and (49), the constant term of the Taylor expansion for $h_{j,j}(w)$ is 1. Comparing the Taylor expansions for both sides of (51), we obtain

$$\left(\frac{q_{j+1,j}}{q_{j,j}}\right)^{A_{j,j}(0)-1} \frac{p_{j+1,j}}{p_{j,j}} = 1 \quad (1 \leq j \leq 2). \tag{64}$$

Upon the specialization (64), the compatibility condition of the equations in (51) is

$$(p_{j,j} - p_{j+1,j})(p_{j,j}p_{j+1,j} - q_{j,j}q_{j+1,j}) = 0 \quad (1 \leq j \leq 2). \tag{65}$$

First, we study the case of $A_{j,j}(0) = 1$. We obtain $p_{j,j} = p_{j+1,j}$ from (64). Solving (51) upon $p_{j,j} = p_{j+1,j}$, we obtain $h_{j,j}(w) = 1-w$. Considering (38) and (49), we obtain $s_j(m)A_{j,j}(m)s_j(-m) = -1$ $(m > 0)$. We obtain (63).

Next, we study the case of $A_{j,j}(0) \neq 1$. We obtain $p_{j,j} \neq p_{j+1,j}$ from (7) and (64). We obtain $p_{j,j}p_{j+1,j} = q_{j,j}q_{j+1,j}$ from $p_{j,j} \neq p_{j+1,j}$ and (65). Combining $p_{j,j}p_{j+1,j} = q_{j,j}q_{j+1,j}$ and (64), we obtain the first part of (62). Solving (51), we obtain

$$h_{j,j}(w) = \exp\left(-\sum_{m=1}^{\infty} \frac{1}{m} \frac{\left[\frac{1}{2}A_{j,j}(0)rm\right]_x [(2-A_{j,j}(0))rm]_x}{\left[\frac{1}{2}(2-A_{j,j}(0))rm\right]_x [rm]_x} w^m\right).$$

We used $|q_{j+1,j}/q_{j,j}| \neq 1$ $(1 \leq j \leq 2)$ in (7) and $q_{j+1,j}/q_{j,j} = x^{2r}$ $(1 \leq j \leq 2)$ from (52). Considering (38) and (49), we obtain the second part of (62). $\square$

Proposition II.10 *The relations (3), (4), and (5) hold if the parameters $p_{i,j}, q_{i,j}$, $A_{i,j}(m)$, $s_i(m)$, g_i, and $\lambda_{i,j}(m)$ are determined by (40), (41), (45), (52), (62), (63), $p_{1,2} = q_{1,2}$, and $p_{3,1} = q_{3,1}$.*

Proof. The proof is divided into Cases 1, 2, and 3 classified in (12) according to the values of $A_{1,1}(0)$ and $A_{2,2}(0)$. For instance, for Case1, we will obtain the formulae (13), (14), (15), (16), (17), (18), and (19) in Theorem II.1 from the conditions (40), (41), (45), (52), (62), (63), $p_{1,2} = q_{1,2}$, and $p_{3,1} = q_{3,1}$. As a by-product of this calculation, we will show that there is no indeterminacy in the free field realization except for (10) and (11) upon the conditions (6) and (7). Because $\lambda_{i,j}(m)$ are determined by (40) and (41), the mutual locality (3) holds from Lemma II.5. The relations (6) and (44) are obtained by direct calculation using (15), (16), (18), and (19). Hence, commutativity (4) holds with (14) because of Lemma II.6. Symmetry (5) holds because of (60). Other cases are shown in the same way.

• Case 1 : From $A_{1,1}(0) \neq 1$ and (8), we set $A_{1,2}(0) = A_{2,1}(0) = (1-r)/r$. From (52) and (62) we obtain $p_{2,1} = sx^{2(1+A_{1,2}(0))r} = sx^{(2-A_{1,1}(0))r}$ and $p_{2,2} = sx^{(1-A_{1,2}(0))r} = sx^{(1+A_{1,2}(0)+A_{2,2}(0))r}$. Hence, we obtain $A_{1,1}(0) = A_{2,2}(0) = 2(r-1)/r$ in (15). Using the first half of (15), (52), and (62) we obtain $q_{i,i} = sx^{i-1}$, $q_{i+1,i} = sx^{2r+i-1}$, $p_{i,i} = sx^{2r+i-3}$, and $p_{i+1,i} = sx^{i+1}$ $(1 \leq i \leq 2)$, where $s = q_{1,1}$. Upon the specialization $s = 1$, we obtain (16). Considering generic $s = q_{1,1}$, we obtain the second half of (10) for Case 1. Using the first half of (15), (16), and (45), we obtain $g_1 = g_2 = g_3$ in the second half of (15). From (15), (52), and (62) we obtain

$$s_i(m)A_{i,j}(m)s_j(-m) = \begin{cases} -\dfrac{[(r-1)m]_x[2m]_x}{[rm]_x[m]_x} & (1 \leq i = j \leq 2), \\ \dfrac{[(r-1)m]_x}{[rm]_x} & (1 \leq i \neq j \leq 2) \end{cases} \qquad (m > 0).$$

Hence, we obtain

$$s_j(-m) = -\frac{[(r-1)m]_x[2m]_x}{[rm]_x[m]_x}\frac{1}{s_j(m)} \quad (m > 0, 1 \leq j \leq 2),$$
$$A_{i,j}(m) = A_{j,i}(-m) = -\frac{[m]_x}{[2m]_x}\frac{s_j(m)}{s_i(m)} \quad (m > 0, 1 \leq i \neq j \leq 2).$$

Thus, setting $s_j(m) = 1$ $(m > 0, 1 \leq j \leq 2)$ provides (17). Setting $s_j(m) = \alpha_j(m) \neq 0$ $(m > 0, 1 \leq j \leq 2)$ provides the scaling of the free field (11) for Case1. Solving the linear equations (40) and (41) for $\lambda_{i,j}(m)$ upon (13), (15), (16), and (17), we obtain (18) and (19). The first half of (10) for Case 1 is obtained in the same way. We obtained (13), (14) for Case 1, and (15)–(19) in Theorem II.1. We obtained (10) and (11) for Case 1 from necessary conditions.

• Case 2 : From $A_{1,1}(0) \neq 1$ and (8), we set $A_{1,2}(0) = A_{2,1}(0) = (1-r)/r$. From (52) and (62), we obtain $p_{2,1} = sx^{2(1+A_{1,2}(0))r} = sx^{(2-A_{1,1}(0))r}$. Hence, we obtain $A_{1,1}(0) = 2(r-1)/r$ in (20). Using the first half of (20), (52), and (62), we obtain $q_{i,i} = sx^{i-1}$, $q_{i+1,i} = sx^{2r+i-1}$, $p_{i,i} = sx^{2r+i-3}$, and $p_{i+1,i} = sx^{-2r+5+(2r-3)i}$ $(1 \leq i \leq 2)$ where $s = q_{1,1}$. Upon the specialization $s = 1$, we obtain (21). Considering generic $s = q_{1,1}$, we obtain the second half of (10) for Case 2. Using the first half of (20), (21), and (45), we obtain $g_1 = g_2$ and $g_3 = [r-1]_x g_1$ in the second half of (20). From (20), (52), (62), and (63), we obtain

$$s_i(m)A_{i,j}(m)s_j(-m) = \begin{cases} -\dfrac{[(r-1)m]_x[2m]_x}{[rm]_x[m]_x} & (i = j = 1), \\ -1 & (i = j = 2), \\ \dfrac{[(r-1)m]_x}{[rm]_x} & (1 \leq i \neq j \leq 2) \end{cases} \qquad (m > 0).$$

Hence, we obtain

$$s_1(-m)=-\frac{[(r-1)m]_x[2m]_x}{[rm]_x[m]_x}\frac{1}{s_1(m)},\quad s_2(-m)=\frac{1}{s_2(m)}\quad (m>0),$$
$$A_{1,2}(m)=-\frac{[(r-1)m]_x}{[rm]_x}\frac{s_2(m)}{s_1(m)},\quad A_{2,1}(m)=-\frac{[m]_x}{[2m]_x}\frac{s_1(m)}{s_2(m)}\quad (m>0).$$

Thus, setting $s_j(m)=1$ $(m>0, 1\le j\le 2)$ provides (22). Setting $s_j(m)=\alpha_j(m)\neq 0$ $(m>0, 1\le j\le 2)$ provides the scaling of the free field (11) for Case 2. Solving the linear equations (40) and (41) for $\lambda_{i,j}(m)$ upon (13), (20), (21), and (22), we obtain (23) and (24). The second half of (10) for Case 2 is obtained in the same way. We obtained (13), (14) for Case 2 and (20)–(24) in Theorem II.1. We obtained (10) and (11) for Case 2 from necessary conditions.

• Case 3 : From $A_{1,1}(0)=1$ and (8), we set $A_{1,2}(0)=A_{2,1}(0)=-1/r$. Using the first half of (25), (52), and (63), we obtain $q_{i,i}=sx^{(r-1)(i-1)}$, $q_{i+1,i}=sx^{r+1+(r-1)i}$, $p_{i,i}=p_{i+1,i}=sx^{3r-5+(-r+3)i}$ $(1\le i\le 2)$, where $s=q_{1,1}$. Upon the specialization $s=1$, we obtain (26). Considering generic $s=q_{1,1}$, we obtain the second half of (10) for Case 3. Using the first half of (25), (26), and (45), we obtain $g_2=[r-1]_x g_1$ and $g_3=g_1$ in the second half of (25). From (25), (52), and (63), we obtain

$$s_i(m)A_{i,j}(m)s_j(-m)=\begin{cases}-1 & (1\le i=j\le 2),\\ \dfrac{[m]_x}{[rm]_x} & (1\le i\neq j\le 2)\end{cases}\quad (m>0).$$

Hence, we obtain

$$s_j(-m)=-\frac{1}{s_j(m)}\quad (m>0, 1\le j\le 2),\quad A_{i,j}(m)=A_{j,i}(-m)=-\frac{[m]_x}{[rm]_x}\frac{s_j(m)}{s_i(m)}\quad (m>0, 1\le i\neq j\le 2).$$

Thus, setting $s_j(m)=1$ $(m>0, 1\le j\le 2)$ provides (27). Setting $s_j(m)=\alpha_j(m)\neq 0$ $(m>0, 1\le j\le 2)$ provides the scaling of the free field (11) for Case 3. Solving the linear equations (40) and (41) for $\lambda_{i,j}(m)$ upon (13), (25), (26), and (27), we obtain (28) and (29). The first half of (10) is derived in the same way. We obtained (13), (14) for Case 3 and (25)–(29) in Theorem II.1. We obtained (10) and (11) for Case 3 from necessary conditions. □

In the proof of Proposition II.10 we proved Theorem II.1 at the same time. As a by-product, we proved that there in no determinacy in the realization except for (9), (10), and (11) upon the conditions (6) and (7).

Proof of Proposition II.3. Using $h_{k,l}(w)$ in (49) we obtain

$$S_k(w_1)S_l(w_2)=\left(\frac{w_1}{w_2}\right)^{A_{k,l}(0)}\frac{h_{k,l}\left(\frac{w_2}{w_1}\right)}{h_{l,k}\left(\frac{w_1}{w_2}\right)}S_l(w_2)S_k(w_1)\quad (1\le k,l\le 2).$$

Using (52), (62) and (63) we obtain (31), (32), (33), (34), and (35). □

III Quadratic relations

In this section we introduce the higher W-currents $T_i(z)$ and obtain a set of quadratic relations of $T_i(z)$ for the deformed W-superalgebra $\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$.

A Quadratic relations

We define the functions $\Delta_i(z)$ $(i=0,1,2,\cdots)$ as

$$\Delta_i(z)=\frac{(1-x^{2r-i}z)(1-x^{-2r+i}z)}{(1-x^iz)(1-x^{-i}z)}. \tag{66}$$

We have

$$\Delta_i(z)-\Delta_i(z^{-1})=\frac{[r]_x[r-i]_x}{[i]_x}(x-x^{-1})(\delta(x^{-i}z)-\delta(x^iz)) \quad (i=1,2,3,\cdots).$$

We define the structure functions $f_{i,j}(z)$ $(i,j=0,1,2,\cdots)$ as

$$f_{i,j}(z)=\exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}\frac{[(r-1)m]_x[rm]_x[\mathrm{Min}(i,j)m]_x[(s-\mathrm{Max}(i,j))m]_x}{[m]_x[sm]_x}(x-x^{-1})^2z^m\right), \tag{67}$$

where

$$s=\begin{cases} 3 & for\ Case\ 1, \\ r+1 & for\ Cases\ 2\ and\ 3. \end{cases}$$

The ratio of the structure function

$$\frac{f_{1,1}(z^{-1})}{f_{1,1}(z)}=-z\frac{\Theta_{x^{2s}}(x^2z,x^{2s-2r}z,x^{2s+2r-2}z)}{\Theta_{x^{2s}}(x^2z^{-1},x^{2s-2r}z^{-1},x^{2s+2r-2}z^{-1})}$$

coincides with those of (30).

We introduce the higher W-currents $T_i(z)$ and give the quadratic relations. From now on, we set $g_1=1$ for all cases, but this is not an essential limitation. Hereafter, we use the abbreviations

$$c(r,x)=[r]_x[r-1]_x(x-x^{-1}),\quad d_0(r,x)=1,\quad d_j(r,x)=\prod_{l=1}^{j}\frac{[r-l]_x}{[l]_x}\quad (j\geq 1). \tag{68}$$

For Case 1, we set the W-currents as

$$T_1(z)=\Lambda_1(z)+\Lambda_2(z)+\Lambda_3(z),$$
$$T_2(z)=:\Lambda_1(x^{-1}z)\Lambda_2(xz):+:\Lambda_1(x^{-1}z)\Lambda_3(xz):+:\Lambda_2(x^{-1}z)\Lambda_3(xz):.$$

They satisfy the quadratic relations of $\mathcal{W}_{q,t}(\mathfrak{sl}(3))^{2,3}$

$$\begin{aligned}
&f_{1,1}\left(\frac{z_2}{z_1}\right)T_1(z_1)T_1(z_2)-f_{1,1}\left(\frac{z_1}{z_2}\right)T_1(z_2)T_1(z_1)=c(r,x)\left(\delta\left(\frac{x^{-2}z_2}{z_1}\right)T_2(x^{-1}z_2)-\delta\left(\frac{x^2z_2}{z_1}\right)T_2(xz_2)\right),\\
&f_{1,2}\left(\frac{z_2}{z_1}\right)T_1(z_1)T_2(z_2)-f_{2,1}\left(\frac{z_1}{z_2}\right)T_2(z_2)T_1(z_1)=c(r,x)\left(\delta\left(\frac{x^{-3}z_2}{z_1}\right)-\delta\left(\frac{x^3z_2}{z_1}\right)\right),\\
&f_{2,2}\left(\frac{z_2}{z_1}\right)T_2(z_1)T_2(z_2)-f_{2,2}\left(\frac{z_1}{z_2}\right)T_2(z_2)T_2(z_1)=c(r,x)\left(\delta\left(\frac{x^{-2}z_2}{z_1}\right)T_1(x^{-1}z_2)-\delta\left(\frac{x^2z_2}{z_1}\right)T_1(xz_2)\right),
\end{aligned} \tag{69}$$

with $f_{1,1}(z)=f_{2,2}(z)$ and $f_{1,2}(z)=f_{2,1}(z)$. Here, we omit the proof. The deformed W-algebra $\mathcal{W}_{q,t}(\mathfrak{sl}(3))$ is the associative algebra over $\mathbf{C}$ with the generators $T_i[m]$ $(m\in\mathbf{Z}, i=1,2)$ and defining relations (69). Here we set $T_i(z)=\sum_{m\in\mathbf{Z}}T_i[m]z^{-m}$ $(i=1,2)$. The parameters q and t in $\mathcal{W}_{q,t}(\mathfrak{sl}(3))$ are given as $q=x^{2r}$ and $t=x^{2(r-1)}$.

For Case 2, we introduce the W-currents $T_i(z)$ $(i=1,2,3,\cdots)$ as

$$
\begin{aligned}
&T_1(z)=\Lambda_1(z)+\Lambda_2(z)+d_1(r,x)\Lambda_3(z),\\
&T_2(z)=:\Lambda_1(x^{-1}z)\Lambda_2(xz):+d_1(r,x):\Lambda_1(x^{-1}z)\Lambda_3(xz):+d_1(r,x):\Lambda_2(x^{-1}z)\Lambda_3(xz):\\
&\qquad+d_2(r,x):\Lambda_3(x^{-1}z)\Lambda_3(xz):,\\
&T_i(z)=d_{i-2}(r,x):\Lambda_1(x^{-i+1}z)\Lambda_2(x^{-i+3}z)\prod_{j=1}^{i-2}\Lambda_3(x^{-i+2j+3}z):\\
&\qquad+d_{i-1}(r,x):\Lambda_1(x^{-i+1}z)\prod_{j=1}^{i-1}\Lambda_3(x^{-i+2j+1}z):+d_{i-1}(r,x):\Lambda_2(x^{-i+1}z)\prod_{j=1}^{i-1}\Lambda_3(x^{-i+2j+1}z):\\
&\qquad+d_i(r,x):\prod_{j=1}^{i}\Lambda_3(x^{-i+2j-1}z):\quad (i=3,4,5,\cdots).
\end{aligned}
\tag{70}
$$

We use $d_j(r,x)$ defined in (68). We have $T_i(z)\neq 1$ $(i=1,2,3,\cdots)$ and $T_i(z)\neq T_j(z)$ $(i\neq j)$. The definition of $T_i(z)$ for Case 3 is similar as those for Case 2. See the definition of $T_i(z)$ for Case 3 in (80). On the other hand, the definitions of $T_i(z)$ for Cases 2 and 3 are different from those of Case 1. See discussion in Section IV.

The following is **the main theorem** of this paper which holds for Cases 2 and 3.

Theorem III.1 *For the deformed W-superalgebra $\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$ the W-currents $T_i(z)$ satisfy the set of quadratic relations*

$$
\begin{aligned}
&f_{i,j}\left(\frac{z_2}{z_1}\right)T_i(z_1)T_j(z_2)-f_{j,i}\left(\frac{z_1}{z_2}\right)T_j(z_2)T_i(z_1)\\
&=c(r,x)\sum_{k=1}^{i}\prod_{l=1}^{k-1}\Delta_1(x^{2l+1})\left(\delta\left(\frac{x^{-j+i-2k}z_2}{z_1}\right)f_{i-k,j+k}(x^{j-i})T_{i-k}(x^kz_1)T_{j+k}(x^{-k}z_2)\right.\\
&\left.-\delta\left(\frac{x^{j-i+2k}z_2}{z_1}\right)f_{i-k,j+k}(x^{-j+i})T_{i-k}(x^{-k}z_1)T_{j+k}(x^kz_2)\right)\quad (j\geq i\geq 1).
\end{aligned}
\tag{71}
$$

Here we use $f_{i,j}(z)$ in (67). $T_0(z)$ in the right hand side is understood as $T_0(z)=1$.

In view of Theorem III.1 we arrive at the following definition.

Definition III.2 *Set $T_i(z)=\sum_{m\in\mathbf{Z}}T_i[m]z^{-m}$ $(i=1,2,3,\cdots)$. The deformed W-superalgebra $\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$ is the associative algebra over $\mathbf{C}$ with the generators $T_i[m]$ $(m\in\mathbf{Z}, i=1,2,3,\cdots)$ and defining relations (71).*

B Proof of Theorem III.1

Lemma III.3 *For Cases 2 and 3, $\Delta_i(z)$ and $f_{i,j}(z)$ satisfy the following fusion relations.*

$$f_{i,j}(z)=f_{j,i}(z)=\prod_{k=1}^{i}f_{1,j}(z^{-i-1+2k}z)\quad(1\le i\le j),\tag{72}$$

$$f_{1,i}(z)=\left(\prod_{k=1}^{i-1}\Delta_1(x^{-i+2k}z)\right)^{-1}\prod_{k=1}^{i}f_{1,1}(x^{-i-1+2k}z)\quad(i\ge 2),\tag{73}$$

$$f_{1,i}(z)f_{j,i}(x^{\pm(j+1)}z)=\begin{cases}f_{j+1,i}(x^{\pm j}z)\Delta_1(x^{\pm i}w) & (1\le i\le j),\\ f_{j+1,i}(x^{\pm j}z) & (1\le j<i),\end{cases}\tag{74}$$

$$f_{1,i}(z)f_{1,j}(x^{\pm(i+j)}z)=f_{1,i+j}(x^{\pm j}z)\Delta_1(x^{\pm i}z)\quad(i,j\ge 1),\tag{75}$$

$$f_{1,i}(z)f_{1,j}(x^{\pm(i-j-2k)}z)=f_{1,i-k}(x^{\mp k}z)f_{1,j+k}(x^{\pm(i-j-k)}z)\quad(i,j,i-k,j+k\ge 1),\tag{76}$$

$$\Delta_{i+1}(z)=\left(\prod_{k=1}^{i-1}\Delta_1(x^{-i+2k}z)\right)^{-1}\prod_{k=1}^{i}\Delta_2(x^{-i-1+2k}z)\quad(i\ge 2).\tag{77}$$

Proof. We obtain (72) and (77) by straightforward calculation from the definitions. We show (73) here. From definitions (66) and (67), we have

$$\left(\prod_{k=1}^{i-1}\Delta_1(x^{-i+2k}z)\right)^{-1}\prod_{k=1}^{i}f_{1,1}(x^{-i-1+2k}z)$$
$$=\exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}\frac{[rm]_x[(r-1)m]_x}{[(r+1)m]_x}(x-x^{-1})^2\left([rm]_x\sum_{k=1}^{i}x^{(-i+2k-1)m}-[(r+1)m]_x\sum_{k=1}^{i-1}x^{(-i+2k)m}\right)z^m\right).$$

Using the relation

$$[rm]_x\sum_{k=1}^{i}x^{(-i+2k-1)m}-[(r+1)m]_x\sum_{k=1}^{i-1}x^{(-i+2k)m}=[(r+1-i)m]_x,$$

we have $f_{1,i}(z)$. Using (72) and (73), we obtain the relations (74), (75), and (76). □

Proposition III.4 $\Lambda_i(z)$*'s satisfy*

$$f_{1,1}\left(\frac{z_2}{z_1}\right)\Lambda_k(z_1)\Lambda_l(z_2)=\Delta_1\left(\frac{x^{-1}z_2}{z_1}\right):\Lambda_k(z_1)\Lambda_l(z_2):,$$
$$f_{1,1}\left(\frac{z_2}{z_1}\right)\Lambda_l(z_1)\Lambda_k(z_2)=\Delta_1\left(\frac{xz_2}{z_1}\right):\Lambda_l(z_1)\Lambda_k(z_2):\quad(1\le k<l\le 3),\tag{78}$$
$$f_{1,1}\left(\frac{z_2}{z_1}\right)\Lambda_1(z_1)\Lambda_1(z_2)=:\Lambda_1(z_1)\Lambda_1(z_2):,$$
$$f_{1,1}\left(\frac{z_2}{z_1}\right)\Lambda_i(z_1)\Lambda_i(z_2)=\Delta_2\left(\frac{s_iz_2}{z_1}\right):\Lambda_i(z_1)\Lambda_i(z_2):(2\le i\le 3),\tag{79}$$

where

$$s_2=\begin{cases}0 & for\ Cases\ 1\ and\ 2,\\ 1 & for\ Case\ 3,\end{cases}\qquad s_3=\begin{cases}0 & for\ Cases\ 1\ and\ 3,\\ 1 & for\ Case\ 2.\end{cases}$$

Here we use $f_{1,1}(z)$ in (67).

Proof. Substituting (19), (24), and (29) into $\varphi_{\Lambda_k,\Lambda_l}(z_1,z_2)$ in (39) we obtain (78) and (79). □

Proof of Proposition II.2. Using $\varphi_{\Lambda_k,\Lambda_l}(z_1,z_2)$ in (39) we obtain

$$\Lambda_k(z_1)\Lambda_l(z_2)=\frac{\varphi_{\Lambda_k,\Lambda_l}(z_1,z_2)}{\varphi_{\Lambda_l,\Lambda_k}(z_2,z_1)}\Lambda_l(z_2)\Lambda_k(z_1)\quad(1\le k,l\le 3).$$

Using the explicit formulae of $\varphi_{\Lambda_k,\Lambda_l}(z_1,z_2)$, we obtain (30). □

Here, we set the higher W-currents $T_i(z)$ for Case 3. To avoid confusion, we write $\Lambda_i(z)$ for Case 2 (resp. Case 3) as $\Lambda_i^{\rm II}(z)$ (resp. $\Lambda_i^{\rm III}(z)$). We write $T_i(z)$ for Case 2 (resp. Case 3) as $T_i^{\rm II}(z)$ (resp. $T_i^{\rm III}(z)$). From Proposition III.4, $\Lambda_i^{\rm II}(z)$ and $\Lambda_i^{\rm III}(z)$ satisfy the same relations (78), (79), where $i\to i'$ is the permutations $(1,2,3)\to(3,1,2)$. If the expression of the W-currents $T_i^{\rm II}(z)$ in (70) is abbreviated as

$$T_i^{\rm II}(z)=P_i(\Lambda_1^{\rm II}(z),\Lambda_2^{\rm II}(z),\Lambda_3^{\rm II}(z))\quad(i=1,2,3,\cdots),$$

the W-currents $T_i^{\rm III}(z)$ is defined as

$$T_i^{\rm III}(z)=P_i(\Lambda_3^{\rm III}(z),\Lambda_1^{\rm III}(z),\Lambda_2^{\rm III}(z))\quad(i=1,2,3,\cdots).\tag{80}$$

The algebras generated by $T_i(z)$ $(i=1,2,3,\cdots)$ are the same as for Case 2 and Case 3.

The following lemmas III.5, III.6, and III.7 give special cases of (71).

Lemma III.5 *For Cases 2 and 3 we have the fusion relation of $T_i(z)$ as*

$$\begin{aligned}&\lim_{z_1\to x^{\pm(i+j)}z_2}\left(1-x^{\pm(i+j)}\frac{z_2}{z_1}\right)f_{i,j}\left(\frac{z_2}{z_1}\right)T_i(z_1)T_j(z_2)\\&=\mp c(r,x)\prod_{k=1}^{{\rm Min}(i,j)-1}\Delta_1(x^{2k+1})T_{i+j}(x^{\pm i}z_2)\quad(i,j\ge 1).\end{aligned}\tag{81}$$

Proof. Summing up the relations (A 2)–(A 12) in Appendix A gives (81). □

Lemma III.6 *For Cases 2 and 3 we have the exchange relation as meromorphic functions*

$$f_{i,j}\left(\frac{z_2}{z_1}\right)T_i(z_1)T_j(z_2)=f_{j,i}\left(\frac{z_1}{z_2}\right)T_j(z_2)T_i(z_1)\quad(j\ge i\ge 1).\tag{82}$$

Proof. Using the commutation relations (78) and (79) repeatedly, (82) is obtained except for poles in both sides. □

Lemma III.7 *For Cases 2 and 3 $T_i(z)$ satisfy the quadratic relation.*

$$\begin{aligned}&f_{1,i}\left(\frac{z_2}{z_1}\right)T_1(z_1)T_i(z_2)-f_{i,1}\left(\frac{z_1}{z_2}\right)T_i(z_2)T_1(z_1)\\&=c(r,x)\left(\delta\left(\frac{x^{-i-1}z_2}{z_1}\right)T_{i+1}(x^{-1}z_2)-\delta\left(\frac{x^{i+1}z_2}{z_1}\right)T_{i+1}(xz_2)\right)\quad(i\ge 1).\end{aligned}\tag{83}$$

Proof. Summing up the relations (B 1)–(B 7) in Appendix B gives the quadratic relation (83). □

Proof of Theorem III.1. We prove Theorem III.1 by induction. Lemma III.7 is the basis of induction for the proof. We define $\mathrm{LHS}_{i,j}$ and $\mathrm{RHS}_{i,j}(k)$ with $(1 \le k \le i \le j)$ as

$$\begin{aligned}
\mathrm{LHS}_{i,j} &= f_{i,j}\left(\frac{z_2}{z_1}\right) T_i(z_1)T_j(z_2) - f_{j,i}\left(\frac{z_1}{z_2}\right) T_j(z_2)T_i(z_1),\\
\mathrm{RHS}_{i,j}(k) &= c(r,x)\prod_{l=1}^{k-1}\Delta_1(x^{2l+1})\left(\delta\left(\frac{x^{-j+i-2k}z_2}{z_1}\right) f_{i-k,j+k}(x^{j-i})T_{i-k}(x^k z_1)T_{j+k}(x^{-k}z_2)\right.\\
&\quad \left. -\,\delta\left(\frac{x^{j-i+2k}z_2}{z_1}\right) f_{i-k,j+k}(x^{-j+i})T_{i-k}(x^{-k}z_1)T_{j+k}(x^k z_2)\right) \quad (1 \le k \le i-1),\\
\mathrm{RHS}_{i,j}(i) &= c(r,x)\prod_{l=1}^{i-1}\Delta_1(x^{2l+1})\left(\delta\left(\frac{x^{-j-i}z_2}{z_1}\right) T_{j+i}(x^{-i}z_2) - \delta\left(\frac{x^{j+i}z_2}{z_1}\right) T_{j+i}(x^i z_2)\right).
\end{aligned}$$

We prove the following relation by induction on i $(1 \le i \le j)$.

$$\mathrm{LHS}_{i,j} = \sum_{k=1}^{i} \mathrm{RHS}_{i,j}(k). \tag{84}$$

The starting point of $i = 1 \le j$ was previously proven in Lemma III.7. We assume that relation (84) holds for some i $(1 \le i < j)$, and we show $\mathrm{LHS}_{i+1,j} = \sum_{k=1}^{i+1} \mathrm{RHS}_{i+1,j}(k)$ from this assumption. Multiplying $\mathrm{LHS}_{i,j}$ by $f_{1,i}(z_1/z_3)\, f_{1,j}(z_2/z_3)\, T_1(z_3)$ on the left and using the quadratic relation (84) with $i = 1$ along with fusion relation (74) gives

$$\begin{aligned}
& f_{1,j}\left(\frac{z_2}{z_3}\right) f_{i,j}\left(\frac{z_2}{z_1}\right) f_{1,i}\left(\frac{z_1}{z_3}\right) T_1(z_3)T_i(z_1)T_j(z_2) - f_{j,1}\left(\frac{z_3}{z_2}\right) f_{j,i}\left(\frac{z_1}{z_2}\right) T_j(z_2) f_{1,i}\left(\frac{z_1}{z_3}\right) T_1(z_3)T_i(z_1)\\
& - c(r,x)\delta\left(\frac{x^{-j-1}z_2}{z_3}\right)\Delta_1\left(\frac{x^{-i}z_1}{z_3}\right) f_{j+1,i}\left(\frac{x^{-j}z_1}{z_3}\right) T_{j+1}(x^j z_3)T_i(z_1)\\
& + c(r,x)\delta\left(\frac{x^{j+1}z_2}{z_3}\right)\Delta_1\left(\frac{x^{i}z_1}{z_3}\right) f_{j+1,i}\left(\frac{x^{j}z_1}{z_3}\right) T_{j+1}(x^{-j} z_3)T_i(z_1).
\end{aligned} \tag{85}$$

Taking the limit $z_3 \to x^{-i-1}z_1$ of (85) multiplied by $c(r,x)^{-1}\left(1 - x^{-i-1}z_1/z_3\right)$ and using fusion relation (81) along with the relation $\lim_{z_3 \to x^{-i-1}z_1}\left(1 - x^{-i-1}z_1/z_3\right)\Delta_1\left(x^{-i}z_1/z_3\right) = c(r,x)$ gives

$$\begin{aligned}
& f_{1,j}\left(\frac{x^{i+1}z_2}{z_1}\right) f_{i,j}\left(\frac{z_2}{z_1}\right) T_{i+1}(x^{-1}z_1)T_j(z_2) - f_{j,1}\left(\frac{x^{-i-1}z_1}{z_2}\right) f_{j,i}\left(\frac{z_1}{z_2}\right) T_j(z_2)T_{i+1}(x^{-1}z_1)\\
& - c(r,x)\delta\left(\frac{x^{i-j}z_2}{z_1}\right) f_{j+1,i}(x^{i-j+1})T_{j+1}(x^{j-i-1}z_1)T_i(z_1)\\
& + c(r,x)\delta\left(\frac{x^{i+j+2}z_2}{z_1}\right)\prod_{l=1}^{i}\Delta_1(x^{2l+1})T_{i+j+1}(x^{i+1}z_2).
\end{aligned}$$

Using fusion relation (74) and $f_{j+1,i}(x^{i-j+1})T_{j+1}(x^{j-i-1}z_1)T_i(z_1) = f_{i,j+1}(x^{j-i-1})T_i(z_1)T_{j+1}(x^{j-i-1}z_1)$ in (84) gives

$$\begin{aligned}
& f_{i+1,j}\left(\frac{xz_2}{z_1}\right) T_{i+1}(x^{-1}z_1)T_j(z_2) - f_{j,i+1}\left(\frac{x^{-1}z_1}{z_2}\right) T_j(z_2)T_{i+1}(x^{-1}z_1)\\
& - c(r,x)\delta\left(\frac{x^{i-j}z_2}{z_1}\right) f_{i,j+1}(x^{-i+j+1})T_i(z_1)T_{j+1}(x^{-1}z_2)\\
& + c(r,x)\delta\left(\frac{x^{i+j+2}z_2}{z_1}\right)\prod_{l=1}^{i}\Delta_1(x^{2l+1})T_{i+j+1}(x^{i+1}z_2).
\end{aligned} \tag{86}$$

Multiplying $\mathrm{RHS}_{i,j}(i)$ by $f_{1,i}\left(z_1/z_3\right) f_{1,j}\left(z_2/z_3\right) T_1(z_3)$ from the left and using fusion relation (75) gives

$$c(r,x)\prod_{l=1}^{i-1}\Delta_1(x^{2l+1})\left(\delta\left(\frac{x^{-i-j}z_2}{z_1}\right)f_{1,i+1}\left(\frac{x^j z_1}{z_3}\right)\Delta_1\left(\frac{x^i z_1}{z_3}\right)T_1(z_3)T_{i+j}(x^j z_1)\right.$$
$$\left. - \ \delta\left(\frac{x^{i+j}z_2}{z_1}\right)f_{1,i+1}\left(\frac{x^{-j}z_1}{z_3}\right)\Delta_1\left(\frac{x^{-i}z_1}{z_3}\right)T_1(z_3)T_{i+j}(x^{-j}z_1)\right). \tag{87}$$

Taking the limit $z_3 \to x^{-i-1}z_1$ of (87) multiplied by $c(r,x)^{-1}\left(1-x^{-i-1}z_1/z_3\right)$ and using fusion relation (81) along with the relation $\lim_{z_3\to x^{-i-1}z_1}\left(1-x^{-i-1}z_1/z_3\right)\Delta_1\left(x^{-i}z_1/z_3\right)=c(r,x)$ gives

$$c(r,x)\delta\left(\frac{x^{-i-j}z_2}{z_1}\right)\prod_{l=1}^{i}\Delta_1(x^{2l+1})T_{i+j+1}(x^{-i-1}z_2)$$
$$-\ c(r,x)\delta\left(\frac{x^{i+j}z_2}{z_1}\right)\prod_{l=1}^{i-1}\Delta_1(x^{2l+1})f_{1,i+j}(x^{i-j+1})T_1(x^{-i-1}z_1)T_{i+j}(x^i z_2). \tag{88}$$

Multiplying $\mathrm{RHS}_{i,j}(k)$ $(1\le k\le i-1)$ by $f_{1,i}\left(z_1/z_3\right) f_{1,j}\left(z_2/z_3\right) T_1(z_3)$ from the left and using fusion relation (76) along with $f_{i-k,j+k}(x^{j-i})T_{i-k}(x^k z_1)T_{j+k}(x^{j-i+k}z_1) = f_{j+k,i-k}(x^{i-j})T_{j+k}(x^{j-i+k}z_1)T_{i-k}(x^k z_1)$ in (84) gives

$$c(r,x)\prod_{l=1}^{k-1}\Delta_1(x^{2l+1}) \tag{89}$$
$$\times\left(\delta\left(\frac{x^{-j+i-2k}z_2}{z_1}\right)f_{1,i-k}\left(\frac{x^k z_1}{z_3}\right)f_{j+k,i-k}(x^{i-j})f_{1,j+k}\left(\frac{x^{-i+j+k}z_1}{z_3}\right)T_1(z_3)T_{j+k}(x^{j-i+k}z_1)T_{i-k}(x^k z_1)\right.$$
$$\left. -\ \delta\left(\frac{x^{j-i+2k}z_2}{z_1}\right)f_{1,i-k}\left(\frac{x^{-k}z_1}{z_3}\right)f_{i-k,j+k}(x^{i-j})f_{1,j+k}\left(\frac{x^{i-j-k}z_1}{z_3}\right)T_1(z_3)T_{i-k}(x^{-k}z_1)T_{j+k}(x^k z_2)\right).$$

Taking the limit $z_3 \to x^{-i-1}z_1$ of (89) multiplied by $c(r,x)^{-1}\left(1-x^{-i-1}z_1/z_3\right)$ and using fusion relations (74) and (81) along with

$$f_{i-k+1,j+k}(x^{i-j+1})T_{i-k+1}(x^{-k-1}z_1)T_{j+k}(x^{-j+i-k}z_1) = f_{j+k,i-k+1}(x^{j-i-1})T_{j+k}(x^{-j+i-k}z_1)T_{i-k+1}(x^{-k-1}z_1)$$

in (84) gives

$$c(r,x)\prod_{l=1}^{k}\Delta_1(x^{2l+1})\delta\left(\frac{x^{-j+i-2k}z_2}{z_1}\right)f_{j+k-1,i-k}(x^{i-j+1})T_{i-k}(x^k z_1)T_{j+k+1}(x^{-k-1}z_2)$$
$$-\ c(r,x)\prod_{l=1}^{k-1}\Delta_1(x^{2l+1})\delta\left(\frac{x^{j-i+2k}z_2}{z_1}\right)f_{i-k+1,j+k}(x^{i-j+1})T_{i-k+1}(x^{-k-1}z_1)T_{j+k}(x^k z_2). \tag{90}$$

Summing (86), (88), and (90) for $1\le k\le i-1$ and shifting the variable $z_1 \mapsto xz_1$ gives $\mathrm{LHS}_{i+1,j} = \sum_{k=1}^{i+1}\mathrm{RHS}_{i+1,j}(k)$. By induction on i, we have shown quadratic relation (71). □

C Classical limit

The deformed W-algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ yields a q-Poisson W-algebra in the classical limit. We set parameters $q = x^{2r}$ and $\beta = (r-1)/r$. We obtain a q-Poisson W-algebra [11–13] in the classical limit $\beta \to 0$ with q

fixed. The defining relations of the deformed W-superalgebra $\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$ are given by

$$[T_i[m],T_j[n]]=-\sum_{l=1}^{\infty}f_{i,j}^{l}\left(T_i[m-l]T_j[n+l]-T_j[n-l]T_i[m+l]\right)$$
$$+c(r,x)\sum_{k=1}^{i}\prod_{l=1}^{k-1}\Delta_1(x^{2l+1})$$
$$\times\sum_{l\in\mathbf{Z}}\Big(f_{i-k,j+k}(x^{j-i})x^{(j-i)l+k(m-n)+4kl}T_{i-k}[m-l]T_{j+k}[n+l]$$
$$-f_{i-k,j+k}(x^{-j+i})x^{(i-j)l+k(n-m)-4kl}T_{i-k}[m-l]T_{j+k}[n+l]\Big).$$

We define $f_{i,j}^{l}$ by $f_{i,j}(z)=\sum_{l=0}^{\infty}f_{i,j}^{l}z^{l}$, where the structure functions $f_{i,j}(z)$ are given in (67). We define the q-Poisson bracket $\{,\}$ by taking the classical limit $\beta\to 0$ with q fixed as

$$\{T_i^{\rm PB}[m],T_j^{\rm PB}[n]\}=-\lim_{\beta\to 0}\frac{1}{\beta\log q}[T_i[m],T_j[n]].$$

Here, we set $T_i^{PB}[m]$ as

$$T_i(z)=\sum_{m\in\mathbf{Z}}T_i[m]z^{-m}\longrightarrow T_i^{PB}(z)=\sum_{m\in\mathbf{Z}}T_i^{PB}[m]z^{-m}\quad(\beta\to 0,\ q\ \text{fixed}).$$

The β-expansions of the structure functions are given as

$$f_{i,j}(z)=1+\beta\log q\sum_{m=1}^{\infty}\frac{\left[\frac{1}{2}{\rm Min}(i,j)m\right]_q\left[\left(\frac{1}{2}{\rm Max}(i,j)-1\right)m\right]_q}{[m]_q}(q-q^{-1})+O(\beta^2)\quad(i,j\geq 1),$$
$$c(r,x)=-\beta\log q+O(\beta^2).$$

We obtain the following Proposition.

Proposition III.8 *For the q-Poisson W-superalgebra for $\mathfrak{sl}(2|1)$ the generating function $T_i^{PB}(z)$ satisfies*

$$\{T_i^{PB}(z_1),T_j^{PB}(z_2)\}=(q-q^{-1})C_{i,j}\left(\frac{z_2}{z_1}\right)T_i^{PB}(z_1)T_j^{PB}(z_2)$$
$$+\sum_{k=1}^{i}\delta\left(\frac{q^{\frac{-j+i}{2}-k}z_2}{z_1}\right)T_{i-k}^{PB}(q^{\frac{k}{2}}z_1)T_{j+k}^{PB}(q^{-\frac{k}{2}}z_2)$$
$$-\sum_{k=1}^{i}\delta\left(\frac{q^{\frac{j-i}{2}+k}z_2}{z_1}\right)T_{i-k}^{PB}(q^{-\frac{k}{2}}z_1)T_{j+k}^{PB}(q^{\frac{k}{2}}z_2)\quad(1\leq i\leq j).$$

Here $T_0^{PB}(z)$ in the right hand side is understood as $T_0^{PB}(z)=1$. Here we set the structure functions $C_{i,j}(z)$ $(i,j\geq 1)$ as

$$C_{i,j}(z)=\sum_{m\in\mathbf{Z}}\frac{\left[\frac{1}{2}{\rm Min}(i,j)m\right]_q\left[\left(\frac{1}{2}{\rm Max}(i,j)-1\right)m\right]_q}{[m]_q}z^{m}\quad(i,j\geq 1).$$

The structure functions satisfy $C_{i,2}(z)=C_{2,i}(z)=0$ $(i=1,2)$.

IV Conclusion and Discussion

We revisited Ding-Feigin's construction of the deformed W-algebras $\mathcal{W}_{q,t}(\mathfrak{sl}(3))$ and $\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$. Using Ding-Feigin realization we introduced the higher W-currents $T_i(z)$ for the deformed W-superalgebra

$\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$. We obtained a set of quadratic relations of $T_i(z)$, which is independent of the choice of Dynkin diagrams for the superalgebra $\mathfrak{sl}(2|1)$, though the screening currents depend on it. There is an infinite number of quadratic relations for an infinite number of the currents $T_i(z)$ for $\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$. The higher W-currents $T_i(z)$ can also be constructed by repeating fusion from $T_1(z)$ as

$$T_i(z) = \frac{1}{c(r,x)} \lim_{w \to x^{-1}z} \left(1 - \frac{x^{-1}z}{w}\right) f_{i-1,1}\left(\frac{x^{i-1}z}{w}\right) T_{i-1}(w) T_1(x^{i-1}z) \quad (i \geq 2), \tag{91}$$

which is a special case of lemma III.5. For $\mathcal{W}_{q,t}(\mathfrak{sl}(3))$, as for $\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$, the higher W-currents $T_i(z)$ can be formally defined by (91). However, with the truncations

$$\lim_{z_1 \to x^{-2}z_2} \left(1 - \frac{x^{-2}z_2}{z_1}\right) f_{1,1}\left(\frac{z_2}{z_1}\right) \Lambda_k(z_1)\Lambda_k(z_2) = 0 \quad (1 \leq k \leq 3), \quad :\Lambda_1(x^{-2}z)\Lambda_2(z)\Lambda_3(x^2z) := 1,$$

we have $T_3(z) = 1$ and $T_i(z) = 0$ $(i \geq 4)$. We get only $T_1(z)$ and $T_2(z)$ from (91) for $\mathcal{W}_{q,t}(\mathfrak{sl}(3))$.

It seems to be possible to extend Ding-Feigin construction to the case of many fermions which will give a higher rank generalization $\mathcal{W}_{q,t}(\mathfrak{sl}(M|N))$, and obtain its quadratic relations. We expect to report it in the near future. It is still an open problem to find quadratic relations of the deformed W-algebra $\mathcal{W}_{q,t}(\mathfrak{g})$, except for $\mathfrak{g} = \mathfrak{sl}(N)$, $A_2^{(2)}$, and $\mathfrak{sl}(2|1)$. It seems to be possible to extend Ding-Feigin construction to other superalgebras and obtain their quadratic relations.

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Appendix

A Fusion relations

In this appendix we summarize the fusion relations of $\Lambda_i(z)$ for Case 2 which are obtained from Proposition III.4. The relations for Case 3 are given in the same way. We use the abbreviations

$$\Lambda_3^{(i)}(z) =: \prod_{l=1}^{i} \Lambda_3(x^{-i-1+2l}z) : \quad (i \geq 1),$$

$$\Lambda_{k,3}^{(i)}(z) = \begin{cases} :\Lambda_k(x^{-i+1}z) \prod_{l=1}^{i-1} \Lambda_3(x^{-i+1+2l}z) : & (i \geq 2), \\ \Lambda_k(z) & (i = 1) \end{cases} \quad (1 \leq k \leq 2), \tag{A 1}$$

$$\Lambda_{1,2,3}^{(i)}(z)=\begin{cases} :\Lambda_1(x^{-i+1}z)\Lambda_2(x^{-i+3}z)\prod_{l=1}^{i-2}\Lambda_3(x^{-i+3+2l}z): & (i\geq 3),\\ :\Lambda_1(x^{-1}z)\Lambda_2(xz): & (i=2),\\ 0 & (i=1).\end{cases}$$

We set

$$F_{i,j}^{(\pm)}(z)=(1-x^{\pm(i+j)}z)f_{i,j}(z).$$

- For $k=1,2$, we have

$$\lim_{z_1\to x^{\pm(i+j)}z_2}F_{i,j}^{(\pm)}\left(\frac{z_2}{z_1}\right)\Lambda_3^{(i)}(z_1)\Lambda_3^{(j)}(z_2)=\mp\frac{c(r,x)d_{i+j}(r,x)}{d_i(r,x)d_j(r,x)}\prod_{l=1}^{\mathrm{Min}(i,j)-1}\Delta_1(x^{2l+1})\Lambda_3^{(i+j)}(x^{\pm i}z_2)$$
$$(i,j\geq 1),\quad \text{(A 2)}$$

$$\lim_{z_1\to x^{i+j}z_2}F_{i,j}^{(+)}\left(\frac{z_2}{z_1}\right)\Lambda_3^{(i)}(z_1)\Lambda_{1,2,3}^{(j)}(z_2)=-\frac{c(r,x)d_{i+j-2}(r,x)}{d_i(r,x)d_{j-2}(r,x)}\prod_{l=1}^{\mathrm{Min}(i,j)-1}\Delta_1(x^{2l+1})\Lambda_{1,2,3}^{(i+j)}(x^{i}z_2)$$
$$(i\geq 1,j\geq 2),\quad \text{(A 3)}$$

$$\lim_{z_1\to x^{i+j}z_2}F_{i,j}^{(+)}\left(\frac{z_2}{z_1}\right)\Lambda_3^{(i)}(z_1)\Lambda_{k,3}^{(j)}(z_2)=-\frac{c(r,x)d_{i+j-1}(r,x)}{d_i(r,x)d_{j-1}(r,x)}\prod_{l=1}^{\mathrm{Min}(i,j)-1}\Delta_1(x^{2l+1})\Lambda_{k,3}^{(i+j)}(x^{i}z_2)$$
$$(i,j\geq 1),\quad \text{(A 4)}$$

$$\lim_{z_1\to x^{-(i+j)}z_2}F_{i,j}^{(-)}\left(\frac{z_2}{z_1}\right)\Lambda_{1,2,3}^{(i)}(z_1)\Lambda_3^{(j)}(z_2)=\frac{c(r,x)d_{i+j-2}(r,x)}{d_{i-2}(r,x)d_j(r,x)}\prod_{l=1}^{\mathrm{Min}(i,j)-1}\Delta_1(x^{2l+1})\Lambda_{1,2,3}^{(i+j)}(x^{-i}z_2)$$
$$(i\geq 2,j\geq 1),\quad \text{(A 5)}$$

$$\lim_{z_1\to x^{-(i+j)}z_2}F_{i,j}^{(-)}\left(\frac{z_2}{z_1}\right)\Lambda_{k,3}^{(i)}(z_1)\Lambda_3^{(j)}(z_2)=\frac{c(r,x)d_{i+j-1}(r,x)}{d_{i-1}(r,x)d_j(r,x)}\prod_{l=1}^{\mathrm{Min}(i,j)-1}\Delta_1(x^{2l+1})\Lambda_{k,3}^{(i+j)}(x^{-i}z_2)$$
$$(i,j\geq 1).\quad \text{(A 6)}$$

- As exceptional formulae of $\Lambda_3^{(i)}(z)$, $\Lambda_{k,3}^{(i)}(z)$, and $\Lambda_{1,2,3}^{(i)}(z)$ for small i, we have

$$\lim_{z_1\to x^{(j+1)}z_2}F_{j,1}^{(+)}\left(\frac{z_2}{z_1}\right)\Lambda_{2,3}^{(j)}(z_1)\Lambda_1(z_2)=-c(r,x)\Lambda_{1,2,3}^{(j+1)}(x^jz_2)\quad (j\geq 2),\quad \text{(A 7)}$$

$$\lim_{z_1\to x^{-(j+1)}z_2}F_{1,j}^{(-)}\left(\frac{z_2}{z_1}\right)\Lambda_1(z_1)\Lambda_{2,3}^{(j)}(z_2)=c(r,x)\Lambda_{1,2,3}^{(j+1)}(x^{-1}z_2)\quad (j\geq 2),\quad \text{(A 8)}$$

$$\lim_{z_1\to x^{\pm 2}z_2}F_{1,1}^{(\pm)}\left(\frac{z_2}{z_1}\right)\Lambda_k(z_1)\Lambda_k(z_2)=0\quad (k=1,2).\quad \text{(A 9)}$$

For $1\leq k<l\leq 3$, we have

$$\lim_{z_1\to x^2z_2}F_{1,1}^{(+)}\left(\frac{z_2}{z_1}\right)\Lambda_k(w_1)\Lambda_l(w_2)=0,\quad \lim_{z_1\to x^{-2}z_2}F_{1,1}^{(-)}\left(\frac{z_2}{z_1}\right)\Lambda_l(z_1)\Lambda_k(z_2)=0,\quad \text{(A 10)}$$

$$\lim_{z_1\to x^2z_2}F_{1,1}^{(+)}\left(\frac{z_2}{z_1}\right)\Lambda_l(z_1)\Lambda_k(z_2)=-c(r,x):\Lambda_k(z_2)\Lambda_l(x^2z_2):,\quad \text{(A 11)}$$

$$\lim_{z_1\to x^{-2}z_2}F_{1,1}^{(-)}\left(\frac{z_2}{z_1}\right)\Lambda_k(z_1)\Lambda_l(z_2)=c(r,x):\Lambda_k(x^{-2}z_2)\Lambda_l(z_2):.\quad \text{(A 12)}$$

The remaining fusions vanish.

B Exchange relations

In this appendix we give the exchange relations of $\Lambda_i(z)$ for Case 2 which are obtained from Proposition III.4, (73), and (77). The relations for Case 3 are obtained in the same way. We use the abbreviations (A 1) and set

$$f_{i,j}\left(\mathcal{A}(z_1),\mathcal{B}(z_2)\right)_{1\leftrightarrow 2}=f_{i,j}\left(\frac{z_2}{z_1}\right)\mathcal{A}(z_1)\mathcal{B}(z_2)-f_{j,i}\left(\frac{z_1}{z_2}\right)\mathcal{B}(z_2)\mathcal{A}(z_1).$$

For $i\geq 1$ and $k=1,2$, we have

$$f_{1,i}\left(\Lambda_k(z_1),\Lambda^{(i)}_{1,2,3}(z_2)\right)_{1\leftrightarrow 2}=0,\quad f_{1,i}\left(\Lambda_k(z_1),\Lambda^{(i)}_{k,3}(z_2)\right)_{1\leftrightarrow 2}=0, \tag{B 1}$$

$$f_{1,i}\left(\Lambda_1(z_1),\Lambda^{(i)}_{2,3}(z_2)\right)_{1\leftrightarrow 2}=c(r,x)\left(\delta\left(\frac{x^{-i-1}z_2}{z_1}\right)-\delta\left(\frac{x^{-i+1}z_2}{z_1}\right)\right):\Lambda_1(z_1)\Lambda^{(i)}_{2,3}(z_2):, \tag{B 2}$$

$$f_{1,i}\left(\Lambda_k(z_1),\Lambda^{(i)}_{3}(z_2)\right)_{1\leftrightarrow 2}=c(r,x)\left(\delta\left(\frac{x^{-i-1}z_2}{z_1}\right)-\delta\left(\frac{x^{-i+1}z_2}{z_1}\right)\right):\Lambda_k(z_1)\Lambda^{(i)}_{3}(z_2):, \tag{B 3}$$

$$f_{1,i}\left(\Lambda_3(z_1),\Lambda^{(i)}_{3}(z_2)\right)_{1\leftrightarrow 2}=\frac{c(r,x)d_{i+1}(r,x)}{d_1(r,x)d_i(r,x)}\left(\delta\left(\frac{x^{-i-1}z_2}{z_1}\right)-\delta\left(\frac{x^{i+1}z_2}{z_1}\right)\right):\Lambda_3(z_1)\Lambda^{(i)}_{3}(z_2):, \tag{B 4}$$

$$f_{1,i}\left(\Lambda_2(z_1),\Lambda^{(i)}_{1,3}(z_2)\right)_{1\leftrightarrow 2}=c(r,x)\left(\delta\left(\frac{x^{-i+1}z_2}{z_1}\right)-\delta\left(\frac{x^{-i+3}z_2}{z_1}\right)\right):\Lambda_2(z_1)\Lambda^{(i)}_{1,3}(z_2):, \tag{B 5}$$

$$f_{1,i}\left(\Lambda_3(z_1),\Lambda^{(i)}_{k,3}(z_2)\right)_{1\leftrightarrow 2}=\frac{c(r,x)d_{i}(r,x)}{d_1(r,x)d_{i-1}(r,x)}\left(\delta\left(\frac{x^{-i+1}z_2}{z_1}\right)-\delta\left(\frac{x^{i+1}z_2}{z_1}\right)\right):\Lambda_3(z_1)\Lambda^{(i)}_{k,3}(z_2):. \tag{B 6}$$

For $i\geq 2$, we have

$$f_{1,i}\left(\Lambda_3(z_1),\Lambda^{(i)}_{1,2,3}(z_2)\right)_{1\leftrightarrow 2}=\frac{c(r,x)d_{i-1}(r,x)}{d_1(r,x)d_{i-2}(r,x)}\left(\delta\left(\frac{x^{-i+3}z_2}{z_1}\right)-\delta\left(\frac{x^{i+1}z_2}{z_1}\right)\right):\Lambda_3(z_1)\Lambda^{(i)}_{1,2,3}(z_2): \tag{B 7}$$

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