Wakimoto realization of the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$

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Abstract A bosonization of the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$ is presented for an arbitrary level $k \in \mathbf{C}$. The Wakimoto realization is given by using $\xi-\eta$ system. The screening operators that commute with $U_q(\widehat{sl}(M|N))$ are presented for the level $k \neq -M+N$. New bosonization of the affine superalgebra $\widehat{sl}(M|N)$ is obtained in the limit $q \to 1$.

1 Introduction

Bosonization is a powerful method to study representation theory and its application to mathematical physics [1]. Wakimoto realization is the bosonization that provides a bridge between representation theory of affine algebras and the geometry of the semi-infinite flag manifold. The Wakimoto realizations have been constructed for the affine Lie algebra $g = (ADE)^{(r)}$ $(r=1,2)$, $(BCFG)^{(1)}$ and $\widehat{sl}(M|N)$, $osp(2|2)^{(2)}$, $D(2,1,a)^{(1)}$ [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. They have been used to construct correlation functions of WZW models, in the study of Drinfeld-Sokolov reduction and W-algebras. It's nontrivial to give quantum deformation of Wakimoto realization as the same as quantum Drinfeld-Sokolov reduction and quantum W-algebras. The quantum Wakimoto realizations have been constructed only for $U_q(\widehat{sl}(N))$ and $U_q(\widehat{sl}(2|1))$ [13, 14, 15, 16, 17]. In this paper we study a higher-rank generalization of the previous works for the quantum affine superalgebra $U_q(\widehat{sl}(2|1))$. We give a bosonization of the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$ for an arbitrary level $k \in \mathbf{C}$, and give the Wakimoto realization using $\xi-\eta$ system. We give the screening operators that commute with $U_q(\widehat{sl}(M|N))$ for the level $k \neq -M+N$. Taking the limit $q \to 1$, we

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obtain new bosonization of the affine superalgebra $\widehat{sl}(M|N)$. This paper is a shorter review of the papers [18, 19, 20, 21].

2 Quantum affine superalgebra $U_q(\widehat{sl}(M|N))$

In this Section we recall the definition of the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$ for $M,N=1,2,3,\cdots$. Throughout this paper, $q\in\mathbf{C}$ is assumed to be $0<|q|<1$. For any integer n, define $[n]_q=\frac{q^n-q^{-n}}{q-q^{-1}}$. We set $\nu_i=+1$ $(1\leq i\leq M)$, $\nu_i=-1$ $(M+1\leq i\leq M+N)$ and $\nu_0=-1$. The Cartan matrix $(A_{i,j})_{0\leq i,j\leq M+N-1}$ of the affine Lie superalgebra $\widehat{sl}(M|N)$ is given by

$$A_{i,j}=(\nu_i+\nu_{i+1})\delta_{i,j}-\nu_i\delta_{i,j+1}-\nu_{i+1}\delta_{i+1,j}.$$

The quantum affine superalgebra $U_q(\widehat{sl}(M|N))$ [22] is the associative algebra over $\mathbf{C}$ with the generators $X_m^{\pm,i}$ $(i=1,2,\cdots,M+N-1, m\in\mathbf{Z})$, H_n^i $(i=1,2,\cdots,M+N-1, n\in\mathbf{Z}_{\neq 0})$, H^i $(i=1,2,\cdots,M+N-1)$, and c. The $\mathbf{Z}_2$-grading of the generators is given by $p(X_m^{\pm,M})\equiv 1 \pmod 2$ for $m\in\mathbf{Z}$ and zero otherwise. The defining relations of the generators are given as follows.

$$\begin{aligned}
&c : \text{central element},\\
&[H^i,H_m^j]=0,\quad [H_m^i,H_n^j]=\frac{[A_{i,j}m]_q[cm]_q}{m}\delta_{m+n,0},\\
&[H^i,X^{\pm,j}(z)]=\pm A_{i,j}X^{\pm,j}(z),\\
&[H_m^i,X^{\pm,j}(z)]=\pm\frac{[A_{i,j}m]_q}{m}q^{\mp\frac{c}{2}|m|}z^mX^{\pm,j}(z),\\
&(z_1-q^{\pm A_{i,j}}z_2)X^{\pm,i}(z_1)X^{\pm,j}(z_2)=(q^{\pm A_{j,i}}z_1-z_2)X^{\pm,j}(z_2)X^{\pm,i}(z_1),\quad \text{for } |A_{i,j}|\neq 0,\\
&[X^{\pm,i}(z_1),X^{\pm,j}(z_2)]=0\quad \text{for } |A_{i,j}|=0,\\
&[X^{+,i}(z_1),X^{-,j}(z_2)]=\frac{\delta_{i,j}}{(q-q^{-1})z_1z_2}\left(\delta(q^cz_2/z_1)\Psi_+^i(q^{\frac{c}{2}}z_2)-\delta(q^{-c}z_2/z_1)\Psi_-^i(q^{-\frac{c}{2}}z_2)\right),\\
&[X^{\pm,i}(z_1),[X^{\pm,i}(z_2),X^{\pm,j}(z)]_{q^{-1}}]_q+(z_1\leftrightarrow z_2)=0\quad \text{for } |A_{i,j}|=1,\ i\neq M,\\
&[X^{\pm,M}(z_1),[X^{\pm,M+1}(w_1),[X^{\pm,M}(z_2),X^{\pm,M-1}(w_2)]_{q^{-1}}]_q]+(z_1\leftrightarrow z_2)=0,
\end{aligned}$$

where we use

$$[X,Y]_a=XY-(-1)^{p(X)p(Y)}aYX,$$

for homogeneous elements $X,Y\in U_q(\widehat{sl}(M|N))$. For simplicity we write $[X,Y]=[X,Y]_1$. Here we set $\delta(z)=\sum_{m\in\mathbf{Z}}z^m$ and the generating functions

$$X^{\pm,j}(z) = \sum_{m\in\mathbf{Z}} X_m^{\pm,j} z^{-m-1},$$

$$\Psi_\pm^i(q^{\pm\frac{c}{2}}z) = q^{\pm h_i} \exp\left(\pm(q-q^{-1})\sum_{m>0} H_{\pm m}^i z^{\mp m}\right).$$

The multiplication rule for the tensor product is $\mathbf{Z}_2$-graded and is defined for homogeneous elements $X_1, X_2, Y_1, Y_2 \in U_q(\widehat{sl}(M|N))$ by $(X_1 \otimes Y_1)(X_2 \otimes Y_2) = (-1)^{p(Y_1)p(X_2)}(X_1X_2 \otimes Y_1Y_2)$, which extends to inhomogeneous elements through linearity.

Let $\bar{\alpha}_i$, $\bar{\Lambda}_i$ $(1 \le i \le M+N-1)$ be the classical simple roots, the classical fundamental weights, respectively. Let $(\cdot|\cdot)$ be the symmetric bilinear form satisfying $(\bar{\alpha}_i|\bar{\alpha}_j) = A_{i,j}$ and $(\bar{\Lambda}_i|\bar{\alpha}_j) = \delta_{i,j}$ for $1 \le i,j \le M+N-1$. Let us introduce the affine weight Λ_0 and the null root δ satisfying $(\Lambda_0|\Lambda_0) = (\delta|\delta) = 0$, $(\Lambda_0|\delta) = 1$, and $(\Lambda_0|\bar{\alpha}_i) = (\Lambda_0|\bar{\Lambda}_i) = 0$ for $1 \le i \le M+N-1$. The other affine weights and the affine roots are given by $\Lambda_i = \bar{\Lambda}_i + \Lambda_0$, $\alpha_i = \bar{\alpha}_i$ for $1 \le i \le M+N-1$, and $\alpha_0 = \delta - \sum_{i=1}^{M+N-1}\alpha_i$. Let $V(\lambda)$ be the highest-weight module over $U_q(\widehat{sl}(M|N))$ generated by the highest weight vector $|\lambda\rangle \neq 0$ such that

$$H_m^i|\lambda\rangle = X_m^{\pm,i}|\lambda\rangle = 0 \quad (m>0),$$
$$X_0^{+,i}|\lambda\rangle = 0, \quad H^i|\lambda\rangle = l_i|\lambda\rangle,$$

where the classical part of the highest weight is $\bar{\lambda} = \sum_{i=1}^{M+N-1} l_i\bar{\Lambda}_i$.

3 Bosonization of $U_q(\widehat{sl}(M|N))$

In this Section we give a bosonization of $U_q(\widehat{sl}(M|N))$ for an arbitrary level $k \in \mathbf{C}$.

3.1 Boson

We introduce bosons a_m^i $(m \in \mathbf{Z}, 1 \le i \le M+N-1)$, $b_m^{i,j}$ $(m \in \mathbf{Z}, 1 \le i < j \le M+N)$, $c_m^{i,j}$ $(m \in \mathbf{Z}, 1 \le i < j \le M+N)$, and zero mode operators Q_a^i $(1 \le i \le M+N-1)$, $Q_b^{i,j}$ $(1 \le i < j \le M+N)$, $Q_c^{i,j}$ $(1 \le i < j \le M+N)$. Their commutation relations are

$$[a_m^i, a_n^j] = \frac{1}{m}[(k+g)m]_q[A_{i,j}m]_q\delta_{m+n,0}, \quad [a_0^i, Q_a^j] = (k+g)A_{i,j},$$

$$[b_m^{i,j}, b_n^{i',j'}] = -\nu_i\nu_j\frac{1}{m}[m]_q^2\delta_{i,i'}\delta_{j,j'}\delta_{m+n,0}, \quad [b_0^{i,j}, Q_b^{i',j'}] = -\nu_i\nu_j\delta_{i,i'}\delta_{j,j'},$$

$$[c_m^{i,j}, c_n^{i',j'}] = \nu_i \nu_j \frac{1}{m} [m]_q^2 \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, \quad [c_0^{i,j}, Q_c^{i',j'}] = \nu_i \nu_j \delta_{i,i'} \delta_{j,j'},$$
$$[Q_b^{i,j}, Q_b^{i',j'}] = \pi\sqrt{-1} \quad (\nu_i \nu_j = \nu_{i'} \nu_{j'} = -1).$$

The remaining commutators vanish. Here $g = M - N$ stands for the dual Coxeter number. We define free boson fields $b_\pm^{i,j}(z), b^{i,j}(z)$ as follows.

$$b_\pm^{i,j}(z) = \pm(q - q^{-1}) \sum_{m>0} b_{\pm m}^{i,j} z^{\mp m} \pm b_0^{i,j} \log q,$$
$$b^{i,j}(z) = -\sum_{m \neq 0} \frac{b_m^{i,j}}{[m]_q} z^{-m} + Q_b^{i,j} + b_0^{i,j} \log z.$$

Free boson fields $a_\pm^i(z), c^{i,j}(z)$ are defined in the same way. We define free boson fields $(\Delta_L^\varepsilon b_\pm^{i,j})(z), (\Delta_R^\varepsilon b_\pm^{i,j})(z)$ $(\varepsilon = \pm, 0)$ as follows.

$$(\Delta_L^\varepsilon b_\pm^{i,j})(z) = \begin{cases} b_\pm^{i+1,j}(q^\varepsilon z) - b_\pm^{i,j}(z) & (\varepsilon = \pm), \\ b_\pm^{i+1,j}(z) + b_\pm^{i,j}(z) & (\varepsilon = 0), \end{cases}$$
$$(\Delta_R^\varepsilon b_\pm^{i,j})(z) = \begin{cases} b_\pm^{i,j+1}(q^\varepsilon z) - b_\pm^{i,j}(z) & (\varepsilon = \pm), \\ b_\pm^{i,j+1}(z) + b_\pm^{i,j}(z) & (\varepsilon = 0). \end{cases}$$

We define free boson fields with parameters $L_1, \cdots, L_r, M_1, \cdots, M_r, \alpha$ as follows.

$$\left(\frac{L_1}{M_1} \frac{L_2}{M_2} \cdots \frac{L_r}{M_r} \ a^i \right)(z; \alpha)$$
$$= -\sum_{m \neq 0} \frac{[L_1 m]_q [L_2 m]_q \cdots [L_r m]_q}{[M_1 m]_q [M_2 m]_q \cdots [M_r m]_q} \frac{a_m^i}{[m]_q} q^{-\alpha |m|} z^{-m} + \frac{L_1 L_2 \cdots L_r}{M_1 M_2 \cdots M_r} (Q_a^i + a_0^i \log z).$$

Normal ordering rules are defined as follows.

$$: b_m^{i,j} b_n^{i',j'} :=: b_n^{i',j'} b_m^{i,j} := \begin{cases} b_m^{i,j} b_n^{i',j'} & (m < 0), \\ b_n^{i',j'} b_m^{i,j} & (m > 0), \end{cases}$$
$$: Q_b^{i,j} Q_b^{i',j'} :=: Q_b^{i',j'} Q_b^{i,j} := Q_b^{i,j} Q_b^{i',j'} \quad (i > i' \ \text{ or } \ i = i', j > j').$$

Normal ordering rules of a_m^i, $c_m^{i,j}$ and $Q_c^{i,j}$ are defined in the same way.

3.2 Bosonization

We define bosonic operators $\Psi_\pm^i(z)$ $(1 \leq i \leq M + N - 1)$ as follows.

$$\Psi_\pm^i(q^{\pm \frac{k}{2}} z) = : e^{a_\pm^i (q^{\pm \frac{g}{2}} z) + \sum_{l=1}^i (\Delta_R^\mp b_\pm^{l,i})(q^{\pm(\frac{k}{2}+l)} z) - \sum_{l=i+1}^M (\Delta_L^\mp b_\pm^{i,l})(q^{\pm(\frac{k}{2}+l)} z)}$$

$$\times\, e^{-\sum_{l=M+1}^{M+N}(\Delta_L^{\mp}b_\pm^{i,l})(q^{\pm(\frac{k}{2}+2M+1-l)}z)}: \quad (1\le i\le M-1), \tag{1}$$

$$\Psi_\pm^M(q^{\pm\frac{k}{2}}z)=:e^{a_\pm^M(q^{\pm\frac{g}{2}}z)-\sum_{l=1}^{M-1}(\Delta_R^0 b_\pm^{l,M})(q^{\pm(\frac{k}{2}+l)}z)+\sum_{l=M+2}^{M+N}(\Delta_L^0 b_\pm^{M,l})(q^{\pm(\frac{k}{2}+2M+1-l)}z)}:, \tag{2}$$

$$\Psi_\pm^i(q^{\pm\frac{k}{2}}z)=:e^{a_\pm^i(q^{\pm\frac{g}{2}}z)-\sum_{l=1}^{M}(\Delta_R^{\pm}b_\pm^{l,i})(q^{\pm(\frac{k}{2}+l-1)}z)-\sum_{l=M+1}^{i}(\Delta_R^{\pm}b_\pm^{l,i})(q^{\pm(\frac{k}{2}+2M-l)}z)}$$
$$\times\, e^{\sum_{l=i+1}^{M+N}(\Delta_L^{\pm}b_\pm^{i,l})(q^{\pm(\frac{k}{2}+2M-l)}z)}: \quad (M+1\le i\le M+N-1). \tag{3}$$

We define bosonic operators $X^{\pm,i}(z)$ $(1\le i\le M+N-1)$ as follows.

$$X^{+,i}(z)=\sum_{j=1}^{i}\frac{c_{i,j}}{(q-q^{-1})z}(E_{i,j}^+(z)-E_{i,j}^-(z)) \quad (1\le i\le M-1), \tag{4}$$

$$X^{+,M}(z)=\sum_{j=1}^{M}c_{M,j}E_{M,j}(z), \tag{5}$$

$$X^{+,i}(z)=\sum_{j=1}^{M}c_{i,j}E_{i,j}(z)+\sum_{j=M+1}^{i}\frac{c_{i,j}}{(q-q^{-1})z}(E_{i,j}^+(z)-E_{i,j}^-(z))$$
$$(M+1\le i\le M+N-1), \tag{6}$$

$$X^{-,i}(z)=\sum_{j=1}^{i-1}\frac{d_{i,j}^1}{(q-q^{-1})z}(F_{i,j}^{1,-}(z)-F_{i,j}^{1,+}(z))+\frac{d_{i,i}^2}{(q-q^{-1})z}(F_{i,i}^{2,-}(z)-F_{i,i}^{2,+}(z))$$
$$+\sum_{j=i+2}^{M}\frac{d_{i,j}^3}{(q-q^{-1})z}(F_{i,j}^{3,-}(z)-F_{i,j}^{3,+}(z))+\sum_{j=M+1}^{M+N}d_{i,j}^3F_{i,j}^3(z)$$
$$(1\le i\le M-1), \tag{7}$$

$$X^{-,M}(z)=\sum_{j=1}^{M-1}\frac{d_{M,j}^1}{(q-q^{-1})z}(F_{M,j}^{1,-}(z)-F_{M,j}^{1,+}(z))+\frac{d_{M,M}^2}{(q-q^{-1})z}(F_{M,M}^{2,-}(z)-F_{M,M}^{2,+}(z))$$
$$+\sum_{j=M+2}^{M+N}\frac{d_{M,j}^3}{(q-q^{-1})z}(F_{M,j}^{3,-}(z)-F_{M,j}^{3,+}(z)), \tag{8}$$

$$X^{-,i}(z)=\sum_{j=1}^{M}d_{i,j}^1F_{i,j}^1(z)+\sum_{j=M+1}^{i-1}\frac{d_{i,j}^1}{(q-q^{-1})z}(F_{i,j}^{1,-}(z)-F_{i,j}^{1,+}(z))$$
$$+\frac{d_{i,i}^2}{(q-q^{-1})z}(F_{i,i}^{2,-}(z)-F_{i,i}^{2,+}(z))+\sum_{j=i+2}^{M+N}\frac{d_{i,j}^3}{(q-q^{-1})z}(F_{i,j}^{3,-}(z)-F_{i,j}^{3,+}(z))$$
$$(M+1\le i\le M+N-1). \tag{9}$$

We set $E_{i,j}^\pm(z)$ as follows.

$$E_{i,j}^\pm(z)=:e^{(b+c)^{j,i}(q^{j-1}z)+b_\pm^{j,i+1}(q^{j-1}z)-(b+c)^{j,i+1}(q^{j-1\pm1}z)+\sum_{l=1}^{j-1}(\Delta_R^{-}b_+^{l,i})(q^lz)}:$$

$$(1 \leq j < i \leq M-1),$$

$$\begin{aligned}
E^{\pm}_{i,i}(z) &= : e^{b^{i,i+1}_{\pm}(q^{i-1}z)-(b+c)^{i,i+1}(q^{i-1\pm 1}z)+\sum_{l=1}^{i-1}(\Delta^-_R b^{l,i}_+)(q^l z)} : \quad (1 \leq j < i \leq M-1),\\
E^{\pm}_{i,i}(z) &= : e^{-b^{i,i+1}_{\pm}(q^{2M+1-i}z)-(b+c)^{i,i+1}(q^{2M+1\mp 1-i}z)}\\
&\quad \times e^{-\sum_{l=1}^{M}(\Delta^+_R b^{l,i}_+)(q^{l-1}z)-\sum_{l=M+1}^{i-1}(\Delta^+_R b^{l,i}_+)(q^{2M-l}z)} : \quad (M+1 \leq i \leq M+N-1),\\
E^{\pm}_{i,j}(z) &= : e^{(b+c)^{j,i}(q^{2M+1-j}z)-b^{j,i+1}_{\pm}(q^{2M+1-j}z)-(b+c)^{j,i+1}(q^{2M+1\mp 1-j}z)}\\
&\quad \times e^{-\sum_{l=1}^{M}(\Delta^+_R b^{l,i}_+)(q^{l-1}z)-\sum_{l=M+1}^{j-1}(\Delta^+_R b^{l,i}_+)(q^{2M-l}z)} : \quad (M+1 \leq j < i \leq M+N-1).
\end{aligned}$$

We set $E_{i,j}(z)$ as follows.

$$\begin{aligned}
E_{M,j}(z) &= : e^{(b+c)^{j,M}(q^{j-1}z)+b^{j,M+1}(q^{j-1}z)-\sum_{l=1}^{j-1}(\Delta^0_R b^{l,M}_+)(q^l z)} : \quad (1 \leq j \leq M-1),\\
E_{M,M}(z) &= : e^{b^{M,M+1}(q^{M-1}z)-\sum_{l=1}^{M-1}(\Delta^0_R b^{l,M}_+)(q^l z)} : \quad (1 \leq j \leq M-1),\\
E_{i,j}(z) &= : e^{b^{j,i}_+(q^{j-1}z)-b^{j,i}(q^j z)+b^{j,i+1}(q^{j-1}z)-\sum_{l=1}^{j-1}(\Delta^+_R b^{l,i}_+)(q^{l-1}z)} :\\
&\qquad (M+1 \leq i \leq M+N-1, 1 \leq j \leq M).
\end{aligned}$$

We set $F^{1,\pm}_{i,j}(z), F^1_{i,j}(z)$ as follows.

$$\begin{aligned}
F^{1,\pm}_{i,j}(z) &= : e^{a^i_-(q^{-\frac{k+g}{2}}z)+(b+c)^{j,i+1}(q^{-k-j}z)-b^{j,i}_{\pm}(q^{-k-j}z)-(b+c)^{j,i}(q^{-k-j\mp 1}z)}\\
&\quad \times e^{\sum_{l=j+1}^{i}(\Delta^+_R b^{l,i}_-)(q^{-k-l}z)-\sum_{l=i+1}^{M}(\Delta^+_L b^{i,l}_-)(q^{-k-l}z)-\sum_{l=M+1}^{M+N}(\Delta^+_L b^{i,l}_-)(q^{-k-2M-1+l}z)} :\\
&\qquad (1 \leq j < i \leq M-1),\\
F^{1,\pm}_{M,j}(z) &= : e^{a^M_-(q^{-\frac{k+g}{2}}z)-b^{j,M}_{\pm}(q^{-k-j}z)-(b+c)^{j,M}(q^{-k-j\mp 1}z)-b^{j,M+1}_-(q^{-k-j}z)-b^{j,M+1}(q^{-k-j+1}z)}\\
&\quad \times e^{-\sum_{l=j+1}^{M-1}(\Delta^0_R b^{l,M}_-)(q^{-k-l}z)+\sum_{l=M+2}^{M+N}(\Delta^0_L b^{M,l}_-)(q^{-k-2M-1+l}z)} : \quad (1 \leq j \leq M-1),\\
F^1_{i,j}(z) &= : e^{a^i_-(q^{-\frac{k+g}{2}}z)-b^{j,i+1}_-(q^{-k-j}z)-b^{j,i+1}(q^{-k-j+1}z)+b^{j,i}(q^{-k-j}z)-\sum_{l=j+1}^{M}(\Delta^-_R b^{l,i}_-)(q^{-k-l+1}z)}\\
&\quad \times e^{-\sum_{l=M+1}^{i}(\Delta^-_R b^{l,i}_-)(q^{-k-2M+l}z)+\sum_{l=i+1}^{M+N}(\Delta^-_L b^{i,l}_-)(q^{-k-2M+l}z)} :\\
&\qquad (M+1 \leq i \leq M+N-1, 1 \leq j \leq M),\\
F^{1,\pm}_{i,j}(z) &= : e^{a^i_-(q^{-\frac{k+g}{2}}z)+(b+c)^{j,i+1}(q^{-k-2M+j}z)+b^{j,i}_{\pm}(q^{-k-2M+j}z)-(b+c)^{j,i}(q^{-k-2M\pm 1+j}z)}\\
&\quad \times e^{-\sum_{l=j+1}^{i}(\Delta^-_R b^{l,i}_-)(q^{-k-2M+l}z)+\sum_{l=i+1}^{M+N}(\Delta^-_L b^{i,l}_-)(q^{-k-2M+l}z)} :\\
&\qquad (M+1 \leq j < i \leq M+N-1).
\end{aligned}$$

We set $F^{2,\pm}_{i,i}(z)$ as follows.

$$\begin{aligned}
F^{2,\pm}_{i,i}(z) &= : e^{a^i_{\pm}(q^{\pm\frac{k+g}{2}}z)+b^{i,i+1}_{\pm}(q^{\pm(k+i+1)}z)+(b+c)^{i,i+1}(q^{\pm(k+i)}z)}\\
&\quad \times e^{-\sum_{l=i+2}^{M}(\Delta^{\mp}_L b^{i,l}_{\pm})(q^{\pm(k+l)}z)-\sum_{l=M+1}^{M+N}(\Delta^{\mp}_L b^{i,l}_{\pm})(q^{\pm(k+2M+1-l)}z)} : \quad (1 \leq i \leq M-1),\\
F^{2,\pm}_{M,M}(z) &= : e^{a^M_{\pm}(q^{\pm\frac{k+g}{2}}z)-b^{M,M+1}(q^{\pm(k+M-1)}z)+\sum_{l=M+2}^{M+N}(\Delta^0_L b^{M,l}_{\pm})(q^{\pm(k+2M+1-l)}z)} :,
\end{aligned}$$

$$F_{i,i}^{2,\pm}(z) = : e^{a_\pm^i(q^{\pm\frac{k+g}{2}}z) - b_\pm^{i,i+1}(q^{\pm(k+2M-1-i)}z) + (b+c)^{i,i+1}(q^{\pm(k+2M-i)}z)} \times e^{\sum_{l=i+2}^{M+N}(\Delta_L^\pm b_\pm^{i,l})(q^{\pm(k+2M-l)}z)} : \quad (M+1 \le i \le M+N-1).$$

We set $F_{i,j}^{3,\pm}(z), F_{i,j}^3(z)$ as follows.

$$F_{i,j}^{3,\pm}(z) = : e^{a_+^i(q^{\frac{k+g}{2}}z) + (b+c)^{i,j}(q^{k+j-1}z) + b_\pm^{i+1,j}(q^{k+j-1}z) - (b+c)^{i+1,j}(q^{k-1\pm 1+j}z)} \times e^{-\sum_{l=j}^{M}(\Delta_L^- b_+^{i,l})(q^{k+l}z) - \sum_{l=M+1}^{M+N}(\Delta_L^- b_+^{i,l})(q^{k+2M+1-l}z)} : \quad (1 \le i < j \le M-1),$$

$$F_{i,j}^{3}(z) = : e^{a_+^i(q^{\frac{k+g}{2}}z) - b^{i,j}(q^{k+2M-j}z) - b_+^{i+1,j}(q^{k+2M-j}z) + b^{i+1,j}(q^{k+2M+1-j}z)} \times e^{-\sum_{l=j+1}^{M+N}(\Delta_L^- b_+^{i,l})(q^{k+2M+1-l}z)} : \quad (1 \le i \le M-1, M+1 \le j \le M+N),$$

$$F_{M,j}^{3,\pm}(z) = : e^{a_+^M(q^{\frac{k+g}{2}}z) - b^{M,j}(q^{k+2M-j}z) - b_\pm^{M+1,j}(q^{k+2M+1-j}z) - (b+c)^{M+1,j}(q^{k+2M+1\mp 1-j}z)} \times e^{b_+^{M+1,j}(q^{k+2M+1-j}z) + \sum_{l=j+1}^{M+N}(\Delta_L^0 b_+^{M,l})(q^{k+2M+1-l}z)} : \quad (M+2 \le j \le M+N),$$

$$F_{i,j}^{3,\pm}(z) = : e^{a_+^i(q^{\frac{k+g}{2}}z) + (b+c)^{i,j}(q^{k+2M+1-j}z) - b_\pm^{i+1,j}(q^{k+2M+1-j}z) - (b+c)^{i+1,j}(q^{k+2M+1\mp 1-j}z)} \times e^{\sum_{l=j+1}^{M+N}(\Delta_L^+ b_+^{i,l})(q^{k+2M-l}z)} : \quad (M+1 \le i < j-1 \le M+N-1).$$

The coefficients $c_{i,j} \in \mathbf{C}$ and $d_{i,j}^1, d_{i,i}^2, d_{i,j}^3 \in \mathbf{C}$ satisfy the following conditions.

$$d_{i,j}^1 = \nu_{i+1}\frac{1}{c_{i,j}} \times \begin{cases} 1 & (1 \le i \le M-1, 1 \le j \le i-1), \\ q^{j-1} & (i = M, 1 \le j \le M-1), \\ q^{-k-1} & (M+1 \le i \le M+N-1, 1 \le j \le M), \\ 1 & (M+1 \le i \le M+N-1, M+1 \le j \le i-1), \end{cases}$$

$$d_{i,i}^2 = \nu_{i+1}\frac{1}{c_{i,i}} \times \begin{cases} 1 & (1 \le i \ne M \le M+N-1), \\ q^{M-1} & (i = M), \end{cases}$$

$$d_{i,j}^3 = \nu_{i+1}\frac{1}{c_{i,i}}\prod_{l=1}^{j-i-1}\frac{c_{i+l,i+1}}{c_{i+l,i}} \times \begin{cases} 1 & (1 \le i \le M-1, i+2 \le j \le M), \\ q^{k+3M+1-2j} & (1 \le i \le M-1, M+1 \le j \le M+N), \\ q^{(M-1)(j-M)} & (i = M, M+2 \le j \le M+N), \\ 1 & (M+1 \le i \le M+N-1, i+2 \le j \le M+N). \end{cases}$$

Theorem 1. *The bosonic operators $\Psi_\pm^i(z)$ defined in (1)-(3), and $X^{\pm,i}(z)$ defined in (4)-(6) and (7)-(9) satisfy the defining relations of the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$ with the central element $c = k \in \mathbf{C}$.*

3.3 Wakimoto realization

In this Section we introduce the $\xi - \eta$ system and give the Wakimoto realization. We set the boson Fock space $F(p_a, p_b, p_c)$ as follows. The vacuum state $|0\rangle \ne 0$ is defined by $a_m^i|0\rangle = b_m^{i,j}|0\rangle = c_m^{i,j}|0\rangle = 0$ $(m \ge 0)$. Let $|p_a, p_b, p_c\rangle$ be

$$|p_a,p_b,p_c\rangle$$
$$=\exp\left(\sum_{i,j=1}^{M+N-1}\frac{(A^{-1})_{i,j}}{k+g}p_a^iQ_a^i-\sum_{1\leq i<j\leq M+N}\nu_i\nu_jp_b^{i,j}Q_b^{i,j}+\sum_{\substack{1\leq i<j\leq M+N\\ \nu_i\nu_j=+1}}p_c^{i,j}Q_c^{i,j}\right)|0\rangle,$$

then $|p_a,p_b,p_c\rangle$ is the highest weight state of the boson Fock space $F(p_a,p_b,p_c)$. The boson Fock space $F(p_a,p_b,p_c)$ is generated by the bosons $a_m^i,b_m^{i,j},c_m^{i,j}$ on the highest weight state $|p_a,p_b,p_c\rangle$. We set the space $F(p_a)$ by

$$F(p_a)=\bigoplus_{\substack{p_b^{i,j}=-p_c^{i,j}\in\mathbf{Z}\ (\nu_i\nu_j=+)\\ p_b^{i,j}\in\mathbf{Z}\ (\nu_i\nu_j=-)}}F(p_a,p_b,p_c).$$

Here we impose the restriction $p_b^{i,j}=-p_c^{i,j}$ $(\nu_i\nu_j=+)$, because $X_m^{\pm,i}$ change $Q_b^{i,j}+Q_c^{i,j}$. $F(p_a)$ is $U_q(\widehat{sl}(M|N))$-module. Let $|\lambda\rangle=|p_a,0,0\rangle$ where $p_a^i=l_i$ $(1\leq i\leq M+N-1)$. The generators $H^i,H_m^i,X_m^{\pm,i}$ act on $|\lambda\rangle$ as follows.

$$H_m^i|\lambda\rangle=X_m^{\pm,i}|\lambda\rangle=0\quad(m>0),$$
$$X_0^{+,i}|\lambda\rangle=0,\quad H^i|\lambda\rangle=l_i|\lambda\rangle.$$

We have the level-k highest weight module $V(\lambda)$ of $U_q(\widehat{sl}(M|N))$.

$$V(\lambda)\subset F(p_a).$$

Here the classical part of the highest weight is $\bar{\lambda}=\sum_{i=1}^{M+N-1}l_i\bar{\Lambda}_i$.

We introduce the $\xi-\eta$ system We set bosonic operators $\xi_m^{i,j},\eta_m^{i,j}$ $(\nu_i\nu_j=+1,1\leq i<j\leq M+N)$ as follows.

$$\eta^{i,j}(z)=\sum_{m\in\mathbf{Z}}\eta_m^{i,j}z^{-m-1}=:e^{c^{i,j}(z)}:,\quad \xi^{i,j}(z)=\sum_{m\in\mathbf{Z}}\xi_m^{i,j}z^{-m}=:e^{-c^{i,j}(z)}:.$$

Fourier components

$$\eta_m^{i,j}=\oint\frac{dz}{2\pi\sqrt{-1}}z^m\eta^{i,j}(z),\quad \xi_m^{i,j}=\oint\frac{dz}{2\pi\sqrt{-1}}z^{m-1}\xi^{i,j}(z)$$

are well-defined on the module $F(p_a)$. The $\mathbf{Z}_2$-grading is given by $p(\xi_m^{i,j})=p(\eta_m^{i,j})=+1$. We have direct sum decomposition.

$$F(p_a)=\eta_0^{i,j}\xi_0^{i,j}F(p_a)\oplus\xi_0^{i,j}\eta_0^{i,j}F(p_a),$$

where $\mathrm{Ker}(\eta_0^{i,j})=\eta_0^{i,j}\xi_0^{i,j}F(p_a)$, $\mathrm{Coker}(\eta_0^{i,j})=\xi_0^{i,j}\eta_0^{i,j}F(p_a)$. We set

$$\eta_0=\prod_{\substack{1\leq i<j\leq M+N\\ \nu_i\nu_j=+1}}\eta_0^{i,j},\quad \xi_0=\prod_{\substack{1\leq i<j\leq M+N\\ \nu_i\nu_j=+1}}\xi_0^{i,j}.$$

We introduce the subspace $\mathcal{F}(p_a)$ by

$$\mathcal{F}(p_a) = \eta_0 \xi_0 F(p_a).$$

The operators $\eta_0^{i,j}, \xi_0^{i,j}$ commute with $X^{\pm,i}(z), \Psi_\pm^i(z)$ up to sign ± 1.

Proposition 1. *$\mathcal{F}(p_a)$ is $U_q(\widehat{sl}(M|N))$-module.*

We call $\mathcal{F}(p_a)$ the Wakimoto realization of $U_q(\widehat{sl}(M|N))$.

4 Screening operator

In this Section we give the screening operators Q_i $(1 \le i \le M+N-1)$ that commute with $U_q(\widehat{sl}(M|N))$ for the level $c = k \ne -g$. We define bosonic operators $S_i(z)$ $(1 \le i \le M+N-1)$ that we call the screening currents as follows.

$$S_i(z) = \sum_{j=i+1}^{M} \frac{e_{i,j}}{(q-q^{-1})z} (S_{i,j}^-(z) - S_{i,j}^+(z)) + \sum_{j=M+1}^{M+N} e_{i,j} S_{i,j}(z) \quad (1 \le i \le M-1), \tag{10}$$

$$S_M(z) = \sum_{j=M+1}^{M+N} e_{M,j} S_{M,j}(z), \tag{11}$$

$$S_i(z) = \sum_{j=i+1}^{M+N} \frac{e_{i,j}}{(q-q^{-1})z} (S_{i,j}^-(z) - S_{i,j}^+(z)) \quad (M+1 \le i \le M+N-1). \tag{12}$$

We set $S_{i,j}^\pm(z)$ as follows.

$$\begin{aligned} S_{i,j}^\pm(z) &= : e^{-(\frac{1}{k+g}a^i)(z;\frac{k+g}{2}) + (b+c)^{i+1,j}(q^{M-N-j}z) - b_\pm^{i,j}(q^{M-N-j}z) - (b+c)^{i,j}(q^{M-N-j\mp 1}z)} \\ &\quad \times e^{\sum_{l=j+1}^{M}(\Delta_L^+ b_-^{i,l})(q^{M-N-l}z) + \sum_{l=M+1}^{M+N}(\Delta_L^+ b_-^{i,l})(q^{-M-N+l-1}z)} : \quad (1 \le i < j \le M), \\ S_{i,j}^\pm(z) &= : e^{-(\frac{1}{k+g}a^i)(z;\frac{k+g}{2}) + (b+c)^{i+1,j}(q^{-M-N+j}z) + b_\pm^{i,j}(q^{-M-N+j}z) - (b+c)^{i,j}(q^{-M-N+j\pm 1}z)} \\ &\quad \times e^{-\sum_{l=j+1}^{M+N}(\Delta_L^- b_-^{i,l})(q^{-M-N+l}z)} : \quad (M+1 \le i < j \le M+N). \end{aligned}$$

We set $S_{i,j}(z)$ as follows.

$$\begin{aligned} S_{i,j}(z) &= : e^{-(\frac{1}{k+g}a^i)(z;\frac{k+g}{2}) + b^{i,j}(q^{-M-N+j}z) + b_+^{i+1,j}(q^{-M-N+j}z) - b^{i+1,j}(q^{-M-N+j+1}z)} \\ &\quad \times e^{\sum_{l=j+1}^{M+N}(\Delta_L^+ b_-^{i,l})(q^{-M-N-1+l}z)} : \quad (1 \le i \le M-1, M+1 \le j \le M+N), \\ S_{M,j}(z) &= : e^{-(\frac{1}{k+g}a^i)(z;\frac{k+g}{2}) + (b+c)^{M+1,j}(q^{-M-N+j}z) + b^{M,j}(q^{-M-N+j}z)} \end{aligned}$$

$$\times\, e^{-\sum_{l=j+1}^{M+N}(\Delta_L^0 b_-^{M,l})(q^{-M-N-1+l}z)} : \quad (M+1 \leq j \leq M+N).$$

Here we set $e_{i,j}$ as follows.

$$e_{i,i+1} = \begin{cases} 1/d_{i,i}^2 & (1 \leq i \leq M-1), \\ -q^{-N+1}/d_{M,M}^2 & (i=M), \\ -1/d_{i,i}^2 & (M+1 \leq i \leq M+N-1), \end{cases}$$

$$e_{i,j} = \begin{cases} 1/d_{i,j}^3 & (1 \leq i \leq M-1, i+2 \leq j \leq M), \\ q^{k+1+M-N}/d_{i,j}^3 & (1 \leq i \leq M-1, M+1 \leq j \leq M+N), \\ -q^{j-M-N}/d_{M,j}^3 & (i=M, M+2 \leq j \leq M+N), \\ -1/d_{i,j}^3 & (M+1 \leq i \leq M+N-1, i+2 \leq j \leq M+N). \end{cases}$$

The $\mathbf{Z}_2$-grading of the screening current is given by $p(S_{M,j}(z)) \equiv 1 \pmod 2$ for $M+1 \leq j \leq M+N$ and zero otherwise. The Jackson integral with parameters $q \in \mathbf{C}$ and $s \in \mathbf{C}^*$ is defined by

$$\int_0^{s\infty} f(w) d_q w = s(1-q) \sum_{n \in \mathbf{Z}} f(sq^n) q^n.$$

We define the screening operators Q_i $(1 \leq i \leq M+N-1)$ as follows, when the Jackson integrals are convergent.

$$Q_i = \int_0^{s\infty} S_i(w) d_{q^{2(k+g)}} w. \tag{13}$$

Theorem 2. *The screening operators Q_i $(1 \leq i \leq M+N-1)$ defined in (10), (11), (12), (13) commute with the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$.*

$$[Q_i, U_q(\widehat{sl}(M|N))] = 0.$$

5 Limit $q \to 1$

Bosonization of the affine superalgebra $\widehat{sl}(M|N)$ for an arbitrary level k have been studied in [9, 10, 11]. We obtain new bosonization of the affine superalgebra $\widehat{sl}(M|N)$ in the limit $q \to 1$.

In what follows we set

$$H^i(z) = \sum_{m \in \mathbf{Z}} H_m^i z^{-m-1} \quad (1 \leq i \leq M+N-1).$$

We set the parameters $c_{i,j} = 1$ in (4)-(6), (7)-(9), (10)-(12) for simplicity. In the limit $q \to 1$ we introduce operators $\alpha_i(z)$ $(1 \leq i \leq M+N-1)$,

$\beta_{i,j}(z), \widehat{\beta}_{i,j}(z), \gamma_{i,j}(z)$ $(1\leq i<j\leq M+N, \nu_i\nu_j=+)$, and $\psi_{i,j}(z), \psi_{i,j}^\dagger(z)$ $(1\leq i<j\leq M+N, \nu_i\nu_j=-)$ as follows.

$$\alpha_i(z)=\partial_z\left(a^i(z)\right), \quad \gamma_{i,j}(z)=: e^{(b+c)^{i,j}(z)}:,$$
$$\beta_{i,j}(z)=:\partial_z\left(e^{-c^{i,j}(z)}\right)e^{-b^{i,j}(z)}:, \ \widehat{\beta}_{i,j}(z)=:\partial_z\left(e^{-b^{i,j}(z)}\right)e^{-c^{i,j}(z)}:,$$
$$\psi_{i,j}(z)=: e^{b^{i,j}(z)}:, \quad \psi_{i,j}^\dagger(z)=: e^{-b^{i,j}(z)}: .$$

They satisfy the following relations.

$$\alpha_i(z)\alpha_j(w)=\frac{(k+g)A_{i,j}}{(z-w)^2}+\cdots,$$
$$\beta_{i,j}(z)\gamma_{i',j'}(w)=\frac{\delta_{i,i'}\delta_{j,j'}}{z-w}+\cdots, \quad \gamma_{i,j}(z)\beta_{i',j'}(w)=-\frac{\delta_{i,i'}\delta_{j,j'}}{z-w}+\cdots,$$
$$\widehat{\beta}_{i,j}(z)\gamma_{i',j'}(w)=-\frac{\delta_{i,i'}\delta_{j,j'}}{z-w}+\cdots, \quad \gamma_{i,j}(z)\widehat{\beta}_{i',j'}(w)=\frac{\delta_{i,i'}\delta_{j,j'}}{z-w}+\cdots,$$
$$\psi_{i,j}(z)\psi_{i',j'}^\dagger(w)=\frac{\delta_{i,i'}\delta_{j,j'}}{z-w}+\cdots, \quad \psi_{i,j}^\dagger(z)\psi_{i',j'}(w)=\frac{\delta_{i,i'}\delta_{j,j'}}{z-w}+\cdots.$$

In the limit $q\to 1$ the operators $a_\pm^i(z)$, $b_\pm^{i,j}(z)$, $(\varDelta_L^\epsilon b_\pm^{i,j})(z)$ and $(\varDelta_R^\epsilon b_\pm^{i,j})(z)$ disappear. We obtain the following.

$$H^i(z)=\alpha_i(z)+\sum_{j=1}^{i}:(\widehat{\beta}_{j,i}(z)\gamma_{j,i}(z)-\widehat{\beta}_{j,i+1}(z)\gamma_{j,i+1}(z)):$$
$$+\sum_{j=i+1}^{M}:(\widehat{\beta}_{i+1,j}(z)\gamma_{i+1,j}(z)-\widehat{\beta}_{i,j}(z)\gamma_{i,j}(z)):$$
$$+\sum_{j=M+1}^{M+N}:((\partial_z\psi_{i+1,j})(z)\psi_{i+1,j}^\dagger(z)-(\partial_z\psi_{i,j})(z)\psi_{i,j}^\dagger(z)): \quad (1\leq i\leq M-1),$$

$$H^M(z)=\alpha_M(z)+\sum_{j=1}^{M-1}:((\partial_z\psi_{j,M+1})(z)\psi_{j,M+1}^\dagger(z)+\widehat{\beta}_{j,M}(z)\gamma_{j,M}(z)):$$
$$-\sum_{j=M+2}^{M+N}:(\widehat{\beta}_{M+1,j}(z)\gamma_{M+1,j}(z)+(\partial_z\psi_{M,j})(z)\psi_{M,j}^\dagger(z)):,$$
$$H^i(z)=\alpha_i(z)+\sum_{j=1}^{M}:((\partial_z\psi_{j,i+1})(z)\psi_{j,i+1}^\dagger(z)-(\partial_z\psi_{j,i})(z)\psi_{j,i}^\dagger(z)):$$
$$+\sum_{j=M+1}^{i}:(\widehat{\beta}_{j,i+1}(z)\gamma_{j,i+1}(z)-\widehat{\beta}_{j,i}(z)\gamma_{j,i}(z)):$$

$$+\sum_{j=i+1}^{M+N}:(\widehat{\beta}_{i,j}(z)\gamma_{i,j}(z)-\widehat{\beta}_{i+1,j}(z)\gamma_{i+1,j}(z)):\quad (M+1\leq i\leq M+N-1).$$

$$X^{+,i}(z)=\sum_{j=1}^{i}:\beta_{j,i+1}(z)\gamma_{j,i}(z):\quad (1\leq i\leq M-1),$$
$$X^{+,M}(z)=\sum_{j=1}^{M}:\gamma_{j,M}(z)\psi_{j,M+1}(z):,$$
$$X^{+,i}(z)=\sum_{j=1}^{M}:\psi_{j,i+1}(z)\psi^{\dagger}_{j,i}(z):-\sum_{j=M+1}^{i}:\beta_{j,i+1}(z)\gamma_{j,i}(z):\ (M+1\leq i\leq M+N-1).$$

$$\begin{aligned}X^{-,i}(z)=&-:\alpha_i(z)\gamma_{i,i+1}(z):-\kappa_i:\partial_z\gamma_{i,i+1}(z):\\&+\sum_{j=1}^{i-1}:\beta_{j,i}(z)\gamma_{j,i+1}(z):-\sum_{j=i+2}^{M}:\beta_{i+1,j}(z)\gamma_{i,j}(z):-\sum_{j=M+1}^{M+N}:\psi_{i+1,j}(z)\psi^{\dagger}_{i,j}(z):\\&+\sum_{j=i+1}^{M}:(\widehat{\beta}_{i,j}(z)\gamma_{i,j}(z)-\widehat{\beta}_{i+1,j}(z)\gamma_{i+1,j}(z))\gamma_{i,i+1}(z):\\&+\sum_{j=M+1}^{M+N}:((\partial_z\psi_{i,j})(z)\psi^{\dagger}_{i,j}(z)-(\partial_z\psi_{i+1,j})(z)\psi^{\dagger}_{i+1,j}(z))\gamma_{i,i+1}(z):\\&\quad(1\leq i\leq M-1),\end{aligned}$$
$$\begin{aligned}X^{-,M}(z)=&:\alpha_M(z)\psi^{\dagger}_{M,M+1}(z):+\kappa_M:\partial_z\psi^{\dagger}_{M,M+1}(z):\\&-\sum_{j=1}^{M-1}:\beta_{j,M}(z)\psi^{\dagger}_{j,M+1}(z):-\sum_{j=M+2}^{M+N}:\beta_{M+1,j}(z)\psi^{\dagger}_{M,j}(z):\\&-\sum_{j=M+2}^{M+N}:(\widehat{\beta}_{M+1,j}(z)\gamma^{\dagger}_{M+1,j}(z)+(\partial_z\psi_{M,j})(z)\psi^{\dagger}_{M,j}(z))\psi^{\dagger}_{M,M+1}(z):,\end{aligned}$$
$$\begin{aligned}X^{-,i}(z)=&:\alpha_i(z)\gamma_{i,i+1}(z):+\kappa_i:\partial_z\gamma_{i,i+1}(z):\\&-\sum_{j=1}^{M}:\psi_{j,i}(z)\psi^{\dagger}_{j,i+1}(z):+\sum_{j=M+1}^{i-1}:\beta_{j,i}(z)\gamma_{j,i+1}(z):-\sum_{j=i+2}^{M+N}:\beta_{i+1,j}(z)\gamma_{i,j}(z):\\&+\sum_{j=i+1}^{M+N}:(\widehat{\beta}_{i,j}(z)\gamma_{i,j}(z)-\widehat{\beta}_{i+1,j}(z)\gamma_{i+1,j}(z))\gamma_{i,j}(z):\\&\quad(M+1\leq i\leq M+N-1).\end{aligned}$$

Here we have set the coefficients κ_i by

$$\kappa_i = \begin{cases} k+i & (1\leq i\leq M-1) \\ k+M-1 & (i=M) \\ k+2M-i & (M+1\leq i\leq M+N-1) \end{cases}.$$

In what follows we assume $k\neq -g$. In the limit $q\to 1$ we have the following.

$$S_i(z)=\sum_{j=i+1}^{M} :\tilde{s}_i(z)\beta_{i,j}(z)\gamma_{i+1,j}(z): + \sum_{j=M+1}^{M+N} :\tilde{s}_i(z)\psi_{i,j}(z)\psi^\dagger_{i+1,j}(z):$$
$$(1\leq i\leq M-1),$$
$$S_M(z)=\sum_{j=M+1}^{M+N} :\tilde{s}_M(z)\gamma_{M+1,j}(z)\psi_{M,j}(z):,$$
$$S_i(z)=\sum_{j=i+1}^{M+N} :\tilde{s}_i(z)\beta_{i,j}(z)\gamma_{i+1,j}(z): \quad (M+1\leq i\leq M+N-1).$$

Here we have set the boson operator

$$\tilde{s}_i(z) =: e^{-\left(\frac{1}{k+g}a^i\right)(z;0)}: .$$

Our bosonization is different from [9, 10, 11].

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