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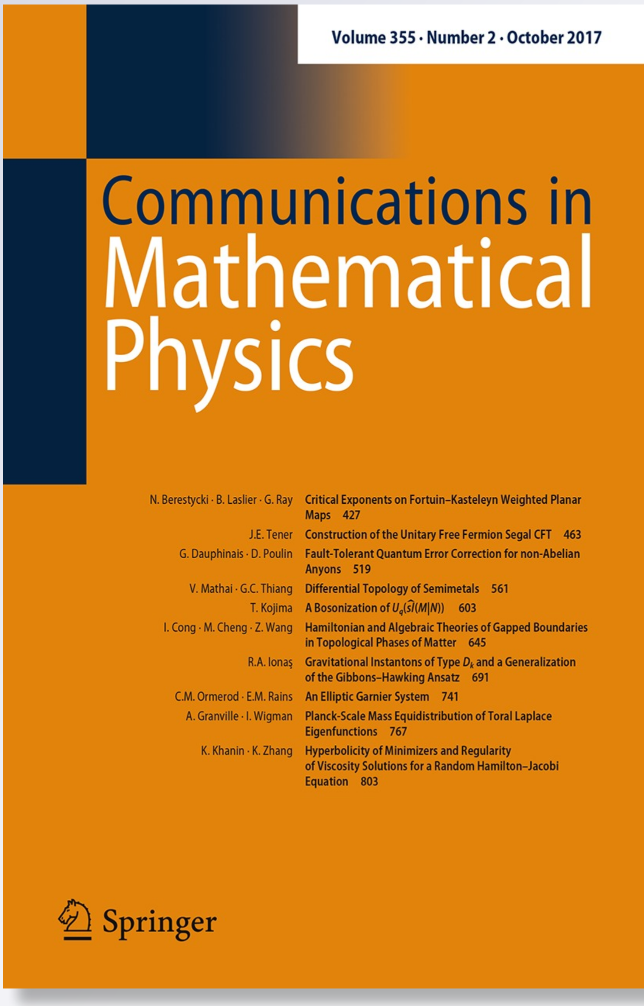

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A Bosonization of $U_q(\widehat{sl}(M|N))$

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Dedicated to Professor Michio Jimbo on the occasion of his 65th birthday

Abstract: A bosonization of the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$ is presented for an arbitrary level $k \in \mathbf{C}$. Screening operators that commute with $U_q(\widehat{sl}(M|N))$ are presented for the level $k \neq -M+N$.

1. Introduction

Bosonization is a powerful method to study representation theory of infinite-dimensional algebras [1] and its application to mathematical physics, such as calculation of correlation functions of exactly solvable models [2]. In this article we give a bosonization of the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$ $(M, N = 1, 2, 3, \ldots)$ for an arbitrary level $k \in \mathbf{C}$, and give screening operators that commute with $U_q(\widehat{sl}(M|N))$ for the level $k \neq -M+N$. For level $k = 1$, bosonization has been constructed for the quantum affine algebra $U_q(g)$ in many cases $g = (ADE)^{(r)}, (BC)^{(1)}, G_2^{(1)}, \widehat{sl}(M|N), osp(2|2)^{(2)}$ [3–11]. Bosonization of an arbitrary level $k \in \mathbf{C}$ is completely different from those of level $k = 1$. For level $k \in \mathbf{C}$, bosonization has been constructed only for $U_q(\widehat{sl}(N))$ and $U_q(\widehat{sl}(N|1))$ [12–20]. In this article we give a higher-rank generalization of the previous works for the quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ including the construction of screening operators [19–23]. Representation theory of the superalgebra is much more complicated than non-superalgebra and has rich structures [24–29].

The text is organized as follows. In Sect. 2, we recall the Chevalley generators and the Drinfeld generators of the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$. In Sect. 3, we introduce bosons and give a bosonization of the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$ for an arbitrary level $k \in \mathbf{C}$. We realize the Wakimoto module as a submodule of this bosonization using the $\xi - \eta$ system. In Sect. 4, we give screening operators that commute with $U_q(\widehat{sl}(M|N))$ for the level $k \neq -M+N$. In Sect. 5, we prove the main results. In Sect. 6, we give concluding remarks. In "Appendix A", we summarize normal ordering rules of bosonic operators. In “Appendix B”, we recall a q-difference realization of $U_q(sl(M|N))$. In “Appendix C”, we summarize useful formulae to take the limit $q \to 1$.

2. $U_q(\widehat{sl}(M|N))$

In this Section we recall the definition of the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$.

2.1. Chevalley generator. Throughout this paper, $q \in \mathbf{C}$ is assumed to be $0 < |q| < 1$. For any integer n, define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{2.1}$$

We begin with the definition of the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$ for $M, N = 1, 2, 3, \ldots$ in terms of Chevalley generators. The Cartan matrix of the affine Lie superalgebra $\widehat{sl}(M|N)$ is

$$(A_{i,j})_{0\le i,j\le M+N-1} = \begin{pmatrix} 0 & -1 & 0 & \cdots & & & & & & \cdots & 0 & 1 \\ -1 & 2 & -1 & \cdots & & & & & & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & & & & & & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & -1 & \cdots & & & & & & \\ & & \cdots & -1 & 2 & -1 & \cdots & & & & & \\ & & & \cdots & -1 & 0 & 1 & \cdots & & & & \\ & & & & \cdots 1 & -2 & 1 & \cdots & & & & \\ & & & & & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & & & & & & \cdots & -2 & 1 & 0 \\ 0 & 0 & \cdots & & & & & & \cdots & 1 & -2 & 1 \\ 1 & 0 & \cdots & & & & & & \cdots & 0 & 1 & -2 \end{pmatrix}. \tag{2.2}$$

where the diagonal part is $(A_{i,i})_{0\le i\le M+N-1} = (0, \overbrace{2, 2, \ldots, 2}^{M-1}, 0, \overbrace{-2, -2, \ldots, -2}^{N-1})$. Let $\bar{\alpha}_i$, $\bar{\Lambda}_i$ $(1 \le i \le M + N - 1)$ be the classical simple roots, the classical fundamental weights, respectively. Let $(\cdot|\cdot)$ be the symmetric bilinear form satisfying $(\bar{\alpha}_i|\bar{\alpha}_j) = A_{i,j}$ and $(\bar{\Lambda}_i|\bar{\alpha}_j) = \delta_{i,j}$ for $1 \le i, j \le M+N-1$. Let us introduce the affine weight Λ_0 and the null root δ satisfying $(\Lambda_0|\Lambda_0) = (\delta|\delta) = 0$, $(\Lambda_0|\delta) = 1$, and $(\Lambda_0|\bar{\alpha}_i) = (\Lambda_0|\bar{\Lambda}_i) = 0$ for $1 \le i \le M + N - 1$. The other affine weights and the affine roots are given by $\Lambda_i = \bar{\Lambda}_i + \Lambda_0$, $\alpha_i = \bar{\alpha}_i$ for $1 \le i \le M + N - 1$, and $\alpha_0 = \delta - \sum_{i=1}^{M+N-1} \alpha_i$.

The quantum affine superalgebra $U_q(\widehat{sl}(M|N))$ [28] is the associative algebra over $\mathbf{C}$ with the Chevalley generators e_i, f_i, h_i, d $(i = 0, 1, 2, \ldots, M + N - 1)$. The $\mathbf{Z}_2$-grading of the Chevalley generators is given by $p(e_0) \equiv p(f_0) \equiv p(e_M) \equiv p(f_M) \equiv 1$ (mod 2) and zero otherwise. The defining relations of the Chevalley generators are given as follows.

$$[h_i, h_j] = 0, \tag{2.3}$$

$$[h_i, e_j] = A_{i,j} e_j, \quad [h_i, f_j] = -A_{i,j} f_j, \tag{2.4}$$

$$[e_i, f_j] = \delta_{i,j} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \tag{2.5}$$

$$[e_j,[e_j,e_i]_{q^{-1}}]_q=0,\quad [f_j,[f_j,f_i]_{q^{-1}}]_q=0 \quad \text{for } |A_{i,j}|=1, i\neq 0, M, \tag{2.6}$$
$$[e_i,e_j]=0,\quad [f_i,f_j]=0 \quad \text{for } |A_{i,j}|=0, \tag{2.7}$$
$$[e_M,[e_{M+1},[e_M,e_{M-1}]_{q^{-1}}]_q]=0,\quad [f_M,[f_{M+1},[f_M,f_{M-1}]_{q^{-1}}]_q]=0, \tag{2.8}$$
$$[e_0,[e_1,[e_0,e_{M+N-1}]_q]_{q^{-1}}]=0,\quad [f_0,[f_1,[f_0,f_{M+N-1}]_q]_{q^{-1}}]=0, \tag{2.9}$$

where we use the notation

$$[X,Y]_a=XY-(-1)^{p(X)p(Y)}aYX, \tag{2.10}$$

for homogeneous elements $X, Y \in U_q(\widehat{sl}(M|N))$. For simplicity we write $[X,Y]=[X,Y]_1$. If $M=1$ or $N=1$, we have extra fifth order Serre relations. As for the explicit forms of the extra Serre relations, we refer the reader to [23,28]. Moreover, $U_q(\widehat{sl}(M|N))$ is a Hopf algebra over $\mathbf{C}$ with coproduct

$$\Delta(h_i)=h_i\otimes 1+1\otimes h_i, \tag{2.11}$$
$$\Delta(e_i)=e_i\otimes q^{h_i}+1\otimes e_i, \tag{2.12}$$
$$\Delta(f_i)=f_i\otimes 1+q^{-h_i}\otimes f_i, \tag{2.13}$$

and antipode

$$S(h_i)=-h_i,\quad S(e_i)=-e_iq^{-h_i},\quad S(f_i)=-q^{h_i}f_i. \tag{2.14}$$

The multiplication rule for the tensor product is $\mathbf{Z}_2$-graded and is defined for homogeneous elements $X_1, X_2, Y_1, Y_2 \in U_q(\widehat{sl}(M|N))$ by $(X_1\otimes Y_1)(X_2\otimes Y_2)=(-1)^{p(Y_1)p(X_2)}(X_1X_2\otimes Y_1Y_2)$, which extends to inhomogeneous elements through linearity. The coproduct is an algebra automorphism $\Delta(XY)=\Delta(X)\Delta(Y)$ and the antipode S is a graded algebra anti-automorphism $S(XY)=(-1)^{p(X)p(Y)}S(Y)S(X)$.

2.2. Drinfeld generator. In [28] was given the second realization of the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$ which is more convenient for the concrete realization given in this article. We recall this realization that we call the Drinfeld realization [30]. The quantum affine superalgebra $U_q(\widehat{sl}(M|N))$ is isomorphic to the associative algebra over $\mathbf{C}$ with the Drinfeld generators $X_m^{\pm,i}$ ($i=1,2,\ldots,M+N-1, m\in\mathbf{Z}$), H_n^i ($i=1,2,\ldots,M+N-1, n\in\mathbf{Z}_{\neq 0}$), H^i ($i=1,2,\ldots,M+N-1$), and c. The $\mathbf{Z}_2$-grading of the Drinfeld generators is given by $p(X_m^{\pm,M})\equiv 1 \pmod 2$ for $m\in\mathbf{Z}$ and zero otherwise. The defining relations of the Drinfeld generators are given as follows.

$$c: \text{central element}, \tag{2.15}$$
$$[H^i,H_m^j]=0, \tag{2.16}$$
$$[H_m^i,H_n^j]=\frac{[A_{i,j}m]_q[cm]_q}{m}\delta_{m+n,0}, \tag{2.17}$$
$$[H^i,X^{\pm,j}(z)]=\pm A_{i,j}X^{\pm,j}(z), \tag{2.18}$$
$$[H_m^i,X^{\pm,j}(z)]=\pm\frac{[A_{i,j}m]_q}{m}q^{\mp\frac{c}{2}|m|}z^mX^{\pm,j}(z), \tag{2.19}$$
$$(z_1-q^{\pm A_{i,j}}z_2)X^{\pm,i}(z_1)X^{\pm,j}(z_2)=(q^{\pm A_{j,i}}z_1-z_2)X^{\pm,j}(z_2)X^{\pm,i}(z_1),\quad \text{for } |A_{i,j}|\neq 0, \tag{2.20}$$
$$[X^{\pm,i}(z_1),X^{\pm,j}(z_2)]=0,\quad \text{for } |A_{i,j}|=0, \tag{2.21}$$

$$[X^{+,i}(z_1), X^{-,j}(z_2)] = \frac{\delta_{i,j}}{(q-q^{-1})z_1z_2} \left(\delta(q^c z_2/z_1)\Psi_+^i(q^{\frac{c}{2}}z_2) - \delta(q^{-c}z_2/z_1)\Psi_-^i(q^{-\frac{c}{2}}z_2)\right), \quad (2.22)$$

$$[X^{\pm,i}(z_1), [X^{\pm,i}(z_2), X^{\pm,j}(z)]_{q^{-1}}]_q + (z_1 \leftrightarrow z_2) = 0, \text{ for } |A_{i,j}| = 1,\ i \neq M, \quad (2.23)$$

$$[X^{\pm,M}(z_1), [X^{\pm,M+1}(w_1), [X^{\pm,M}(z_2), X^{\pm,M-1}(w_2)]_{q^{-1}}]_q] + (z_1 \leftrightarrow z_2) = 0, \quad (2.24)$$

where we set $\delta(z) = \sum_{m\in\mathbf{Z}} z^m$. Here we use the generating functions

$$X^{\pm,j}(z) = \sum_{m\in\mathbf{Z}} X_m^{\pm,j} z^{-m-1}, \quad (2.25)$$

$$\Psi_\pm^i(q^{\pm\frac{c}{2}}z) = q^{\pm h_i} \exp\left(\pm(q-q^{-1})\sum_{m>0} H_{\pm m}^i z^{\mp m}\right). \quad (2.26)$$

The Chevalley generators are obtained by

$$h_i = H^i \quad (i = 1, 2, \ldots, M+N-1), \quad (2.27)$$

$$e_i = X_0^{+,i}, \quad f_i = X_0^{-,i} \quad (i = 1, 2, \ldots, M+N-1), \quad (2.28)$$

$$h_0 = c - (H^1 + H^2 + \cdots + H^{M+N-1}), \quad (2.29)$$

$$e_0 = (-1)^N [X_0^{-,M+N-1}, \ldots, [X_0^{-,M+1}, [X_0^{-,M}, \ldots, [X_0^{-,2}, X_0^{-,1}]_{q^{-1}} \cdots]_{q^{-1}}]_q \cdots]_q, \quad (2.30)$$

$$f_0 = q^{H^1+H^2+\cdots+H^{M+N-1}} \times [\cdots[[\cdots[X_{-1}^{+,1}, X_0^{+,2}]_q, \ldots, X_0^{+,M}]_q, X_0^{+,M+1}]_{q^{-1}}, \ldots, X_0^{+,M+N-1}]_{q^{-1}}. \quad (2.31)$$

Let $V(\lambda)$ be the highest-weight module over $U_q(\widehat{sl}(M|N))$ generated by the highest weight vector $|\lambda\rangle \neq 0$ such that

$$H_m^i|\lambda\rangle = X_m^{\pm,i}|\lambda\rangle = 0 \quad (m > 0), \quad (2.32)$$

$$X_0^{+,i}|\lambda\rangle = 0, \quad (2.33)$$

$$H^i|\lambda\rangle = l_i|\lambda\rangle, \quad (2.34)$$

where the classical part of the highest weight is $\bar{\lambda} = \sum_{i=1}^{M+N-1} l_i \bar{\Lambda}_i$.

3. Bosonization of $U_q(\widehat{sl}(M|N))$

In this Section we give a bosonization of $U_q(\widehat{sl}(M|N))$ for an arbitrary level $k \in \mathbf{C}$.

3.1. Boson. In order to construct a bosonization of $U_q(\widehat{sl}(M|N))$, we introduce bosons a_m^i ($m \in \mathbf{Z}, 1 \le i \le M+N-1$), $b_m^{i,j}$ ($m \in \mathbf{Z}, 1 \le i < j \le M+N$), $c_m^{i,j}$ ($m \in \mathbf{Z}, \nu_i\nu_j = +1, 1 \le i < j \le M+N$), and zero mode operators Q_a^i ($1 \le i \le M+N-1$), $Q_b^{i,j}$ ($1 \le i < j \le M+N$), $Q_c^{i,j}$ ($\nu_i\nu_j = +1, 1 \le i < j \le M+N$). Their commutation relations are

$$[a_m^i, a_n^j] = \frac{1}{m}[(k+g)m]_q[A_{i,j}m]_q\delta_{m+n,0}, \quad [a_0^i, Q_a^j] = (k+g)A_{i,j}, \tag{3.1}$$

$$[b_m^{i,j}, b_n^{i',j'}] = -\nu_i\nu_j\frac{1}{m}[m]_q^2\delta_{i,i'}\delta_{j,j'}\delta_{m+n,0}, \quad [b_0^{i,j}, Q_b^{i',j'}] = -\nu_i\nu_j\delta_{i,i'}\delta_{j,j'}, \tag{3.2}$$

$$[c_m^{i,j}, c_n^{i',j'}] = \nu_i\nu_j\frac{1}{m}[m]_q^2\delta_{i,i'}\delta_{j,j'}\delta_{m+n,0}, \quad [c_0^{i,j}, Q_c^{i',j'}] = \nu_i\nu_j\delta_{i,i'}\delta_{j,j'} \quad (\nu_i\nu_j = +1), \tag{3.3}$$

$$[Q_b^{i,j}, Q_b^{i',j'}] = \pi\sqrt{-1} \quad (\nu_i\nu_j = \nu_{i'}\nu_{j'} = -1). \tag{3.4}$$

The remaining commutators vanish. Here $g = M - N$ stands for the dual Coxeter number of $sl(M|N)$, and $\nu_i = 1$ for $1 \le i \le M$ and $\nu_i = -1$ for $M+1 \le i \le M+N$. For instance, we have a minus sigh by exchanging operators $e^{\pm Q_b^{i,j}}$,

$$e^{\epsilon Q_b^{i,j}} e^{\epsilon' Q_b^{i',j'}} = -e^{\epsilon' Q_b^{i',j'}} e^{\epsilon Q_b^{i,j}} \quad (\nu_i\nu_j = \nu_{i'}\nu_{j'} = -1, \epsilon, \epsilon' = \pm). \tag{3.5}$$

We define free boson fields $a_\pm^i(z)$, $b_\pm^{i,j}(z)$, $b^{i,j}(z)$, $c^{i,j}(z)$ as follows.

$$a_\pm^i(z) = \pm(q-q^{-1})\sum_{m>0} a_{\pm m}^i z^{\mp m} \pm a_0^i \log q, \tag{3.6}$$

$$b_\pm^{i,j}(z) = \pm(q-q^{-1})\sum_{m>0} b_{\pm m}^{i,j} z^{\mp m} \pm b_0^{i,j} \log q, \tag{3.7}$$

$$b^{i,j}(z) = -\sum_{m\ne 0}\frac{b_m^{i,j}}{[m]_q}z^{-m} + Q_b^{i,j} + b_0^{i,j}\log z, \tag{3.8}$$

$$c^{i,j}(z) = -\sum_{m\ne 0}\frac{c_m^{i,j}}{[m]_q}z^{-m} + Q_c^{i,j} + c_0^{i,j}\log z. \tag{3.9}$$

We define free boson fields $(\Delta_L^\varepsilon b_\pm^{i,j})(z)$, $(\Delta_R^\varepsilon b_\pm^{i,j})(z)$ ($\varepsilon = \pm, 0$) as follows.

$$(\Delta_L^\varepsilon b_\pm^{i.j})(z) = \begin{cases} b_\pm^{i+1,j}(q^\varepsilon z) - b_\pm^{i,j}(z) & (\varepsilon = \pm), \\ b_\pm^{i+1,j}(z) + b_\pm^{i,j}(z) & (\varepsilon = 0), \end{cases} \tag{3.10}$$

$$(\Delta_R^\varepsilon b_\pm^{i.j})(z) = \begin{cases} b_\pm^{i,j+1}(q^\varepsilon z) - b_\pm^{i,j}(z) & (\varepsilon = \pm), \\ b_\pm^{i,j+1}(z) + b_\pm^{i,j}(z) & (\varepsilon = 0). \end{cases} \tag{3.11}$$

We define further free boson fields with parameters $L_1, L_2, \ldots, L_r, M_1, M_2, \ldots, M_r, \alpha$ as follows.

$$\left(\frac{L_1}{M_1}\frac{L_2}{M_2}\cdots\frac{L_r}{M_r}a^i\right)(z;\alpha) = -\sum_{m\ne 0}\frac{[L_1m]_q[L_2m]_q\cdots[L_rm]_q}{[M_1m]_q[M_2m]_q\cdots[M_rm]_q}\frac{a_m^i}{[m]_q}q^{-\alpha|m|}z^{-m} + \frac{L_1L_2\cdots L_r}{M_1M_2\cdots M_r}(Q_a^i + a_0^i\log z). \tag{3.12}$$

Normal ordering rules are defined as follows.

$$: a_m^i a_n^j :=: a_n^j a_m^i := \begin{cases} a_m^i a_n^j \ (m < 0), \\ a_n^j a_m^i \ (m > 0), \end{cases} \tag{3.13}$$

$$: b_m^{i,j} b_n^{i',j'} :=: b_n^{i',j'} b_m^{i,j} := \begin{cases} b_m^{i,j} b_n^{i',j'} \ (m < 0), \\ b_n^{i',j'} b_m^{i,j} \ (m > 0), \end{cases} \tag{3.14}$$

$$: Q_b^{i,j} Q_b^{i',j'} :=: Q_b^{i',j'} Q_b^{i,j} := Q_b^{i,j} Q_b^{i',j'} \quad (i > i' \text{ or } i = i', j > j'). \tag{3.15}$$

Normal ordering rules of $c_m^{i,j}$ and $Q_c^{i,j}$ are defined in the same way. For instance we have

$$: \exp(a^i(z)) := \exp\left(\sum_{m>0} \frac{a_{-m}^i}{[m]_q} z^m\right) \exp\left(-\sum_{m>0} \frac{a_m^i}{[m]_q} z^{-m}\right) e^{Q_a^i} z^{a_0^i}, \tag{3.16}$$

$$: e^{Q_b^{i,j}} e^{Q_b^{i',j'}} :=: e^{Q_b^{i',j'}} e^{Q_b^{i,j}} := e^{Q_b^{i,j}} e^{Q_b^{i',j'}} = \nu_i \nu_j e^{Q_b^{i',j'}} e^{Q_b^{i,j}} \\ (i > i' \text{ or } i = i', j > j'). \tag{3.17}$$

3.2. Bosonization. In this Section we fix a complex number $k \in \mathbf{C}$.

• We define bosonic operators $H^i(z)$ $(1 \leq i \leq M + N - 1)$ as follows.

$$\begin{aligned} H^i(z) = \frac{1}{(q-q^{-1})z} \Bigg\{ \Bigg[& a_+^i(q^{\frac{g}{2}} z) + \sum_{l=1}^{i} (\Delta_R^- b_+^{l,i})(q^{\frac{k}{2}+l} z) \\ & - \sum_{l=i+1}^{M} (\Delta_L^- b_+^{i,l})(q^{\frac{k}{2}+l} z) - \sum_{l=M+1}^{M+N} (\Delta_L^- b_+^{i,l})(q^{\frac{k}{2}+2M+1-l} z) \Bigg] \\ & - \Bigg[a_-^i(q^{-\frac{g}{2}} z) + \sum_{l=1}^{i} (\Delta_R^+ b_-^{l,i})(q^{-\frac{k}{2}-l} z) \\ & - \sum_{l=i+1}^{M} (\Delta_L^+ b_-^{i,l})(q^{-\frac{k}{2}-l} z) - \sum_{l=M+1}^{M+N} (\Delta_L^+ b_-^{i,l})(q^{-\frac{k}{2}-2M-1+l} z) \Bigg] \Bigg\} \\ & (1 \leq i \leq M-1), \end{aligned} \tag{3.18}$$

$$\begin{aligned} H^M(z) = \frac{1}{(q-q^{-1})z} \Bigg\{ \Bigg[& a_+^M(q^{\frac{g}{2}} z) - \sum_{l=1}^{M-1} (\Delta_R^0 b_+^{l,M})(q^{\frac{k}{2}+l} z) \\ & + \sum_{l=M+2}^{M+N} (\Delta_L^0 b_+^{M,l})(q^{\frac{k}{2}+2M+1-l} z) \Bigg] \\ & - \Bigg[a_-^M(q^{-\frac{g}{2}} z) - \sum_{l=1}^{M-1} (\Delta_R^0 b_-^{l,M})(q^{-\frac{k}{2}-l} z) \\ & + \sum_{l=M+2}^{M+N} (\Delta_L^0 b_-^{M,l})(q^{-\frac{k}{2}-2M-1+l} z) \Bigg] \Bigg\}, \end{aligned} \tag{3.19}$$

$$H^i(z)=\frac{1}{(q-q^{-1})z}\left\{\left[a_+^i(q^{\frac{g}{2}}z)-\sum_{l=1}^{M}(\Delta_R^+b_+^{l,i})(q^{\frac{k}{2}+l-1}z)\right.\right.$$
$$\left.-\sum_{l=M+1}^{i}(\Delta_R^+b_+^{l,i})(q^{\frac{k}{2}+2M-l}z)+\sum_{l=i+1}^{M+N}(\Delta_L^+b_+^{i,l})(q^{\frac{k}{2}+2M-l}z)\right]$$
$$-\left[a_-^i(q^{-\frac{g}{2}}z)-\sum_{l=1}^{M}(\Delta_R^-b_-^{l,i})(q^{-\frac{k}{2}-l+1}z)-\sum_{l=M+1}^{i}(\Delta_R^-b_-^{l,i})(q^{-\frac{k}{2}-2M+l}z)\right.$$
$$\left.\left.+\sum_{l=i+1}^{M+N}(\Delta_L^-b_-^{i,l})(q^{-\frac{k}{2}-2M+l}z)\right]\right\}\quad(M+1\le i\le M+N-1).\tag{3.20}$$

We define bosonic operators H_m^i $(m\in\mathbf{Z}, 1\le i\le M+N-1)$ as follows.

$$H^i(z)=\sum_{m\in\mathbf{Z}}H_m^i z^{-m-1}\quad(1\le i\le M+N-1).\tag{3.21}$$

- We define bosonic operators $\Psi_\pm^i(z)$ $(1\le i\le M+N-1)$ as follows.

$$\Psi_\pm^i(q^{\pm\frac{k}{2}}z)=:e^{a_\pm^i(q^{\pm\frac{g}{2}}z)+\sum_{l=1}^{i}(\Delta_R^\mp b_\pm^{l,i})(q^{\pm(\frac{k}{2}+l)}z)-\sum_{l=i+1}^{M}(\Delta_L^\mp b_\pm^{i,l})(q^{\pm(\frac{k}{2}+l)}z)}$$
$$\times e^{-\sum_{l=M+1}^{M+N}(\Delta_L^\mp b_\pm^{i,l})(q^{\pm(\frac{k}{2}+2M+1-l)}z)}:\quad(1\le i\le M-1),\tag{3.22}$$
$$\Psi_\pm^M(q^{\pm\frac{k}{2}}z)=:e^{a_\pm^M(q^{\pm\frac{g}{2}}z)-\sum_{l=1}^{M-1}(\Delta_R^0 b_\pm^{l,M})(q^{\pm(\frac{k}{2}+l)}z)+\sum_{l=M+2}^{M+N}(\Delta_L^0 b_\pm^{M,l})(q^{\pm(\frac{k}{2}+2M+1-l)}z)}:,\tag{3.23}$$
$$\Psi_\pm^i(q^{\pm\frac{k}{2}}z)=:e^{a_\pm^i(q^{\pm\frac{g}{2}}z)-\sum_{l=1}^{M}(\Delta_R^\pm b_\pm^{l,i})(q^{\pm(\frac{k}{2}+l-1)}z)-\sum_{l=M+1}^{i}(\Delta_R^\pm b_\pm^{l,i})(q^{\pm(\frac{k}{2}+2M-l)}z)}$$
$$\times e^{\sum_{l=i+1}^{M+N}(\Delta_L^\pm b_\pm^{i,l})(q^{\pm(\frac{k}{2}+2M-l)}z)}:\quad(M+1\le i\le M+N-1).\tag{3.24}$$

- We define bosonic operators $X^{+,i}(z)$ $(1\le i\le M+N-1)$ as follows.

$$X^{+,i}(z)=\sum_{j=1}^{i}\frac{c_{i,j}}{(q-q^{-1})z}(E_{i,j}^+(z)-E_{i,j}^-(z))\quad(1\le i\le M-1),\tag{3.25}$$
$$X^{+,M}(z)=\sum_{j=1}^{M}c_{M,j}E_{M,j}(z),\tag{3.26}$$
$$X^{+,i}(z)=\sum_{j=1}^{M}c_{i,j}E_{i,j}(z)+\sum_{j=M+1}^{i}\frac{c_{i,j}}{(q-q^{-1})z}(E_{i,j}^+(z)-E_{i,j}^-(z))$$
$$(M+1\le i\le M+N-1).\tag{3.27}$$

Here $c_{i,j}\in\mathbf{C}_{\neq0}$ $(1\le j\le i\le M+N-1)$ and we set $E_{i,j}^\pm(z)$ and $E_{i,j}(z)$ as follows.

For $1\le i\le M-1$ and $1\le j\le i-1$ we set

$$E_{i,j}^\pm(z)=:e^{(b+c)^{j,i}(q^{j-1}z)+b_\pm^{j,i+1}(q^{j-1}z)-(b+c)^{j,i+1}(q^{j-1\pm1}z)+\sum_{l=1}^{j-1}(\Delta_R^-b_+^{l,i})(q^l z)}:,\tag{3.28}$$
$$E_{i,i}^\pm(z)=:e^{b_\pm^{i,i+1}(q^{i-1}z)-(b+c)^{i,i+1}(q^{i-1\pm1}z)+\sum_{l=1}^{i-1}(\Delta_R^-b_+^{l,i})(q^l z)}:.\tag{3.29}$$

For $1 \le j \le M-1$ we set

$$E_{M,j}(z) =: e^{(b+c)^{j,M}(q^{j-1}z)+b^{j,M+1}(q^{j-1}z)-\sum_{l=1}^{j-1}(\Delta_R^0 b_+^{l,M})(q^l z)} :, \tag{3.30}$$

$$E_{M,M}(z) =: e^{b^{M,M+1}(q^{M-1}z)-\sum_{l=1}^{M-1}(\Delta_R^0 b_+^{l,M})(q^l z)} : . \tag{3.31}$$

For $M+1 \le i \le M+N-1$ we set

$$E_{i,i}^{\pm}(z) =: e^{-b_{\pm}^{i,i+1}(q^{2M+1-i}z)-(b+c)^{i,i+1}(q^{2M+1\mp 1-i}z)} \times e^{-\sum_{l=1}^{M}(\Delta_R^+ b_+^{l,i})(q^{l-1}z)-\sum_{l=M+1}^{i-1}(\Delta_R^+ b_+^{l,i})(q^{2M-l}z)} : . \tag{3.32}$$

For $M+1 \le i \le M+N-1$ and $1 \le j \le M$ we set

$$E_{i,j}(z) =: e^{b_+^{j,i}(q^{j-1}z)-b^{j,i}(q^j z)+b^{j,i+1}(q^{j-1}z)-\sum_{l=1}^{j-1}(\Delta_R^+ b_+^{l,i})(q^{l-1}z)} : . \tag{3.33}$$

For $M+1 \le i \le M+N-1$ and $M+1 \le j \le i-1$ we set

$$E_{i,j}^{\pm}(z) =: e^{(b+c)^{j,i}(q^{2M+1-j}z)-b_{\pm}^{j,i+1}(q^{2M+1-j}z)-(b+c)^{j,i+1}(q^{2M+1\mp 1-j}z)} \times e^{-\sum_{l=1}^{M}(\Delta_R^+ b_+^{l,i})(q^{l-1}z)-\sum_{l=M+1}^{j-1}(\Delta_R^+ b_+^{l,i})(q^{2M-l}z)} : . \tag{3.34}$$

- We define bosonic operators $X^{-,i}(z)$ $(1 \le i \le M+N-1)$ as follows.

$$X^{-,i}(z) = \sum_{j=1}^{i-1} \frac{d_{i,j}^1}{(q-q^{-1})z}(F_{i,j}^{1,-}(z) - F_{i,j}^{1,+}(z)) + \frac{d_{i,i}^2}{(q-q^{-1})z}(F_{i,i}^{2,-}(z) - F_{i,i}^{2,+}(z)) + \sum_{j=i+2}^{M} \frac{d_{i,j}^3}{(q-q^{-1})z}(F_{i,j}^{3,-}(z) - F_{i,j}^{3,+}(z)) + \sum_{j=M+1}^{M+N} d_{i,j}^3 F_{i,j}^3(z) \quad (1 \le i \le M-1), \tag{3.35}$$

$$X^{-,M}(z) = \sum_{j=1}^{M-1} \frac{d_{M,j}^1}{(q-q^{-1})z}(F_{M,j}^{1,-}(z) - F_{M,j}^{1,+}(z)) + \frac{d_{M,M}^2}{(q-q^{-1})z}(F_{M,M}^{2,-}(z) - F_{M,M}^{2,+}(z)) + \sum_{j=M+2}^{M+N} \frac{d_{M,j}^3}{(q-q^{-1})z}(F_{M,j}^{3,-}(z) - F_{M,j}^{3,+}(z)), \tag{3.36}$$

$$X^{-,i}(z)=\sum_{j=1}^{M}d_{i,j}^{1}F_{i,j}^{1}(z)+\sum_{j=M+1}^{i-1}\frac{d_{i,j}^{1}}{(q-q^{-1})z}(F_{i,j}^{1,-}(z)-F_{i,j}^{1,+}(z))$$
$$+\frac{d_{i,i}^{2}}{(q-q^{-1})z}(F_{i,i}^{2,-}(z)-F_{i,i}^{2,+}(z))$$
$$+\sum_{j=i+2}^{M+N}\frac{d_{i,j}^{3}}{(q-q^{-1})z}(F_{i,j}^{3,-}(z)-F_{i,j}^{3,+}(z))$$
$$(M+1\le i\le M+N-1). \tag{3.37}$$

Here we set $F_{i,j}^{1,\pm}(z)$, $F_{i,j}^{1}(z)$, $F_{i,i}^{2,\pm}(z)$, $F_{i,j}^{3,\pm}(z)$, $F_{i,j}^{3}(z)$ as follows.

For $1\le i\le M-1$ and $1\le j\le i-1$ we set

$$F_{i,j}^{1,\pm}(z)=:e^{a_-^i(q^{-\frac{k+g}{2}}z)+(b+c)^{j,i+1}(q^{-k-j}z)-b_\pm^{j,i}(q^{-k-j}z)-(b+c)^{j,i}(q^{-k-j\mp1}z)}$$
$$\times e^{\sum_{l=j+1}^{i}(\Delta_R^+b_-^{l,i})(q^{-k-l}z)-\sum_{l=i+1}^{M}(\Delta_L^+b_-^{i,l})(q^{-k-l}z)-\sum_{l=M+1}^{M+N}(\Delta_L^+b_-^{i,l})(q^{-k-2M-1+l}z)}:. \tag{3.38}$$

For $1\le j\le M-1$ we set

$$F_{M,j}^{1,\pm}(z)=:e^{a_-^M(q^{-\frac{k+g}{2}}z)-b_\pm^{j,M}(q^{-k-j}z)-(b+c)^{j,M}(q^{-k-j\mp1}z)-b_-^{j,M+1}(q^{-k-j}z)-b^{j,M+1}(q^{-k-j+1}z)}$$
$$\times e^{-\sum_{l=j+1}^{M-1}(\Delta_R^0b_-^{l,M})(q^{-k-l}z)+\sum_{l=M+2}^{M+N}(\Delta_L^0b_-^{M,l})(q^{-k-2M-1+l}z)}:. \tag{3.39}$$

For $M+1\le i\le M+N-1$ and $1\le j\le M$ we set

$$F_{i,j}^{1}(z)=:e^{a_-^i(q^{-\frac{k+g}{2}}z)-b_-^{j,i+1}(q^{-k-j}z)-b^{j,i+1}(q^{-k-j+1}z)+b^{j,i}(q^{-k-j}z)-\sum_{l=j+1}^{M}(\Delta_R^-b_-^{l,i})(q^{-k-l+1}z)}$$
$$\times e^{-\sum_{l=M+1}^{i}(\Delta_R^-b_-^{l,i})(q^{-k-2M+l}z)+\sum_{l=i+1}^{M+N}(\Delta_L^-b_-^{i,l})(q^{-k-2M+l}z)}:. \tag{3.40}$$

For $M+1\le i\le M+N-1$ and $M+1\le j\le i-1$ we set

$$F_{i,j}^{1,\pm}(z)=:e^{a_-^i(q^{-\frac{k+g}{2}}z)+(b+c)^{j,i+1}(q^{-k-2M+j}z)+b_\pm^{j,i}(q^{-k-2M+j}z)-(b+c)^{j,i}(q^{-k-2M\pm1+j}z)}$$
$$\times e^{-\sum_{l=j+1}^{i}(\Delta_R^-b_-^{l,i})(q^{-k-2M+l}z)+\sum_{l=i+1}^{M+N}(\Delta_L^-b_-^{i,l})(q^{-k-2M+l}z)}:. \tag{3.41}$$

For $1\le i\le M-1$ we set

$$F_{i,i}^{2,\pm}(z)=:e^{a_\pm^i(q^{\pm\frac{k+g}{2}}z)+b_\pm^{i,i+1}(q^{\pm(k+i+1)}z)+(b+c)^{i,i+1}(q^{\pm(k+i)}z)}$$
$$\times e^{-\sum_{l=i+2}^{M}(\Delta_L^\mp b_\pm^{i,l})(q^{\pm(k+l)}z)-\sum_{l=M+1}^{M+N}(\Delta_L^\mp b_\pm^{i,l})(q^{\pm(k+2M+1-l)}z)}:, \tag{3.42}$$
$$F_{M,M}^{2,\pm}(z)=:e^{a_\pm^M(q^{\pm\frac{k+g}{2}}z)-b^{M,M+1}(q^{\pm(k+M-1)}z)+\sum_{l=M+2}^{M+N}(\Delta_L^0b_\pm^{M,l})(q^{\pm(k+2M+1-l)}z)}:. \tag{3.43}$$

For $M+1\le i\le M+N-1$ we set

$$F_{i,i}^{2,\pm}(z)=:e^{a_\pm^i(q^{\pm\frac{k+g}{2}}z)-b_\pm^{i,i+1}(q^{\pm(k+2M-1-i)}z)+(b+c)^{i,i+1}(q^{\pm(k+2M-i)}z)}$$
$$\times e^{\sum_{l=i+2}^{M+N}(\Delta_L^\pm b_\pm^{i,l})(q^{\pm(k+2M-l)}z)}:. \tag{3.44}$$

For $1 \le i \le M-2$ and $i+2 \le j \le M$ we set

$$F_{i,j}^{3,\pm}(z) =: e^{a_+^i(q^{\frac{k+g}{2}}z)+(b+c)^{i,j}(q^{k+j-1}z)+b_\pm^{i+1,j}(q^{k+j-1}z)-(b+c)^{i+1,j}(q^{k-1\pm 1+j}z)} \times e^{-\sum_{l=j}^{M}(\Delta_L^- b_+^{i,l})(q^{k+l}z)-\sum_{l=M+1}^{M+N}(\Delta_L^- b_+^{i,l})(q^{k+2M+1-l}z)} : . \tag{3.45}$$

For $1 \le i \le M-1$ and $M+1 \le j \le M+N$ we set

$$F_{i,j}^{3}(z) =: e^{a_+^i(q^{\frac{k+g}{2}}z)-b^{i,j}(q^{k+2M-j}z)-b_+^{i+1,j}(q^{k+2M-j}z)+b^{i+1,j}(q^{k+2M+1-j}z)} \times e^{-\sum_{l=j+1}^{M+N}(\Delta_L^- b_+^{i,l})(q^{k+2M+1-l}z)} : . \tag{3.46}$$

For $M+2 \le j \le M+N$ we set

$$F_{M,j}^{3,\pm}(z) =: e^{a_+^M(q^{\frac{k+g}{2}}z)-b^{M,j}(q^{k+2M-j}z)-b_\pm^{M+1,j}(q^{k+2M+1-j}z)-(b+c)^{M+1,j}(q^{k+2M+1\mp 1-j}z)} \times e^{b_+^{M+1,j}(q^{k+2M+1-j}z)+\sum_{l=j+1}^{M+N}(\Delta_L^0 b_+^{M,l})(q^{k+2M+1-l}z)} : . \tag{3.47}$$

For $M+1 \le i \le M+N-1$ and $i+2 \le j \le M+N$ we set

$$F_{i,j}^{3,\pm}(z) =: e^{a_+^i(q^{\frac{k+g}{2}}z)+(b+c)^{i,j}(q^{k+2M+1-j}z)-b_\pm^{i+1,j}(q^{k+2M+1-j}z)-(b+c)^{i+1,j}(q^{k+2M+1\mp 1-j}z)} \times e^{\sum_{l=j+1}^{M+N}(\Delta_L^+ b_+^{i,l})(q^{k+2M-l}z)} : . \tag{3.48}$$

Here we set $d_{i,j}^1, d_{i,i}^2, d_{i,j}^3 \in \mathbf{C}$ as follows.

$$d_{i,j}^1 = \nu_{i+1}\frac{1}{c_{i,j}} \times \begin{cases} 1 & (1 \le i \le M-1, 1 \le j \le i-1), \\ q^{j-1} & (i = M, 1 \le j \le M-1), \\ q^{-k-1} & (M+1 \le i \le M+N-1, 1 \le j \le M), \\ 1 & (M+1 \le i \le M+N-1, M+1 \le j \le i-1), \end{cases} \tag{3.49}$$

$$d_{i,i}^2 = \nu_{i+1}\frac{1}{c_{i,i}} \times \begin{cases} 1 & (1 \le i \ne M \le M+N-1), \\ q^{M-1} & (i = M), \end{cases} \tag{3.50}$$

$$d_{i,j}^3 = \nu_{i+1}\frac{1}{c_{i,i}}\prod_{l=1}^{j-i-1}\frac{c_{i+l,i+1}}{c_{i+l,i}} \times \begin{cases} 1 & (1 \le i \le M-1, i+2 \le j \le M), \\ q^{k+3M+1-2j} & (1 \le i \le M-1, M+1 \le j \le M+N), \\ q^{(M-1)(j-M)} & (i = M, M+2 \le j \le M+N), \\ 1 & (M+1 \le i \le M+N-1, i+2 \le j \le M+N). \end{cases} \tag{3.51}$$

The following is **the first main result** of this article.

Theorem 3.1. *The bosonic operators $H^i = H_0^i$, H_m^i ($m \in \mathbf{Z}_{\ne 0}, 1 \le i \le M+N-1$) defined in (3.18)–(3.21), $\Psi_\pm^i(z)$ ($1 \le i \le M+N-1$) defined in (3.22)–(3.24), and $X^{\pm,i}(z)$ ($1 \le i \le M+N-1$) defined in (3.25)–(3.27) and (3.35)–(3.37) satisfy the defining relations of the Drinfeld realization (2.15)–(2.24) with the central element $c = k \in \mathbf{C}$. Here the coefficients $d_{i,j}^1$, $d_{i,i}^2$, and $d_{i,j}^3$ are given in (3.49)–(3.51).*

This bosonization reproduces those of $U_q(\widehat{sl}(M|1))$ upon the specialization $N=1$ [20].

We introduce the boson Fock space $F(p_a,p_b,p_c)$ as follows. The vacuum state $|0\rangle\neq 0$ is defined by

$$a_m^i|0\rangle = b_m^{i,j}|0\rangle = c_m^{i,j}|0\rangle = 0 \quad (m\geq 0). \tag{3.52}$$

Let $|p_a,p_b,p_c\rangle$ be

$$|p_a,p_b,p_c\rangle = \exp\left(\sum_{i,j=1}^{M+N-1}\frac{(A^{-1})_{i,j}}{k+g}p_a^iQ_a^i - \sum_{1\leq i<j\leq M+N-1}\nu_i\nu_j p_b^{i,j}Q_b^{i,j} + \sum_{\substack{1\leq i<j\leq M+N-1\\ \nu_i\nu_j=+1}} p_c^{i,j}Q_c^{i,j}\right)|0\rangle, \tag{3.53}$$

then $|p_a,p_b,p_c\rangle$ is the highest weight state of the boson Fock space $F(p_a,p_b,p_c)$, i.e.,

$$a_m^i|p_a,p_b,p_c\rangle = b_m^{i,j}|p_a,p_b,p_c\rangle = c_m^{i,j}|p_a,p_b,p_c\rangle = 0 \quad (m>0), \tag{3.54}$$

$$a_0^i|p_a,p_b,p_c\rangle = p_a^i|p_a,p_b,p_c\rangle, \quad b_0^{i,j}|p_a,p_b,p_c\rangle = p_b^{i,j}|p_a,p_b,p_c\rangle, \tag{3.55}$$

$$c_0^{i,j}|p_a,p_b,p_c\rangle = p_c^{i,j}|p_a,p_b,p_c\rangle \quad (\nu_i\nu_j=+1). \tag{3.56}$$

The boson Fock space $F(p_a,p_b,p_c)$ is generated by the bosons $a_m^i, b_m^{i,j}, c_m^{i,j}$ on the highest weight state $|p_a,p_b,p_c\rangle$. We set the space $F(p_a)$ by

$$F(p_a) = \bigoplus_{\substack{p_b^{i,j}=-p_c^{i,j}\in\mathbf{Z}\ (\nu_i\nu_j=+)\\ p_b^{i,j}\in\mathbf{Z}\ (\nu_i\nu_j=-)}} F(p_a,p_b,p_c). \tag{3.57}$$

Here we impose the restriction $p_b^{i,j}=-p_c^{i,j}$ $(\nu_i\nu_j=+)$, because $X_m^{\pm,i}$ change $Q_b^{i,j}+Q_c^{i,j}$. $F(p_a)$ is $U_q(\widehat{sl}(M|N))$-module. Let $|\lambda\rangle=|p_a,0,0\rangle$ where $p_a^i=l_i$ $(1\leq i\leq M+N-1)$.

Proposition 3.2. *The Drinfeld generators H^i, H_m^i, $X_m^{\pm,i}$ act on $|\lambda\rangle$ as follows.*

$$H_m^i|\lambda\rangle = X_m^{\pm,i}|\lambda\rangle = 0 \quad (m>0), \tag{3.58}$$

$$X_0^{+,i}|\lambda\rangle = 0, \tag{3.59}$$

$$H^i|\lambda\rangle = l_i|\lambda\rangle. \tag{3.60}$$

This property is just the highest weight state condition of the highest weight module $V(\lambda)$.

Corollary 3.3. *We have the level-k highest weight module $V(\lambda)$ of $U_q(\widehat{sl}(M|N))$:*

$$V(\lambda)\subset F(p_a). \tag{3.61}$$

Here the classical part of the highest weight is $\bar{\lambda}=\sum_{i=1}^{M+N-1}l_i\bar{\Lambda}_i$.

The module $F(p_a)$ is not irreducible. In [17] the irreducible highest weight module of $U_q(\widehat{sl}(2))$ was constructed by two steps from the similar module as $F(p_a)$: the first step is construction of Wakimoto module using $\xi-\eta$ system, and the second step is resolution by Felder complex using screening operators Q_i [31]. Construction of Felder complex is an open problem even for non-superalgebra $U_q(\widehat{sl}(3))$. In this paper we propose Wakimoto module of $U_q(\widehat{sl}(M|N))$ using $\xi-\eta$ system. We would like to report on Felder-complex of $U_q(\widehat{sl}(M|N))$ in future publication.

We set bosonic operators $\xi_m^{i,j}$, $\eta_m^{i,j}$ ($\nu_i\nu_j=+1, 1\leq i<j\leq M+N-1$) as follows.

$$\eta^{i,j}(z)=\sum_{m\in\mathbf{Z}}\eta_m^{i,j}z^{-m-1}=:e^{c^{i,j}(z)}:,$$
$$\xi^{i,j}(z)=\sum_{m\in\mathbf{Z}}\xi_m^{i,j}z^{-m}=:e^{-c^{i,j}(z)}:. \tag{3.62}$$

Fourier components $\eta_m^{i,j}=\oint\frac{dz}{2\pi\sqrt{-1}}z^m\eta^{i,j}(z)$, $\xi_m^{i,j}=\oint\frac{dz}{2\pi\sqrt{-1}}z^{m-1}\xi^{i,j}(z)$ are well-defined on the module $F(p_a)$. The $\mathbf{Z}_2$-grading is given by $p(\xi_m^{i,j})=p(\eta_m^{i,j})=+1$. They satisfy anti-commutation relations.

$$[\eta_m^{i,j},\xi_m^{i,j}]=\delta_{m+n,0},\quad [\eta_m^{i,j},\eta_n^{i,j}]=[\xi_m^{i,j},\xi_n^{i,j}]=0. \tag{3.63}$$

They commute with each other.

$$[\eta_m^{i,j},\xi_m^{i',j'}]=[\eta_m^{i,j},\eta_n^{i',j'}]=[\xi_m^{i,j},\xi_n^{i',j'}]=0\quad ((i,j)\neq(i',j')). \tag{3.64}$$

The operators $\eta_0^{i,j}$, $\xi_0^{i,j}$ satisfy

$$\mathrm{Im}(\eta_0^{i,j})=\mathrm{Ker}(\eta_0^{i,j}),\quad \mathrm{Im}(\xi_0^{i,j})=\mathrm{Ker}(\xi_0^{i,j}), \tag{3.65}$$
$$\eta_0^{i,j}\xi_0^{i,j}+\xi_0^{i,j}\eta_0^{i,j}=1, \tag{3.66}$$
$$(\eta_0^{i,j}\xi_0^{i,j})^2=\eta_0^{i,j}\xi_0^{i,j},\quad (\xi_0^{i,j}\eta_0^{i,j})^2=\xi_0^{i,j}\eta_0^{i,j}, \tag{3.67}$$
$$(\xi_0^{i,j}\eta_0^{i,j})(\eta_0^{i,j}\xi_0^{i,j})=(\eta_0^{i,j}\xi_0^{i,j})(\xi_0^{i,j}\eta_0^{i,j})=0. \tag{3.68}$$

Hence we have direct sum decomposition.

$$F(p_a)=\eta_0^{i,j}\xi_0^{i,j}F(p_a)\oplus\xi_0^{i,j}\eta_0^{i,j}F(p_a), \tag{3.69}$$

where $\mathrm{Ker}(\eta_0^{i,j})=\eta_0^{i,j}\xi_0^{i,j}F(p_a)$, $\mathrm{Coker}(\eta_0^{i,j})=\xi_0^{i,j}\eta_0^{i,j}F(p_a)$. We set

$$\eta_0=\prod_{\substack{1\leq i<j\leq M+N-1\\ \nu_i\nu_j=+1}}\eta_0^{i,j},\quad \xi_0=\prod_{\substack{1\leq i<j\leq M+N-1\\ \nu_i\nu_j=+1}}\xi_0^{i,j}. \tag{3.70}$$

We introduce the subspace $\mathcal{F}(p_a)$ that gives a generalization of the articles [21,22] by

$$\mathcal{F}(p_a)=\eta_0\xi_0F(p_a). \tag{3.71}$$

The operators $\eta_0^{i,j}$, $\xi_0^{i,j}$ commute with $X^{\pm,i}(z)$, $\Psi_\pm^i(z)$ up to sign ±1.

Proposition 3.4. *$\mathcal{F}(p_a)$ is $U_q(\widehat{sl}(M|N))$-module.*

We call $\mathcal{F}(p_a)$ Wakimoto module of $U_q(\widehat{sl}(M|N))$.

4. Screening Operator

In this Section we give the screening operators Q_i $(1 \le i \le M+N-1)$ that commute with $U_q(\widehat{sl}(M|N))$ for the level $c = k \ne -g$. We define bosonic operators $S_i(z)$ $(1 \le i \le M+N-1)$ that we call the screening currents as follows.

$$S_i(z) = \sum_{j=i+1}^{M} \frac{e_{i,j}}{(q-q^{-1})z}(S_{i,j}^-(z) - S_{i,j}^+(z)) + \sum_{j=M+1}^{M+N} e_{i,j} S_{i,j}(z) \quad (1 \le i \le M-1), \tag{4.1}$$

$$S_M(z) = \sum_{j=M+1}^{M+N} e_{M,j} S_{M,j}(z), \tag{4.2}$$

$$S_i(z) = \sum_{j=i+1}^{M+N} \frac{e_{i,j}}{(q-q^{-1})z}(S_{i,j}^-(z) - S_{i,j}^+(z)) \quad (M+1 \le i \le M+N-1), \tag{4.3}$$

where we set

$$S_{i,j}^{\pm}(z) =: e^{-(\frac{1}{k+g}a^i)(z;\frac{k+g}{2})}\widetilde{S}_{i,j}^{\pm}(z) :, \quad S_{i,j}(z) =: e^{-(\frac{1}{k+g}a^i)(z;\frac{k+g}{2})}\widetilde{S}_{i,j}(z) : . \tag{4.4}$$

Here we set $e_{i,j}$ as follows.

$$e_{i,i+1} = \begin{cases} 1/d_{i,i}^2 & (1 \le i \le M-1), \\ -q^{-N+1}/d_{M,M}^2 & (i = M), \\ -1/d_{i,i}^2 & (M+1 \le i \le M+N-1), \end{cases} \tag{4.5}$$

$$e_{i,j} = \begin{cases} 1/d_{i,j}^3 & (1 \le i \le M-1, i+2 \le j \le M), \\ q^{k+1+M-N}/d_{i,j}^3 & (1 \le i \le M-1, M+1 \le j \le M+N), \\ -q^{j-M-N}/d_{M,j}^3 & (i = M, M+2 \le j \le M+N), \\ -1/d_{i,j}^3 & (M+1 \le i \le M+N-1, i+2 \le j \le M+N). \end{cases} \tag{4.6}$$

For $1 \le i \le M-1$ and $i+1 \le j \le M$ we set

$$\tilde{S}_{i,j}^{\pm}(z) =: e^{(b+c)^{i+1,j}(q^{M-N-j}z) - b_{\pm}^{i,j}(q^{M-N-j}z) - (b+c)^{i,j}(q^{M-N-j\mp 1}z)} \times e^{\sum_{l=j+1}^{M}(\Delta_L^+ b_-^{i,l})(q^{M-N-l}z) + \sum_{l=M+1}^{M+N}(\Delta_L^+ b_-^{i,l})(q^{-M-N+l-1}z)} : . \tag{4.7}$$

For $1 \le i \le M-1$ and $M+1 \le j \le M+N$ we set

$$\tilde{S}_{i,j}(z) =: e^{b^{i,j}(q^{-M-N+j}z) + b_+^{i+1,j}(q^{-M-N+j}z) - b^{i+1,j}(q^{-M-N+j+1}z)} \times e^{\sum_{l=j+1}^{M+N}(\Delta_L^+ b_-^{i,l})(q^{-M-N-1+l}z)} : . \tag{4.8}$$

For $M+1 \le j \le M+N$ we set

$$\tilde{S}_{M,j}(z) =: e^{(b+c)^{M+1,j}(q^{-M-N+j}z) + b^{M,j}(q^{-M-N+j}z) - \sum_{l=j+1}^{M+N}(\Delta_L^0 b_-^{M,l})(q^{-M-N-1+l}z)} : . \tag{4.9}$$

For $M+1 \le i \le M+N-1$ and $i+1 \le j \le M+N$ we set

$$\tilde{S}^{\pm}_{i,j}(z) =: e^{(b+c)^{i+1,j}(q^{-M-N+j}z)+b^{i,j}_{\pm}(q^{-M-N+j}z)-(b+c)^{i,j}(q^{-M-N+j\pm1}z)} \times e^{-\sum_{l=j+1}^{M+N}(\Delta_L^- b_-^{i,l})(q^{-M-N+l}z)} :. \tag{4.10}$$

The $\mathbf{Z}_2$-grading of the screening current is given by $p(S_{M,j}(z)) \equiv 1 \pmod 2$ for $M+1 \le j \le M+N$ and zero otherwise. The following is **the second main result** of this article.

Theorem 4.1. *The screening currents $S_i(z)$ $(1 \le i \le M+N-1)$ defined in* (4.1), (4.2), and (4.3) *commute with $U_q(\widehat{sl}(M|N))$ up to total difference.*

$$[S_i(z), H^j] = 0, \tag{4.11}$$

$$[S_i(z), H^j_m] = 0 \quad (m \in \mathbf{Z}), \tag{4.12}$$

$$[S_i(z_1), X^{+,j}(z_2)] = 0, \tag{4.13}$$

$$[S_i(z_1), X^{-,j}(z_2)] = \frac{\delta_{i,j}}{(q-q^{-1})z_1z_2}(\delta(q^{k+g}z_2/z_1) - \delta(q^{-k-g}z_2/z_1)) \times : e^{-(\frac{a^i}{k+g})(z_1|-\frac{k+g}{2})} :. \tag{4.14}$$

These screening currents reproduces those of $U_q(\widehat{sl}(M|1))$ upon the specialization $N = 1$ [21,23].

The q-difference operator with a parameter α is defined by

$$_\alpha\partial_z f(z) = \frac{f(q^\alpha z) - f(q^{-\alpha}z)}{(q-q^{-1})z}. \tag{4.15}$$

The Jackson integral with parameters $p \in \mathbf{C}$ $(|p| < 1)$ and $s \in \mathbf{C}^*$ is defined by

$$\int_0^{s\infty} f(w)d_pw = s(1-p)\sum_{n\in\mathbf{Z}} f(sp^n)p^n. \tag{4.16}$$

The Jackson integral of the q-difference satisfy the following property.

$$\int_0^{s\infty} {}_\alpha\partial_w f(w)d_pw = 0 \quad (p = q^{2\alpha}). \tag{4.17}$$

We define the screening operators Q_i $(1 \le i \le M+N-1)$ as follows, when the Jackson integrals are convergent.

$$Q_i = \int_0^{s\infty} S_i(w)d_{q^{2(k+g)}}w. \tag{4.18}$$

Corollary 4.2. *The screening operators Q_i $(1 \le i \le M+N-1)$ commute with $U_q(\widehat{sl}(M|N))$.*

$$[Q_i, U_q(\widehat{sl}(M|N))] = 0. \tag{4.19}$$

For $r>0$ we define the theta function $[u]_r$ as

$$[u]_r = q^{\frac{u^2}{r}-u}\frac{\Theta_{q^{2r}}(q^{2u})}{(q^{2r};q^{2r})_\infty}, \tag{4.20}$$

where we set

$$\Theta_p(z)=(p;p)_\infty(z;p)_\infty(p/z;p)_\infty, \quad (z;p)_\infty=\prod_{m=0}^{\infty}(1-p^m z). \tag{4.21}$$

The theta function $[u]_r$ satisfies the following quasi-periodicity condition

$$[u+r]_r=-[u]_r, \quad [u+r\tau]_r=-e^{-\pi\sqrt{-1}\tau-\frac{2\pi\sqrt{-1}}{r}u}[u]_r, \tag{4.22}$$

where $\tau\in\mathbf{C}$ is given by $q=e^{-\pi\sqrt{-1}/r\tau}$.

Proposition 4.3. *The screening currents $S_i(z)$ $(1\le i\le M+N-1)$ satisfy*

$$\left[u_1-u_2+\frac{A_{i,j}}{2}\right]_{k+g} S_i(z_1)S_j(z_2)=\left[u_2-u_1+\frac{A_{i,j}}{2}\right]_{k+g} S_j(z_2)S_i(z_1), \tag{4.23}$$

where $q^{2u_j}=z_j$ $(j=1,2)$.

5. Proof of Main Results

In this Section we will show Theorems 3.1 and 4.1.

5.1. Proof of Theorem 3.1.

We will show

$$[X^{+,i}(z_1),X^{-,i}(z_2)] = \frac{1}{(q-q^{-1})z_1z_2}\left(\delta(q^k z_2/z_1)\Psi_+^i(q^{\frac{k}{2}}z_2)-\delta(q^{-k}z_2/z_1)\Psi_-^i(q^{-\frac{k}{2}}z_2)\right) \tag{5.1}$$

for $1\le i\le M+N-1$.

• For $1\le i\le M-1$ we have

$$[E_{i,i}^+(z_1),F_{i,i}^{2,+}(z_2)]=-(q-q^{-1})\delta(q^k z_2/z_1):E_{i,i}^+(z_1)F_{i,i}^{2,+}(z_2):, \tag{5.2}$$

$$[E_{i,j}^{\pm}(z_1),F_{i,j}^{1,\pm}(z_2)]=\mp(q-q^{-1})\delta(q^{-k-2j+1\mp1}z_2/z_1):E_{i,j}^{\pm}(z_1)F_{i,j}^{1,\pm}(z_2): \quad (1\le j\le i-1), \tag{5.3}$$

$$[E_{i,i}^-(z_1),F_{i,i}^{2,-}(z_2)]=(q-q^{-1})\delta(q^{-k-2i+2}z_2/z_1):E_{i,i}^-(z_1)F_{i,i}^{2,-}(z_2):. \tag{5.4}$$

The remaining commutators vanish. Hence we have

$$[X^{+,i}(z_1),X^{-,i}(z_2)]\times(q-q^{-1})z_1z_2 = \sum_{j=1}^{i-2}\delta(q^{-k-2j}z_2/z_1)\Big(c_{i,j}d_{i,j}^1:E_{i,j}^+(z_1)F_{i,j}^{1,+}(z_2): -c_{i,j+1}d_{i,j+1}^1:E_{i,j+1}^-(z_1)F_{i,j+1}^{1,-}(z_2):\Big)$$

$$+\delta(q^{-k-2i+2}z_2/z_1)\Big(c_{i,i-1}d^1_{i,i-1} : E^+_{i,i-1}(z_1)F^{1,-}_{i,j+1}(z_2) :$$
$$-c_{i,i}d^2_{i,i} : E^-_{i,i}(z_1)f^{2,-}_{i,i}(z_2) :\Big) + \delta(q^k z_2/z_1)c_{i,i}d^2_{i,i} : E^+_{i,i}(z_1)F^{2,+}_{i,i}(z_2) :$$
$$-c_{i,1}d^1_{i,1}\delta(q^{-k}z_2/z_1) : E^-_{i,1}(z_1)F^{1,-}_{i,1}(z_2) : . \tag{5.5}$$

For $1 \leq i \leq M-1$ we have

$$: E^+_{i,j}(q^{-k-2j}z)F^{1,+}_{i,j}(z) :=: E^-_{i,j+1}(q^{-k-2j}z)F^{1,-}_{i,j+1}(z) : \quad (1 \leq j \leq i-2), \tag{5.6}$$
$$: E^+_{i,i-1}(q^{-k-2i+2}z)F^{1,+}_{i,i-1}(z) :=: E^-_{i,i}(q^{-k-2i+2}z)F^{2,-}_{i,i}(z) :, \tag{5.7}$$
$$: E^+_{i,i}(q^k z)F^{2,+}_{i,i}(z) := \Psi^i_+(q^{\frac{k}{2}}z), \quad : E^-_{i,1}(q^{-k}z)F^{1,-}_{i,1}(z) := \Psi^i_-(q^{-\frac{k}{2}}z), \tag{5.8}$$

and

$$c_{i,j}d^1_{i,j} = c_{i,j+1}d^1_{i,j+1} \quad (1 \leq j \leq i-2), \tag{5.9}$$
$$c_{i,i-1}d^1_{i,i-1} = c_{i,i}d^2_{i,i}, \tag{5.10}$$
$$c_{i,i}d^2_{i,i} = c_{i,1}d^1_{i,1} = 1. \tag{5.11}$$

Hence we have (5.1) for $1 \leq i \leq M-1$.

• For $i = M$ we have

$$[E_{M,M}(z_1), F^{2,+}_{M,M}(z_2)] = \frac{1}{q^{M-1}z_1}\delta(q^k z_2/z_1) : E_{M,M}(z_1)F^{2,+}_{M,M}(z_2) :, \tag{5.12}$$
$$[E_{M,j}(z_1), F^{1,\pm}_{M,j}(z_2)] = \frac{1}{q^{j-1}z_1}\delta(q^{-k-2j+1\mp 1}z_2/z_1) : E_{M,j}(z_1)F^{1,\pm}_{M,j}(z_2) : \quad (1 \leq j \leq M-1), \tag{5.13}$$
$$[E_{M,M}(z_1), F^{2,-}_{M,M}(z_2)] = \frac{1}{q^{M-1}z_1}\delta(q^{-k-2M+2}z_2/z_1) : E_{M,M}(z_1)F^{2,-}_{M,M}(z_2) : . \tag{5.14}$$

The remaining anti-commutators vanish. Hence we have

$$[X^{+,M}(z_1), X^{-,M}(z_2)] \times (q-q^{-1})z_1z_2$$
$$= \sum_{j=1}^{M-2}\delta(q^{-k-2j}z_2/z_1)\left(\frac{c_{M,j+1}d^1_{M,j+1}}{q^j} : E_{M,j+1}(z_1)F^{1,-}_{M,j+1}(z_2) :\right.$$
$$\left.-\frac{c_{M,j}d^1_{M,j}}{q^{j-1}} : E_{M,j}(z_1)F^{1,+}_{M,j}(z_2) :\right) + \delta(q^{-k-2M+2}z_2/z_1)$$
$$\times\left(\frac{c_{M,M}d^2_{M,M}}{q^{M-1}} : E_{M,M}(z_1)F^{2,-}_{M,M}(z_2) :\right.$$
$$\left.-\frac{c_{M,M-1}d^1_{M,M-1}}{q^{M-2}} : E_{M,M-1}(z_1)F^{1,+}_{M,M-1}(z_2) :\right)$$
$$-\delta(q^k z_2/z_1)\frac{c_{M,M}d^2_{M,M}}{q^{M-1}} : E_{M,M}(z_1)F^{2,+}_{M,M}(z_2) :$$

$$+\delta(q^{-k}z_2/z_1)c_{M,1}d^1_{M,1}:E_{M,1}(z_1)F^{1,-}_{M,1}(z_2):. \quad (5.15)$$

Moreover we have

$$:E_{M,j}(q^{-k-2j}z)F^{1,+}_{M,j}(z):=:E_{M,j+1}(q^{-k-2j}z)F^{1,-}_{M,j+1}(z):$$
$$(1\le j\le M-2), \quad (5.16)$$
$$:E_{M,M-1}(q^{-k-2M+2}z)F^{1,+}_{M,M-1}(z):=:E_{M,M}(q^{-k-2M+2}z)F^{2,-}_{M,M}(z):, \quad (5.17)$$
$$:E_{M,M}(q^{k}z)F^{2,+}_{M,M}(z):=\Psi^M_+(q^{\frac{k}{2}}z),$$
$$:E_{M,1}(q^{-k}z)F^{1,-}_{M,1}(z):=\Psi^M_-(q^{-\frac{k}{2}}z), \quad (5.18)$$

and

$$qc_{M,j}d^1_{M,j}=c_{M,j+1}d^1_{M,j+1}\quad (1\le j\le M-2), \quad (5.19)$$
$$qc_{M,M-1}d^1_{M,M-1}=c_{M,M}d^2_{M,M}, \quad (5.20)$$
$$c_{M,M}d^2_{M,M}=-q^{M-1},\quad c_{M,1}d^1_{M,1}=-1. \quad (5.21)$$

Hence we have (5.1) for $i=M$.
• For $M+1\le i\le M+N-1$ we have

$$[E^+_{i,i}(z_1),F^{2,+}_{i,i}(z_2)]=(q-q^{-1})\delta(q^kz_2/z_1):E^+_{i,i}(z_1)F^{2,+}_{i,i}(z_2):, \quad (5.22)$$
$$[E_{i,j}(z_1),F^1_{i,j}(z_2)]=\frac{q^{k+1}}{(q-q^{-1})z_1z_2}(\delta(q^{-k-2j+2}z_2/z_1)-\delta(q^{-k-2j}z_2/z_1))$$
$$\times :E_{i,j}(z_1)F^1_{i,j}(z_2):\quad (1\le j\le M), \quad (5.23)$$
$$[E^{\pm}_{i,j}(z_1),F^{1,\pm}_{i,j}(z_2)]=\pm(q-q^{-1})\delta(q^{-k-4M+2j-1\pm 1}z_2/z_1):E^{\pm}_{i,j}(z_1)F^{1,\pm}_{i,j}(z_2):$$
$$(M+1\le j\le i-1), \quad (5.24)$$
$$[E^-_{i,i}(z_1),F^{2,-}_{i,i}(z_2)]=-(q-q^{-1})\delta(q^{-k-4M+2i-2}z_2/z_1)$$
$$:E^-_{i,i}(z_1)F^{2,-}_{i,i}(z_2):. \quad (5.25)$$

The remaining commutators vanish. Hence we have

$$[X^{+,i}(z_1),X^{-,i}(z_2)]\times(q-q^{-1})z_1z_2$$
$$=\sum_{j=1}^{M-1}\delta(q^{-k-2j}z_2/z_1)q^{k+1}\Big(c_{i,j+1}d^1_{i,j+1}:E_{i,j+1}(z_1)F^1_{i,j+1}(z_2):$$
$$-c_{i,j}d^1_{i,j}:E_{i,j}(z_1)F^1_{i,j}(z_2):\Big)+\delta(q^{-k-2M}z_2/z_1)$$
$$\Big(c_{i,M+1}d^1_{i,M+1}:E^-_{i,M+1}(z_1)F^{1,-}_{i,M+1}(z_2):-c_{i,M}d^1_{i,M}q^{k+1}:E_{i,M}(z_1)F^1_{i,M}(z_2):\Big)$$
$$+\sum_{j=M+1}^{i-2}\delta(q^{-k-4M+2j}z_2/z_1)\Big(c_{i,j+1}d^1_{i,j+1}:E^-_{i,j+1}(z_1)F^{1,-}_{i,j+1}(z_2):$$
$$-c_{i,j}d^1_{i,j}:E^+_{i,j}(z_1)F^{1,+}_{i,j}(z_2):\Big)+\delta(q^{-k-4M+2i-2}z_2/z_1)$$
$$\Big(c_{i,i}d^2_{i,i}:E^-_{i,i}(z_1)F^{2,-}_{i,i}(z_2):-c_{i,i-1}d^1_{i,i-1}:E^+_{i,i-1}(z_1)F^1_{i,i-1}(z_2):\Big)$$

$$-\delta(q^k z_2/z_1)c_{i,i}d^2_{i,i} : E^+_{i,i}(z_1)F^{2,+}_{i,i}(z_2) :$$
$$+c_{i,1}d^1_{i,1}q^{k+1}\delta(q^{-k}z_2/z_1) : E_{i,1}(z_1)F^1_{i,1}(z_2) : . \quad (5.26)$$

For $M+1 \leq i \leq M+N-1$ we have

$$: E_{i,j+1}(q^{-k-2j}z)F_{i,j+1}(z) :=: E_{i,j}(q^{-k-2j}z)F^1_{i,j}(z) :$$
$$(1 \leq j \leq M-1), \quad (5.27)$$
$$: E^-_{i,M+1}(q^{-k-2M}z)F^{1,-}_{i,M+1}(z) :=: E_{i,M}(q^{-k-2M}z)F^1_{i,M}(z) :, \quad (5.28)$$
$$: E^-_{i,j+1}(q^{-k-4M+2j}z)F^{1,-}_{i,j+1}(z) :=: E^+_{i,j}(q^{-k-4M+2j}z)F^{1,+}_{i,j}(z) :$$
$$(M+1 \leq j \leq i-2), \quad (5.29)$$
$$: E^-_{i,i}(q^{-k-4M+2i-2}z)F^{2,-}_{i,i}(z) :=: E^+_{i,i-1}(q^{-k-4M+2i-2}z)F^{1,+}_{i,i-1}(z) :, \quad (5.30)$$
$$: E^+_{i,i}(q^k z)F^{2,+}_{i,i}(z) := \Psi^i_+(q^{\frac{k}{2}}z), \quad : E_{i,1}(q^{-k}z)F^1_{i,1}(z) := \Psi^i_-(q^{-\frac{k}{2}}z), \quad (5.31)$$

and

$$c_{i,j+1}d^1_{i,j+1} = c_{i,j}d^1_{i,j} \quad (1 \leq j \leq M-1 \text{ or } M+1 \leq i-2), \quad (5.32)$$
$$c_{i,M+1}d^1_{i,M+1} = q^{k+1}c_{i,M}d_{i,M}, \quad (5.33)$$
$$c_{i,i}d^2_{i,i} = c_{i,i-1}d^1_{i,i-1}, \quad (5.34)$$
$$c_{i,i}d^2_{i,i} = q^{k+1}c_{i,1}d^1_{i,1} = -1. \quad (5.35)$$

Hence we have (5.1) for $M+1 \leq i \leq M+N-1$. Now we have shown (5.1) for all $1 \leq i \leq M+N-1$. Other defining relations of the Drinfeld realization (2.15)–(2.24) are shown in the same way. We summarize useful formulae for proof of theorem in "Appendix A".

5.2. Proof of Theorem 4.1. First, we will show

$$[S_i(z_1), X^{-,i}(z_2)]$$
$$= \frac{1}{(q-q^{-1})z_1z_2}\Big(\delta(q^{k+g}z_2/z_1) - \delta(q^{-k-g}z_2/z_1)\Big) : e^{-\left(\frac{1}{k+g}a^i\right)\left(z_1|-\frac{k+g}{2}\right)} : \quad (5.36)$$

for $1 \leq i \leq M+N-1$.

• For $1 \leq i \leq M-1$ we have

$$[S^+_{i,i+1}(z_1), F^{2,+}_{i,i}(z_2)] = (q-q^{-1})\delta(q^{k-M+N+2i+2}z_2/z_1) : S^+_{i,i+1}(z_1)F^{2,+}_{i,i}(z_2) :, \quad (5.37)$$
$$[S^-_{i,i+1}(z_1), F^{2,-}_{i,i}(z_2)] = -(q-q^{-1})\delta(q^{k-M+N}z_2/z_1) : S^-_{i,i+1}(z_1)F^{2,-}_{i,i}(z_2) :, \quad (5.38)$$

$$[S_{i,j}(z_1), F^3_{i,j}(z_2)] = \frac{q^{-k-M+N-1}}{(q-q^{-1})z_1z_2} \Big(\delta(q^{k+3M+N-2j}z_2/z_1) - \delta(q^{k+3M+N+2-2j}z_2/z_1)\Big) \times :S_{i,j}(z_1)F^3_{i,j}(z_2): \quad (M+1 \le j \le M+N). \tag{5.39}$$

For $1 \le i \le M-2$ and $i+2 \le j \le M$ we have

$$[S^{\pm}_{i,j}(z_1), F^{3,\pm}_{i,j}(z_2)] = \pm(q-q^{-1})\delta(q^{k-M+N+2j-1\pm1}z_2/z_1) :S^{\pm}_{i,j}(z_1)F^{3,\pm}_{i,j}(z_2):. \tag{5.40}$$

The remaining commutators vanish. Hence we have

$$\begin{aligned}
&[S_i(z_1), X^{-,i}(z_2)] \times (q-q^{-1})z_1z_2 \\
&= \sum_{j=i+2}^{M-1} \delta(q^{k-M+N+2j}z_2/z_1)\Big(e_{i,j}d^3_{i,j} :S^+_{i,j}(z_1)F^{3,+}_{i,j}(z_2): \\
&\quad -e_{i,j+1}d^3_{i,j+1} :S^-_{i,j+1}(z_1)F^{3,-}_{i,j+1}(z_2):\Big) + \delta(q^{k-M+N+2i+2}z_2/z_1) \\
&\quad \Big(e_{i,i+1}d^2_{i,i} :S^+_{i,i+1}(z_1)F^{2,+}_{i,i}(z_2): -e_{i,i+2}d^3_{i,i+2} :S^-_{i,i+2}(z_1)F^{3,-}_{i,i+2}(z_2):\Big) \\
&\quad +\delta(q^{k+M+N}z_2/z_1) \times \Big(e_{i,M}d^3_{i,M} :S^+_{i,M}(z_1)F^{3,+}_{i,M}(z_2): \\
&\quad -q^{-k-M+N-1}e_{i,M+1}d^3_{i,M+1} :S_{i,M+1}(z_1)F^3_{i,M+1}(z_2):\Big) \\
&\quad + \sum_{j=M+1}^{M+N-1} \delta(q^{k+3M+N-2j}z_2/z_1)q^{-k-M+N-1} \\
&\quad \times \Big(e_{i,j+1}d^3_{i,j+1} :S_{i,j+1}(z_1)F^3_{i,j+1}(z_2): -e_{i,j}d^3_{i,j} :S_{i,j}(z_1)F^3_{i,j}(z_2):\Big) \\
&\quad +\delta(q^{k+g}z_2/z_1)e_{i,M+N}d^3_{i,M+N}q^{-k-M+N-1} :S_{i,M+N}(z_1)F^3_{i,M+N}(z): \\
&\quad -\delta(q^{-k-g}z_2/z_1)e_{i,i+1}d^2_{i,i} :S_{i,i+1}(z_1)F^{2,-}_{i,i}(z_2):.
\end{aligned} \tag{5.41}$$

Moreover we have

$$:S^-_{i,i+1}(q^{-k-M+N}z)F^{2,-}_{i,i}(z): =: S_{i,M+N}(q^{k+M-N}z)F^3_{i,M+N}(z): =: e^{-(\frac{1}{k+g}a^i)(z|-\frac{k+g}{2})}: \quad (1 \le i \le M-1), \tag{5.42}$$

$$:S^+_{i,j}(q^{k-M+N+2j}z)F^{3,+}_{i,j}(z): =: S^-_{i,j+1}(q^{k-M+N+2j}z)F^{3,+}_{i,j+1}(z): \quad (1 \le i \le M-2, i+2 \le j \le M-1), \tag{5.43}$$

$$:S^+_{i,i+1}(q^{k-M+N+2i+2}z)F^{2,+}_{i,i}(z): =: S^-_{i,i+2}(q^{k-M+N+2i+2}z)F^{3,-}_{i,i+2}(z): \quad (1 \le i \le M-2), \tag{5.44}$$

$$:S^+_{i,M}(q^{k+M+N}z)F^{3,+}_{i,M}(z): =: S_{i,M+1}(q^{k+M+N}z)F^3_{i,M+1}(z): \quad (1 \le i \le M-2), \tag{5.45}$$

$$:S_{i,j+1}(q^{k+3M+N-2j}z)F^3_{i,j+1}(z): =: S_{i,j}(q^{k+3M+N-2j}z)F^3_{i,j}(z): \quad (1 \le i \le M-1, M+1 \le j \le M+N-1), \tag{5.46}$$

and

$$e_{i,i+1}d^2_{i,i}=e_{i,i+2}d^3_{i,i+2}\ \ (1\leq i\leq M-1), \tag{5.47}$$

$$e_{i,j}d^3_{i,j}=e_{i,j+1}d^3_{i,j+1}\ \ \left(\begin{array}{c}1\leq i\leq M-2, i+2\leq j\leq M+N-1\\ \text{or}\ \ 1\leq i\leq M-1, M+1\leq j\leq M+N-1\end{array}\right), \tag{5.48}$$

$$e_{i,M}d^3_{i,M}=e_{i,M+1}d^3_{i,M+1}q^{-k-M+N-1},\quad (1\leq i\leq M-1), \tag{5.49}$$

$$e_{i,M+N}d^3_{i,M+N}q^{-k-M+N-1}=e_{i,i+1}d^2_{i,i}=1\quad (1\leq i\leq M-1). \tag{5.50}$$

Hence we have (5.36) for $1\leq i\leq M-1$.

• For $i=M$ and $M+2\leq j\leq M+N$ we have

$$[S_{M,M+1}(z_1), F^{2,\pm}_{M,M}(z_2)]=\frac{q^{N-1}}{z_1}\delta(q^{k+M+N-1\mp 1}z_2/z_1): S_{M,M+1}(z_1)F^{2,\pm}_{M,M}(z_2):, \tag{5.51}$$

$$[S_{M,j}(z_1), F^{3,\pm}_{M,j}(z_2)]=\frac{q^{M+N-j}}{z_1}\delta(q^{k+3M+N-2j+1\mp 1}z_2/z_1): S_{M,j}(z_1)F^{3,\pm}_{M,j}(z_2):. \tag{5.52}$$

The remaining anti-commutators vanish. Hence we have

$$\begin{aligned}&[S_M(z_1), X^{-,M}(z_2)]\times(q-q^{-1})z_1z_2\\ &=\delta(q^{k+M+N-2}z_2/z_1)\\ &\quad\times\Big(q^{N-2}e_{M,M+2}d^3_{M,M+2}: S_{M,M+1}(z_1)F^{2,+}_{M,M}(z_2):\\ &\quad -q^{N-1}e_{M,M+1}d^2_{M,M}: S_{M,M+1}(z_1)F^{2,+}_{M,M}(z_2):\Big)\\ &\quad +\sum_{j=M+2}^{M+N-1}\delta(q^{k+3M+N-2j}z_2/z_1)\times\Big(q^{M+N-j-1}e_{M,j+1}d^3_{M,j+1}:\\ &\quad S_{M,j+1}(z_1)F^{3,-}_{M,j+1}(z_2): -q^{M+N-j}e_{M,j}d^3_{M,j}: S_{M,j}(z_1)F^{3,+}_{M,j}(z_2):\Big)\\ &\quad -\delta(q^{k+g}z_2/z_1)e_{M,M+N}d^3_{M,M+N}: S_{M,M+N}(z_1)F^{3,+}_{M,M+N}(z_2):\\ &\quad +e_{M,M+1}d^2_{M,M}\delta(q^{-k-g}z_2/z_1): S_{M,M+1}(z_1)F^2_{M,M}(z_2):.\end{aligned} \tag{5.53}$$

For $M+2\leq i\leq M+N-1$ we have

$$\begin{aligned}: S_{M,M+1}(q^{-k-M+N}z)F^{2,-}_{M,M}(z) :=: S_{M,M+N}(q^{k+M-N}z)F^{3,+}_{M,M+N}(z):\\ =: e^{-(\frac{1}{k+g}a^M)(z|-\frac{k+g}{2})}:,\end{aligned} \tag{5.54}$$

$$: S_{M,j}(q^{k+3M+N-2j}z)F^{3,+}_{M,j}(z) :=: S_{M,j+1}(q^{k+3M+N-2j}z)F^{3,-}_{M,j+1}(z):, \tag{5.55}$$

$$: S_{M,M+1}(q^{k+M+N-2}z)F^{2,+}_{M,M}(z) :=: S_{M,M+2}(q^{k+M-N-2}z)F^{3,-}_{M,M+2}(z):, \tag{5.56}$$

and

$$e_{M,M+1}d^2_{M,M}q^{N-1}=-1,\ \ e_{M,M+N}d^3_{M,M+N}=-1, \tag{5.57}$$

$$qe_{M,M+1}d^2_{M,M}=e_{M,M+2}d^3_{M,M+2}, \tag{5.58}$$

$$qe_{M,j}d^3_{M,j}=e_{M,j+1}d^3_{M,j+1}. \tag{5.59}$$

Hence we have (5.36) for $i = M$.

- For $M+1 \le i \le M+N-1$ and $i+2 \le j \le M+N$ we have

$$[S^+_{i,i+1}(z_1), F^{2,+}_{i,i}(z_2)] = -(q-q^{-1})\delta(q^{k+3M+N-2i-2}z_2/z_1) : S^+_{i,i+1}(z_1)F^{2,+}_{i,i}(z_2) :, \tag{5.60}$$

$$[S^-_{i,i+1}(z_1), F^{2,-}_{i,i}(z_2)] = (q-q^{-1})\delta(q^{-k-M+N}z_2/z_1) : S^-_{i,i+1}(z_1)F^{2,-}_{i,i}(z_2) :, \tag{5.61}$$

$$[S^\pm_{i,j}(z_1), F^{3,\pm}_{i,j}(z_2)] = \mp(q-q^{-1})\delta(q^{k+3M+N-2j+1\mp 1}z_2/z_1) : S^\pm_{i,j}(z_1)F^{3,\pm}_{i,j}(z_2) : . \tag{5.62}$$

The remaining commutators vanish. Hence we have

$$\begin{aligned}
&[S_i(z_1), X^{-,i}(z_2)] \times (q-q^{-1})z_1z_2 \\
&= \delta(q^{k+3M+N-2i-2}z_2/z_1)\Big(e_{i,i+2}d^3_{i,i+2} : S^-_{i,i+2}(z_1)F^{3,-}_{i,i+2}(z_2) : \\
&\quad -e_{i,i+1}d^2_{i,i} : S^+_{i,i+1}(z_1)F^{2,+}_{i,i}(z_2) :\Big) \\
&\quad + \sum_{j=i+2}^{M+N-1} \delta(q^{k+3M+N-2j}z_2/z_1)\Big(e_{i,j+1}d^3_{i,j+1} : S^-_{i,j+1}(z_1)F^{3,-}_{i,j+1}(z_2) : \\
&\quad -e_{i,j}d^3_{i,j} : S^+_{i,j}(z_1)F^{3,+}_{i,j}(z_2) :\Big) \\
&\quad -\delta(q^{k+g}z_2/z_1)e_{i,M+N}d^3_{i,M+N} : S^+_{i,M+N}(z_1)F^{3,+}_{i,M+N}(z_2) : \\
&\quad +\delta(q^{-k-g}z_2/z_1)e_{i,i+1}d^2_{i,i} : S^-_{i,i+1}(z_1)F^{2,-}_{i,i}(z_2) : .
\end{aligned} \tag{5.63}$$

Moreover we have

$$\begin{aligned}
&: S^-_{i,i+1}(q^{-k-M+N}z)F^{2,-}_{i,i}(z) \\
&\quad :=: S^+_{i,M+N}(q^{k+M-N}z)F^{3,+}_{i,M+N}(z) \\
&\quad :=: e^{-(\frac{1}{k+g}a^i)(z|-\frac{k+g}{2})} : \quad (M+1 \le i \le M+N-1),
\end{aligned} \tag{5.64}$$

$$\begin{aligned}
&: S^+_{i,j}(q^{k+3M+N-2j}z)F^{3,+}_{i,j}(z) \\
&\quad :=: S^-_{i,j+1}(q^{k+3M+N-2j}z)F^{3,-}_{i,j+1}(z) : \\
&\quad (M+1 \le i \le M+N-1, i+2 \le j \le M+N-1),
\end{aligned} \tag{5.65}$$

$$\begin{aligned}
&: S^-_{i,i+2}(q^{k+3M+N-2i-2}z)F^{3,-}_{i,i+2}(z) \\
&\quad :=: S^+_{i,i+1}(q^{k+3M+N-2i-2}z)F^{2,+}_{i,i}(z) : \quad (M+1 \le i \le M+N-2).
\end{aligned} \tag{5.66}$$

For $M+1 \le i \le M+N-1$ we have

$$e_{i,i+1}d^2_{i,i} = -1, \quad e_{i,M+N}d^3_{i,M+N} = -1, \tag{5.67}$$

$$e_{i,i+1}d^2_{i,i} = e_{i,i+2}d^3_{i.i+2}, \quad e_{i,j}d^3_{i,j} = e_{i,j+1}d^3_{i,j+1} \;\; (i+3 \le j \le M+N-1). \tag{5.68}$$

Hence we have (5.36) for $M+1 \le i \le M+N-1$. Now we have shown (5.36) for all $1 \le i \le M+N-1$.

Next, we will show

$$[S_i(z_1), X^{+,i}(z_2)] = 0 \quad (1 \le i \le M+N-1). \tag{5.69}$$

• For $1 \le i \le M-1$ we have

$$[S^{\pm}_{i,i+1}(z_1), E^{\pm}_{i,i}(z_2)] = \mp(q-q^{-1})\delta(q^{2i-M+N}z_2/z_1) : S^{\pm}_{i,i+1}(z_1)E^{\pm}_{i,i}(z_2) : . \tag{5.70}$$

Hence we have

$$[S_i(z_1), X^{+,i}(z_2)] = \delta(q^{2i-M+N}z_2/z_1)e_{i,i+1}c_{i,i} \left(: S^{+}_{i,i+1}(z_1)E^{+}_{i,i}(z_2) : - : S^{-}_{i,i+1}(z_1)E^{-}_{i,i}(z_2) : \right). \tag{5.71}$$

Moreover we have

$$: S^{+}_{i,i+1}(q^{2i-M+N}z)E^{+}_{i,i}(z) :=: S^{-}_{i,i+1}(q^{2i-M+N}z)E^{-}_{i,i}(z) : . \tag{5.72}$$

Hence we have (5.69) for $1 \le i \le M-1$.

• For $i = M$ all anti-commutators vanish. Hence we have (5.69) for $i = M$.

• For $M+1 \le i \le M+N-1$ we have

$$[S^{\pm}_{i,i+1}(z_1), E^{\pm}_{i,i}(z_2)] = \pm(q-q^{-1})\delta(q^{3M+N-2i}z_2/z_1) : S^{\pm}_{i,i+1}(z_1)E^{\pm}_{i,i}(z_2) : . \tag{5.73}$$

Hence we have

$$[S_i(z_1), X^{+,i}(z_2)] = \delta(q^{3M+N-2i}z_2/z_1)e_{i,i+1}c_{i,i} \left(: S^{+}_{i,i+1}(z_1)E^{+}_{i,i}(z_2) : - : S^{-}_{i,i+1}(z_1)E^{-}_{i,i}(z_2) : \right). \tag{5.74}$$

Moreover we have

$$: S^{+}_{i,i+1}(q^{3M+N-2i}z)E^{+}_{i,i}(z) :=: S^{-}_{i,i+1}(q^{3M+N-2i}z)E^{-}_{i,i}(z) : . \tag{5.75}$$

Hence we have (5.69) for $M+1 \le i \le M+N-1$. Now we have shown (5.69) for all $1 \le i \le M+N-1$.

Other commutation relations of the screening currents are shown in the same way. We summarize useful formulae for proof of theorem in "Appendix A".

6. Concluding Remarks

In this article we found a bosonization of $U_q(\widehat{sl}(M|N))$ for an arbitrary level $k \in \mathbf{C}$. Our bosonization is obtained from a q-difference realization of $U_q(sl(M|N))$ (see "Appendix B") by the replacement

$$q^{\vartheta_{i,j}} \to e^{\pm b^{i,j}_{\mp}(z)}, \tag{6.1}$$

$$x_{i,j} \to \begin{cases} : e^{(b+c)^{i,j}(z)} : & (\nu_i\nu_j = +), \\ : e^{-b^{i,j}(z)} : \text{ or } : e^{-b^{i,j}_{\pm}(z) - b^{i,j}(z)} : & (\nu_i\nu_j = -), \end{cases} \tag{6.2}$$

$$q^{\lambda_i} \to e^{\pm a^{i}_{\pm}(z)}, \tag{6.3}$$

$$[\vartheta_{i,j}]_q \to \begin{cases} \frac{e^{\pm b^{i,j}_{+}(z)} - e^{\pm b^{i,j}_{-}(z)}}{(q-q^{-1})z} & (\nu_i\nu_j = +), \\ 1 & (\nu_i\nu_j = -). \end{cases} \tag{6.4}$$

For instance, we have

$$\begin{aligned}
f^1_{M,j} &\to \frac{1}{(q-q^{-1})z} :\left(e^{-b_-^{j,M}(z)-(b+c)^{j,M}(z)} - e^{-b_-^{j,M}(z)-(b+c)^{j,M}(z)}\right) \\
&\quad \times e^{a_-^M(z)-b_-^{j,M+1}(z)-b^{j,M+1}(z)-\sum_{l=j+1}^{M-1}(\Delta_R^0 b_-^{l,M})(z)+\sum_{l=M+2}^{M+N}(\Delta_R^0 b_-^{M,l})(z)} : \\
&\to \frac{1}{(q-q^{-1})z} :\left(e^{-b_+^{j,M}(q^{-k-j}z)-(b+c)^{j,M}(q^{-k-j-1}z)}\right. \\
&\quad \left. -e^{-b_-^{j,M}(q^{-k-j}z)-(b+c)^{j,M}(q^{-k-j+1}z)}\right) \\
&\quad \times e^{a_-^M(q^{-\frac{k+g}{2}}z)-b_-^{j,M+1}(q^{-k-j}z)-b^{j,M+1}(q^{-k-j+1}z)} \\
&\quad \times e^{-\sum_{l=j+1}^{M-1}(\Delta_R^0 b_-^{l,M})(q^{-k-l}z)+\sum_{l=M+2}^{M+N}(\Delta_R^0 b_-^{M,l})(q^{-k-2M-1+l}z)} : \\
&= \frac{1}{(q-q^{-1})z}(F^{1,-}_{M,j}(z) - F^{1,+}_{M,j}(z)) \quad (1 \le j \le M-1).
\end{aligned} \tag{6.5}$$

Taking the limit $q \to 1$ we obtain a bosonization of the affine superalgebra $\widehat{sl}(M|N)$ for an arbitrary level $k \in \mathbf{C}$. Bosonizations of the affine superalgebra $\widehat{sl}(M|N)$ for level k have been studied in [14,32–39]. We compare our bosonization with those of [39]. In the limit $q \to 1$ we introduce operators $\alpha_i(z)$ $(1 \le i \le M+N-1)$, $\beta_{i,j}(z)$, $\widehat{\beta}_{i,j}(z)$, $\gamma_{i,j}(z)$ $(1 \le i < j \le M+N,\ \nu_i\nu_j = +)$, and $\psi_{i,j}(z)$, $\psi^\dagger_{i,j}(z)$ $(1 \le i < j \le M+N,\ \nu_i\nu_j = -)$ as follows.

$$\alpha_i(z) = \partial_z\left(a^i(z)\right), \tag{6.6}$$

$$\beta_{i,j}(z) =: \partial_z\left(e^{-c^{i,j}(z)}\right)e^{-b^{i,j}(z)} :, \quad \widehat{\beta}_{i,j}(z) =: \partial_z\left(e^{-b^{i,j}(z)}\right)e^{-c^{i,j}(z)} :, \tag{6.7}$$

$$\gamma_{i,j}(z) =: e^{(b+c)^{i,j}(z)} :, \tag{6.8}$$

$$\psi_{i,j}(z) =: e^{b^{i,j}(z)} :, \quad \psi^\dagger_{i,j}(z) =: e^{-b^{i,j}(z)} : . \tag{6.9}$$

They satisfy the following relations.

$$\alpha_i(z)\alpha_j(w) = \frac{(k+g)A_{i,j}}{(z-w)^2} + \cdots, \tag{6.10}$$

$$\beta_{i,j}(z)\gamma_{i',j'}(w) = \frac{\delta_{i,i'}\delta_{j,j'}}{z-w} + \cdots, \quad \gamma_{i,j}(z)\beta_{i',j'}(w) = -\frac{\delta_{i,i'}\delta_{j,j'}}{z-w} + \cdots, \tag{6.11}$$

$$\widehat{\beta}_{i,j}(z)\gamma_{i',j'}(w) = -\frac{\delta_{i,i'}\delta_{j,j'}}{z-w} + \cdots, \quad \gamma_{i,j}(z)\widehat{\beta}_{i',j'}(w) = \frac{\delta_{i,i'}\delta_{j,j'}}{z-w} + \cdots, \tag{6.12}$$

$$\psi_{i,j}(z)\psi^\dagger_{i',j'}(w) = \frac{\delta_{i,i'}\delta_{j,j'}}{z-w} + \cdots, \quad \psi^\dagger_{i,j}(z)\psi_{i',j'}(w) = \frac{\delta_{i,i'}\delta_{j,j'}}{z-w} + \cdots. \tag{6.13}$$

In the limit $q \to 1$ our bosonization becomes simpler because the operators $a^i_\pm(z)$, $b^{i,j}_\pm(z)$, $(\Delta^\epsilon_L b^{i,j}_\pm)(z)$ and $(\Delta^\epsilon_R b^{i,j}_\pm)(z)$ disappear. In order to resolve a singularity in denominator $(q-q^{-1})$, we take the limit of $\frac{1}{(q-q^{-1})z}(E^+_{i,j}(z) - E^-_{i,j}(z))$ instead of $\frac{1}{(q-q^{-1})z}E^\pm_{i,j}(z)$. In "Appendix C" we summarize useful formulae to take the limit $q \to 1$

upon the condition $k \neq -g$. Taking the limit $q \to 1$, we obtain the following bosonization of the affine superalgebra $\widehat{sl}(M|N)$. In what follows we set the coefficients $c_{i,j} = 1$ for simplicity.

$$\begin{aligned} H^i(z) = \alpha_i(z) &+ \sum_{j=1}^{i} :(\widehat{\beta}_{j,i}(z)\gamma_{j,i}(z) - \widehat{\beta}_{j,i+1}(z)\gamma_{j,i+1}(z)): \\ &+ \sum_{j=i+1}^{M} :(\widehat{\beta}_{i+1,j}(z)\gamma_{i+1,j}(z) - \widehat{\beta}_{i,j}(z)\gamma_{i,j}(z)): \\ &+ \sum_{j=M+1}^{M+N} :((\partial_z\psi_{i+1,j})(z)\psi^\dagger_{i+1,j}(z) - (\partial_z\psi_{i,j})(z)\psi^\dagger_{i,j}(z)): \\ &(1 \le i \le M-1), \end{aligned} \tag{6.14}$$

$$\begin{aligned} H^M(z) = \alpha_M(z) &+ \sum_{j=1}^{M-1} :((\partial_z\psi_{j,M+1})(z)\psi^\dagger_{j,M+1}(z) + \widehat{\beta}_{j,M}(z)\gamma_{j,M}(z)): \\ &- \sum_{j=M+2}^{M+N} :(\widehat{\beta}_{M+1,j}(z)\gamma_{M+1,j}(z) + (\partial_z\psi_{M,j})(z)\psi^\dagger_{M,j}(z)):, \end{aligned} \tag{6.15}$$

$$\begin{aligned} H^i(z) = \alpha_i(z) &+ \sum_{j=1}^{M} :((\partial_z\psi_{j,i+1})(z)\psi^\dagger_{j,i+1}(z) - (\partial_z\psi_{j,i})(z)\psi^\dagger_{j,i}(z)): \\ &+ \sum_{j=M+1}^{i} :(\widehat{\beta}_{j,i+1}(z)\gamma_{j,i+1}(z) - \widehat{\beta}_{j,i}(z)\gamma_{j,i}(z)): \\ &+ \sum_{j=i+1}^{M+N} :(\widehat{\beta}_{i,j}(z)\gamma_{i,j}(z) - \widehat{\beta}_{i+1,j}(z)\gamma_{i+1,j}(z)): \\ &(M+1 \le i \le M+N-1). \end{aligned} \tag{6.16}$$

$$X^{+,i}(z) = \sum_{j=1}^{i} :\beta_{j,i+1}(z)\gamma_{j,i}(z): \quad (1 \le i \le M-1), \tag{6.17}$$

$$X^{+,M}(z) = \sum_{j=1}^{M} :\gamma_{j,M}(z)\psi_{j,M+1}(z):, \tag{6.18}$$

$$\begin{aligned} X^{+,i}(z) = &\sum_{j=1}^{M} :\psi_{j,i+1}(z)\psi^\dagger_{j,i}(z): - \sum_{j=M+1}^{i} :\beta_{j,i+1}(z)\gamma_{j,i}(z): \\ &(M+1 \le i \le M+N-1). \end{aligned} \tag{6.19}$$

$$\begin{aligned} X^{-,i}(z) \;=\; &- :\alpha_i(z)\gamma_{i,i+1}(z): - \kappa_i :\partial_z\gamma_{i,i+1}(z): \\ &+ \sum_{j=1}^{i-1} :\beta_{j,i}(z)\gamma_{j,i+1}(z): - \sum_{j=i+2}^{M} :\beta_{i+1,j}(z)\gamma_{i,j}(z): \end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=M+1}^{M+N} :\psi_{i+1,j}(z)\psi_{i,j}^{\dagger}(z):\\
&+\sum_{j=i+1}^{M} :(\widehat{\beta}_{i,j}(z)\gamma_{i,j}(z)-\widehat{\beta}_{i+1,j}(z)\gamma_{i+1,j}(z))\gamma_{i,i+1}(z): \qquad (6.20)\\
&+\sum_{j=M+1}^{M+N} :((\partial_z\psi_{i,j})(z)\psi_{i,j}^{\dagger}(z)-(\partial_z\psi_{i+1,j})(z)\psi_{i+1,j}^{\dagger}(z))\gamma_{i,i+1}(z):\\
&(1\le i\le M-1),
\end{aligned}
$$

$$
\begin{aligned}
X^{-,M}(z) =:\ & \alpha_M(z)\psi_{M,M+1}^{\dagger}(z):+\kappa_M:\partial_z\psi_{M,M+1}^{\dagger}(z):\\
&-\sum_{j=1}^{M-1} :\beta_{j,M}(z)\psi_{j,M+1}^{\dagger}(z):-\sum_{j=M+2}^{M+N} :\beta_{M+1,j}(z)\psi_{M,j}^{\dagger}(z):\\
&-\sum_{j=M+2}^{M+N} :(\widehat{\beta}_{M+1,j}(z)\gamma_{M+1,j}^{\dagger}(z)+(\partial_z\psi_{M,j})(z)\psi_{M,j}^{\dagger}(z))\psi_{M,M+1}^{\dagger}(z):,
\end{aligned}
\qquad (6.21)
$$

$$
\begin{aligned}
X^{-,i}(z) =:\ & \alpha_i(z)\gamma_{i,i+1}(z):+\kappa_i:\partial_z\gamma_{i,i+1}(z):\\
&-\sum_{j=1}^{M} :\psi_{j,i}(z)\psi_{j,i+1}^{\dagger}(z):+\sum_{j=M+1}^{i-1} :\beta_{j,i}(z)\gamma_{j,i+1}(z):\\
&-\sum_{j=i+2}^{M+N} :\beta_{i+1,j}(z)\gamma_{i,j}(z):+\sum_{j=i+1}^{M+N} :(\widehat{\beta}_{i,j}(z)\gamma_{i,j}(z)\\
&-\widehat{\beta}_{i+1,j}(z)\gamma_{i+1,j}(z))\gamma_{i,j}(z): \quad (M+1\le i\le M+N-1).
\end{aligned}
\qquad (6.22)
$$

Here we have set the coefficients κ_i by

$$
\kappa_i=\begin{cases} k+i & (1\le i\le M-1)\\ k+M-1 & (i=M)\\ k+2M-i & (M+1\le i\le M+N-1)\end{cases}. \qquad (6.23)
$$

For $k\neq -g$ we have a bosonization of the screening current as follows.

$$
\begin{aligned}
S_i(z) &= \sum_{j=i+1}^{M} :\tilde{s}_i(z)\beta_{i,j}(z)\gamma_{i+1,j}(z):\\
&\quad+\sum_{j=M+1}^{M+N} :\tilde{s}_i(z)\psi_{i,j}(z)\psi_{i+1,j}^{\dagger}(z): \ (1\le i\le M-1),
\end{aligned}
\qquad (6.24)
$$

$$
S_M(z)=\sum_{j=M+1}^{M+N} :\tilde{s}_M(z)\gamma_{M+1,j}(z)\psi_{M,j}(z):, \qquad (6.25)
$$

$$
S_i(z)=\sum_{j=i+1}^{M+N} :\tilde{s}_i(z)\beta_{i,j}(z)\gamma_{i+1,j}(z): \quad (M+1\le i\le M+N-1). \qquad (6.26)
$$

Here we have set the operator $\tilde{s}_i(z) =: e^{-\left(\frac{1}{k+g}a^i\right)(z;0)}:$. Our bosonization is similar as those of [39], because both bosonizations are based on the same differential realization of $sl(M|N)$ (see "Appendix B"). The bosonization of [39] is reproduced from our bosonization by the following "formal replacement"

$$\widehat{\beta}_{i,j}(z) \to \beta_{i,j}(z), \quad \partial_z\psi_{i,j}(z) \to \psi_{i,j}(z),$$
$$\kappa_i \to \begin{cases} k+i-1 & (1 \le i \le M-1) \\ k+M-1 & (i=M) \\ k+M+1-i & (M+1 \le i \le M+N-1) \end{cases}, \tag{6.27}$$

with $\alpha_i(z), \beta_{i,j}(z), \gamma_{i,j}(z), \psi_{i,j}(z), \psi^\dagger_{i,j}(z)$ fixed. Of course the map satisfying both $\partial_z\psi_{i,j}(z) \to \psi_{i,j}(z)$ and $\psi_{i,j}(z) \to \psi_{i,j}(z)$ is impossible. This is the reason why we used the word "formal replacement".

In order to calculate correlation functions of exactly solvable models, we have to prepare the vertex operator $\Phi_{V(\mu)}^{V(\nu)V_\lambda}(z)$ that satisfies intertwining property [2]. We will propose a bosonization of the vertex operator. In what follows we assume $k+g \neq 0$ and $g \neq 0$. We define a bosonic operator $\phi^{\bar{\lambda}}(z)$ for the weight $\bar{\lambda} = \sum_{i=1}^{M+N-1} l_i \bar{\Lambda}_i$ as follows.

$$\phi^{\bar{\lambda}}(z) =: \exp\left(\sum_{i,j=1}^{M+N-1}\left(\frac{l_i}{k+g}\cdot\frac{\alpha_{i,j}}{g}\cdot\frac{\beta_{i,j}}{1}\,a^j\right)\left(z;-\frac{k+g}{2}\right)\right):, \tag{6.28}$$

where we have set

$$\alpha_{i,j} = \begin{cases} \min(i,j) & (\min(i,j) \le M+1), \\ 2(M+1)-\min(i,j) & (\min(i,j) \ge M+2), \end{cases} \tag{6.29}$$

$$\beta_{i,j} = \begin{cases} M-N-\max(i,j) & (\max(i,j) \le M+1), \\ -M-N-2+\max(i,j) & (\max(i,j) \ge M+2). \end{cases} \tag{6.30}$$

The bosonic operator $\phi^{\bar{\lambda}}(z)$ satisfies the following relations for $1 \le i \le M+N-1$.

$$[H_m^i, \phi^{\bar{\lambda}}(z)] = \frac{1}{m}[l_i m]_q q^{-\frac{k}{2}|m|} z^m \phi^{\bar{\lambda}}(z) \quad (m \in \mathbf{Z}_{\neq 0}), \tag{6.31}$$

$$[X^{+,i}(z), \phi^{\bar{\lambda}}(z)] = 0, \tag{6.32}$$

$$(z_1 - q^{l_i} z_2) X^{-,i}(z_1)\phi^{\bar{\lambda}}(z_2) = (q^{l_i} z_1 - z_2)\phi^{\bar{\lambda}}(z_2) X^{-,i}(z_1). \tag{6.33}$$

We set the bosonic operators $\phi^{\bar{\lambda}}_{i_1,i_2,\ldots,i_n}(z)$ $(1 \le i_1, i_2, \cdots, i_n \le M+N-1)$ as follows.

$$\phi^{\bar{\lambda}}_{i_1,i_2,\ldots,i_n}(z) = \left[\phi^{\bar{\lambda}}_{i_1,i_2,\ldots,i_{n-1}}, X_0^{-,i_n}\right]_{q^x}, \tag{6.34}$$

where $x = (\bar{\lambda} - \sum_{s=1}^{n-1}\bar{\alpha}_{i_s}|\bar{\alpha}_{i_n}))$. The $\mathbf{Z}_2$-grading is given by $p(\phi^{\bar{\lambda}}(z)) \equiv 0 \pmod 2$ and $p(\phi^{\bar{\lambda}}_{i_1,i_2,\ldots,i_n}(z)) \equiv \sum_{s=1}^n p(X_m^{-,i_s}) \pmod 2$. Let $\Phi_{V(\mu)}^{V(\nu)V_\lambda}(z)$ be the vertex operator satisfying the intertwining property : $\Phi_{V(\mu)}^{V(\nu)V_\lambda} : V(\mu) \to V(\nu) \otimes V_{\lambda_z}$. Here $V(\mu)$ and $V(\nu)$ are highest weight modules. V_λ and V_{λ_z} are a typical module and its evaluation module, respectively [29]. Let $\Phi_{V(\mu)\ i_1,i_2,\ldots,i_n}^{V(\nu)\ V_\lambda}(z)$ be $\Phi_{V(\mu)}^{V(\nu)V_\lambda}(z) = \sum_{i_1,i_2,\ldots,i_n} \Phi_{V(\mu)\ i_1,i_2,\ldots,i_n}^{V(\nu)\ V_\lambda}(z)$

$v_{i_1,i_2,\ldots,i_n}$ where $\{v_{i_1,i_2,\ldots,i_n}\}$ is a basis of V_λ. We propose a bosonization of the vertex operator as follows.

$$\Phi_{V(\mu)\ i_1,i_2,\ldots,i_n}^{V(\nu)\ V_\lambda}(z) = \eta_0\xi_0 \prod_{j=1}^{M+N-1} : Q_j^{n_j} \cdot \phi_{i_1,i_2,\ldots,i_n}^{\bar{\lambda}}(q^{k+g}z) : \eta_0\xi_0, \tag{6.35}$$

where $n_j \in \mathbf{N}$ $(1 \le j \le M+N-1)$. Here $\eta_0\xi_0$ is the projection operator on the Wakimoto module (see Sect. 3) and Q_j are the screening operators (see Sect. 4). In order to balance "background charge" of the Wakimoto module, we multiply the screening operators. Trace of the vertex operators vanishes if we do not multiply the screening operators. For small rank we have checked this conjecture by direct calculation. For $2 \le M \le 4$ and $N = 1$, the intertwining property of the vertex operator was checked [21,23]. We would like to report on this conjecture in future publication.

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Appendix A: Normal Ordering Rules

In this Appendix we summarize normal ordering rules that are useful for proof of main results.

A.1. $[X^{+,i}(z_1), X^{-,i+1}(z_2)]$ $(1 \le i \le M+N-2)$.

- For $1 \le i \le M-1$ we have

$$[E_{i,j}^{\pm}(z_1), F_{i+1,j}^{1,\pm}(z_2)] = \pm(q-q^{-1})\delta(q^{-k+1-2j}z_2/z_1)$$
$$: E_{i,j}^{\pm}(z_1)F_{i+1,j}^{1,\pm}(z_2) : \quad (1 \le j \le i). \tag{A.1}$$
$$: E_{i,j}^{+}(q^{-k+1-2j}z)F_{i+1,j}^{1,+}(z) :=: E_{i,j}^{-}(q^{-k+1-2j}z)F_{i+1,j}^{1,-}(z) : \quad (1 \le j \le i). \tag{A.2}$$

- For $i = M$ we have

$$[E_{M,j}(z_1), F_{M+1,j}^{1}(z_2)] = 0 \quad (1 \le j \le M). \tag{A.3}$$

- For $M+1 \le i \le M+N-2$ we have

$$[E_{i,j}(z_1), F_{i+1,j}^{1}(z_2)] = 0 \quad (1 \le j \le M), \tag{A.4}$$
$$[E_{i,j}^{\pm}(z_1), F_{i+1,j}^{1,\pm}(z_2)] = \mp(q-q^{-1})\delta(q^{-k-4M-1+2j}z_2/z_1)$$
$$: E_{i,j}^{\pm}(z_1)F_{i+1,j}^{1,\pm}(z_2) : \quad (M+1 \le j \le i). \tag{A.5}$$
$$: E_{i,j}^{+}(q^{-k-4M-1+2j}z)F_{i+1,j}^{1,+}(z) :=: E_{i,j}^{-}(q^{-k-4M-1+2j}z)F_{i+1,j}^{1,-}(z) :$$
$$(M+1 \le j \le i). \tag{A.6}$$

A.2. $[X^{+,i}(z_1), X^{-,i-1}(z_2)]$ $(2 \le i \le M+N-1)$.

- For $2 \le i \le M-1$ we have

$$[E^{+}_{i,i-1}(z_1), F^{3,\pm}_{i-1,i+1}(z_2)] = -(q-q^{-1})\delta(q^{k+1}z_2/z_1) : E^{+}_{i,i-1}(z_1)F^{3,\pm}_{i-1,i+1}(z_2) :, \tag{A.7}$$

$$[E^{\pm}_{i,i}(z_1), F^{2,+}_{i-1,i-1}(z_2)] = (q-q^{-1})\delta(q^{k+1}z_2/z_1) : E^{\pm}_{i,i}(z_1)F^{2,+}_{i-1,i-1}(z_2) :, \tag{A.8}$$

$$[E^{\pm}_{i,i}(z_1), F^{3,\mp}_{i-1,i+1}(z_2)] = \pm(q-q^{-1})\delta(q^{k+1}z_2/z_1) : E^{\pm}_{i,i}(z_1)F^{3,\mp}_{i-1,i+1}(z_2) : . \tag{A.9}$$

$$: E^{+}_{i,i-1}(q^{k+1}z)F^{3,\pm}_{i-1,i+1}(z) :=: E^{\pm}_{i,i}(q^{k+1}z)F^{2,+}_{i-1,i-1}(z) :, \tag{A.10}$$

$$: E^{+}_{i,i}(q^{k+1}z)F^{3,-}_{i-1,i+1}(z) :=: E^{-}_{i,i}(q^{k+1}z)F^{3,+}_{i-1,i+1}(z) : . \tag{A.11}$$

- For $i = M$ we have

$$[E_{M,M-1}(z_1), F^{3}_{M-1,M+1}(z_2)] = \frac{1}{q^{M-2}z_1}\delta(q^{k+1}z_2/z_1) : E_{M,M-1}(z_1)F^{3}_{M-1,M+1}(z_2) :, \tag{A.12}$$

$$[E_{M,M}(z_1), F^{2,+}_{M-1,M-1}(z_2)] = (q-q^{-1})\delta(q^{k+1}z_2/z_1) : E_{M,M}(z_1)F^{2,+}_{M-1,M-1}(z_2) :, \tag{A.13}$$

$$[E_{M,M}(z_1), F^{3}_{M-1,M+1}(z_2)] = 0. \tag{A.14}$$

$$: E_{M,M-1}(q^{k+1}z)F^{3}_{M-1,M+1}(z) :=: E_{M,M}(q^{k+1}z)F^{2,+}_{M-1,M-1}(z) : . \tag{A.15}$$

- For $i = M+1$ we have

$$[E_{M+1,M}(z_1), F^{3,\pm}_{M,M+2}(z_2)] = \frac{1}{q^{M-1}z_1}\delta(q^{k-1}z_2/z_1) : E_{M+1,M}(z_1)F^{3,\pm}_{M,M+2}(z_2) :, \tag{A.16}$$

$$[E^{\pm}_{M+1,M+1}(z_1), F^{2,+}_{M,M}(z_2)] = -(q-q^{-1})\delta(q^{k-1}z_2/z_1) : E^{\pm}_{M+1,M+1}(z_1)F^{2,+}_{M,M}(z_2) :, \tag{A.17}$$

$$[E^{\pm}_{M+1,M+1}(z_1), F^{3,\mp}_{M,M+2}(z_2)] = \mp q(q-q^{-1})\delta(q^{k-1}z_2/z_1) : E^{\pm}_{M+1,M+1}(z_1)F^{3,\mp}_{M,M+2}(z_2) : . \tag{A.18}$$

$$: E_{M+1,M}(q^{k-1}z)F^{3,\pm}_{M,M+2}(z) :=: E^{\pm}_{M+1,M+1}(q^{k-1}z)F^{2,+}_{M,M}(z) :, \tag{A.19}$$

$$: E^{+}_{M+1,M+1}(q^{k-1}z)F^{3,-}_{M,M+2}(z) :=: E^{-}_{M+1,M+1}(q^{k-1}z)F^{3,+}_{M,M+2}(z) : . \tag{A.20}$$

- For $M+2 \le i \le M+N-1$ we have

$$[E^{+}_{i,i-1}(z_1), F^{3,\pm}_{i-1,i+1}(z_2)] = (q-q^{-1})\delta(q^{k-1}z_2/z_1) : E^{+}_{i,i-1}(z_1)F^{3,\pm}_{i-1,i+1}(z_2) :, \tag{A.21}$$

$$[E^{\pm}_{i,i}(z_1), F^{2,+}_{i-1,i-1}(z_2)] = -(q-q^{-1})\delta(q^{k-1}z_2/z_1) : E^{\pm}_{i,i}(z_1)F^{2,+}_{i-1,i-1}(z_2) :, \tag{A.22}$$

$$[E^{\pm}_{i,i}(z_1), F^{3,\mp}_{i-1,i+1}(z_2)] = \mp q(q-q^{-1})\delta(q^{k-1}z_2/z_1) : E^{\pm}_{i,i}(z_1)F^{3,\mp}_{i-1,i+1}(z_2) : . \tag{A.23}$$

$$: E^{+}_{i,i-1}(q^{k-1}z)F^{3,\pm}_{i-1,i+1}(z) :=: E^{\pm}_{i,i}(q^{k-1}z)F^{2,+}_{i-1,i-1}(z) :, \tag{A.24}$$

$$: E^{+}_{i,i}(q^{k-1}z)F^{3,-}_{i-1,i+1}(z) :=: E^{-}_{i,i}(q^{k-1}z)F^{3,+}_{i-1,i+1}(z) : . \tag{A.25}$$

A.3. $[X^{+,i}(z_1), X^{-,j}(z_2)]$ $(3 \le i \le M+N-1,\ 1 \le j \le i-2)$.

- For $3 \le i \le M-1$ and $1 \le j \le i-2$ we have

$$[E^{+}_{i,j}(z_1), F^{3,\pm}_{j,i+1}(z_2)] = -(q-q^{-1})\delta(q^{k+i-j}z_2/z_1) : E^{+}_{i,j}(z_1)F^{3,\pm}_{j,i+1}(z_2) :, \tag{A.26}$$

$$[E^{\pm}_{i,j+1}(z_1), F^{3,+}_{j,i}(z_2)] = (q-q^{-1})\delta(q^{k+i-j}z_2/z_1) : E^{\pm}_{i,j+1}(z_1)F^{3,+}_{j,i}(z_2) :, \tag{A.27}$$

$$[E^{\pm}_{i,j+1}(z_1), F^{3,\mp}_{j,i+1}(z_2)] = \pm(q-q^{-1})\delta(q^{k+i-j}z_2/z_1) : E^{\pm}_{i,j+1}(z_1)F^{3,\mp}_{j,i+1}(z_2) : . \tag{A.28}$$

$$: E^{+}_{i,j}(q^{k+i-j}z)F^{3,\pm}_{j,i+1}(z) :=: E^{\pm}_{i,j+1}(q^{k+i-j}z)F^{3,+}_{j,i}(z) :, \tag{A.29}$$

$$: E^{+}_{i,j+1}(q^{k+i-j}z)F^{3,-}_{j,i+1}(z) :=: E^{-}_{i,j-1}(q^{k+i-j}z)F^{3,+}_{j,i+1}(z) : . \tag{A.30}$$

- For $i = M$ and $1 \le j \le M-2$ we have

$$[E_{M,j+1}(z_1), F^{3,+}_{j,M}(z_2)] = (q-q^{-1})\delta(q^{k+M-j}z_2/z_1) : E_{M,j+1}(z_1)F^{3,+}_{j,M}(z_2) :, \tag{A.31}$$

$$[E_{M,j}(z_1), F^{3}_{j,M+1}(z_2)] = \frac{1}{q^{j-1}z}\delta(q^{k+M-j}z_2/z_1) : E_{M,j}(z_1)F^{3}_{j,M+1}(z_2) :, \tag{A.32}$$

$$[E_{M,j+1}(z_1), F^{3}_{j,M+1}(z_2)] = 0. \tag{A.33}$$

$$: E_{M,j+1}(q^{k+M-j}z)F^{3,+}_{j,M}(z) :=: E_{M,j}(q^{k+M-j}z)F^{3}_{j,M+1}(z) : . \tag{A.34}$$

- For $M+1 \le i \le M+N-1$ and $1 \le j \le M-1$ we have

$$[E_{i,j+1}(z_1), F^{3}_{j,i}(z_2)] = -\frac{1}{q^{j+1}z_1}\delta(q^{k+2M-i-j}z_2/z_1) : E_{i,j+1}(z_1)F^{3}_{j,i}(z_2) :, \tag{A.35}$$

$$[E_{i,j}(z_1), F^{3}_{j,i+1}(z_2)] = \frac{1}{q^{j-1}z_1}\delta(q^{k+2M-i-j}z_2/z_1) : E_{i,j}(z_1)F^{3}_{j,i+1}(z_2) :, \tag{A.36}$$

$$[E_{i,j+1}(z_1), F^{3}_{j,i+1}(z_2)] = 0. \tag{A.37}$$

$$: E_{i,j+1}(q^{k+2M-i-j}z)F^{3}_{j,i}(z) :=: E_{i,j}(q^{k+2M-i-j}z)F^{3}_{j,i+1}(z) : . \tag{A.38}$$

- For $M+2 \le i \le M+N-1$ and $j = M$ we have

$$[E^{\pm}_{i,M+1}(z_1), F^{3,+}_{M,i}(z_2)]$$

$$= -(q-q^{-1})\delta(q^{k+M-i}z_2/z_1) : E^{\pm}_{i,M+1}(z_1)F^{3,+}_{M,i}(z_2) :, \tag{A.39}$$

$$[E_{i,M}(z_1), F^{3,\pm}_{M,i+1}(z_2)] = \frac{1}{q^{M-1}z_1}\delta(q^{k+M-i}z_2/z_1) : E_{i,M}(z_1)F^{3,\pm}_{M,i+1}(z_2) :, \tag{A.40}$$

$$[E^{\pm}_{i,M+1}(z_1), F^{3,\mp}_{M,i+1}(z_2)]$$
$$= \mp q(q-q^{-1})\delta(q^{k+M-i}z_2/z_1) : E^{\pm}_{i,M+1}(z_1)F^{3,\mp}_{M,i+1}(z_2) : . \tag{A.41}$$

$$: E^{\pm}_{i,M+1}(q^{k+M-i}z)F^{3,+}_{M,i}(z) :=: E_{i,M}(q^{k+M-i}z)F^{3,\pm}_{M,i+1}(z) :, \tag{A.42}$$

$$: E^{+}_{i,M+1}(q^{k+M-i}z)F^{3,-}_{M,i+1}(z) :=: E^{-}_{i,M-1}(q^{k+M-i}z)F^{3,-}_{M,i+1}(z) : . \tag{A.43}$$

- For $M+2 \le i \le M+N-1$ and $M+1 \le j \le i-2$ we have

$$[E^{\pm}_{i,j+1}(z_1), F^{3,+}_{j,i}(z_2)]$$
$$= -(q-q^{-1})\delta(q^{k-i+j}z_2/z_1) : E^{\pm}_{i,j+1}(z_1)F^{3,+}_{j,i}(z_2) :, \tag{A.44}$$

$$[E^{+}_{i,j}(z_1), F^{3,\pm}_{j,i+1}(z_2)]$$
$$= (q-q^{-1})\delta(q^{k-i+j}z_2/z_1) : E^{+}_{i,j}(z_1)F^{3,\pm}_{j,i+1}(z_2) :, \tag{A.45}$$

$$[E^{\pm}_{i,j+1}(z_1), F^{3,\mp}_{j,i+1}(z_2)]$$
$$= \mp q(q-q^{-1})\delta(q^{k-i+j}z_2/z_1) : E^{\pm}_{i,j+1}(z_1)F^{3,\mp}_{j,i+1}(z_2) : . \tag{A.46}$$

$$: E^{\pm}_{i,j+1}(q^{k-i+j}z)F^{3,+}_{j,i}(z) :=: E^{+}_{i,j}(q^{k-i+j}z)F^{3,\pm}_{j,i+1}(z) :, \tag{A.47}$$

$$: E^{+}_{i,j+1}(q^{k-i+j}z)F^{3,-}_{j,i+1}(z) :=: E^{-}_{i,j+1}(q^{k-i+j}z)F^{3,+}_{j,i+1}(z) : . \tag{A.48}$$

A.4. $[X^{-,i}(z_1), X^{-,i-1}(z_2)]$ $(2 \le i \le M+N-1)$.

- For $2 \le i \le M-1$ we have

$$[F^{1,\pm}_{i,i-1}(z_1), F^{2,\pm}_{i-1,i-1}(z_2)]$$
$$= \pm(q-q^{-1})\delta(q^{2k+2i-1}z_2/z_1) : F^{1,\pm}_{i,i-1}(z_1)F^{2,\pm}_{i-1,i-1}(z_2) : . \tag{A.49}$$

$$: F^{1,+}_{i,i-1}(q^{2k+2i-1}z)F^{2,+}_{i-1,i-1}(z) :=: F^{1,-}_{i,i-1}(q^{2k+2i-1}z)F^{2,-}_{i-1,i-1}(z) : . \tag{A.50}$$

- For $i = M$ we have

$$[F^{1,+}_{M,M-1}(z_1), F^{2,+}_{M-1,M-1}(z_2)]$$
$$= (q-q^{-1})\delta(q^{2k+2M-1}z_2/z_1) : F^{1,+}_{M,M-1}(z_1)F^{2,+}_{M-1,M-1}(z_2) :, \tag{A.51}$$

$$[F^{2,-}_{M,M}(z_1), F^{3}_{M-1,M+1}(z_2)]$$
$$= -\frac{q^{k+M-1}}{z_1}\delta(q^{2k+2M-1}z_2/z_1) : F^{2,-}_{M,M}(z_1)F^{3}_{M-1,M+1}(z_2) : . \tag{A.52}$$

$$: F^{1,+}_{M,M-1}(q^{2k+2M-1}z)F^{2,+}_{M-1,M-1}(z) :=: F^{2,-}_{M,M}(q^{2k+2M-1}z)F^{3}_{M-1,M+1}(z) : . \tag{A.53}$$

• For $i = M+1$ we have

$$[F^{1}_{M+1,M}(z_1), F^{2,+}_{M,M}(z_2)] = -\frac{q^{k+M}}{z_1}\delta(q^{2k+2M-1}z_2/z_1) : F^{1}_{M+1,M}(z_1)F^{2,+}_{M,M}(z_2) :, \quad (A.54)$$

$$[F^{2,-}_{M+1,M+1}(z_1), F^{3,-}_{M,M+2}(z_2)] = (q-q^{-1})\delta(q^{2k+2M-1}z_2/z_1) : F^{2,-}_{M+1,M+1}(z_1)F^{3,-}_{M,M+2}(z_2) : . \quad (A.55)$$

$$: F^{1}_{M+1,M}(q^{2k+2M-1}z)F^{2,+}_{M,M}(z) :=: F^{2,-}_{M+1,M+1}(q^{2k+2M-1}z)F^{3,-}_{M,M+2}(z) : . \quad (A.56)$$

• For $M+2 \le i \le M+N-1$ we have

$$[F^{1,+}_{i,i-1}(z_1), F^{2,+}_{i-1,i-1}(z_2)] = \mp(q-q^{-1})\delta(q^{2k+4M-2i+1}z_2/z_1) : F^{1,+}_{i,i-1}(z_1)F^{2,+}_{i-1,i-1}(z_2) : . \quad (A.57)$$

$$: F^{1,+}_{i,i-1}(q^{2k+4M-2i+1}z)F^{2,+}_{i-1,i-1}(z) :=: F^{1,-}_{i,i-1}(q^{2k+4M-2i+1}z)F^{2,-}_{i-1,i-1}(z) : . \quad (A.58)$$

A.5. $[X^{-,i}(z_1), X^{-,j}(z_2)]$ $(3 \le i \le M+N-1, 1 \le j \le i-2)$.

• For $3 \le i \le M-1$ and $1 \le j \le i-2$ we have

$$[F^{1,+}_{i,j}(z_1), F^{3,\pm}_{j,i}(z_2)] = q(q-q^{-1})\delta(q^{2k+i+j}z_2/z_1) : F^{1,+}_{i,j}(z_1)F^{3,\pm}_{j,i}(z_2) :, \quad (A.59)$$

$$[F^{1,\pm}_{i,j+1}(z_1), F^{3,-}_{j,i+1}(z_2)] = -(q-q^{-1})\delta(q^{2k+i+j}z_2/z_1) : F^{1,\pm}_{i,j+1}(z_1)F^{3,-}_{j,i+1}(z_2) :, \quad (A.60)$$

$$[F^{1,\pm}_{i,j+1}(z_1), F^{3,\pm}_{j,i}(z_2)] = \mp(q-q^{-1})\delta(q^{2k+i+j}z_2/z_1) : F^{1,\pm}_{i,j+1}(z_1)F^{3,\pm}_{j,i}(z_2) : . \quad (A.61)$$

$$: F^{1,+}_{i,j}(q^{2k+i+j}z)F^{3,\pm}_{j,i}(z) :=: F^{1,\pm}_{i,j+1}(q^{2k+i+j}z)F^{3,-}_{j,i+1}(z) :, \quad (A.62)$$

$$: F^{1,+}_{i,j+1}(q^{2k+i+j}z)F^{3,+}_{j,i}(z) :=: F^{1,-}_{i,j+1}(q^{2k+i+j}z)F^{3,-}_{j,i}(z) : . \quad (A.63)$$

• For $i = M$ and $1 \le j \le M-2$ we have

$$[F^{1,+}_{M,j}(z_1), F^{3,\pm}_{j,M}(z_2)] = q(q-q^{-1})\delta(q^{2k+M+j}z_2/z_1) : F^{1,+}_{M,j}(z_1)F^{3,\pm}_{j,M}(z_2) :, \quad (A.64)$$

$$[F^{1,\pm}_{M,j+1}(z_1), F^{3}_{j,M+1}(z_2)] = -\frac{q^{k+j-1}}{z_1}\delta(q^{2k+M+j}z_2/z_1) : F^{1,\pm}_{M,j+1}(z_1)F^{3}_{j,M+1}(z_2) :, \quad (A.65)$$

$$[F^{1,\pm}_{M,j+1}(z_1), F^{3,\pm}_{j,M}(z_2)] = \mp(q-q^{-1})\delta(q^{2k+M+j}z_2/z_1) : F^{1,\pm}_{M,j+1}(z_1)F^{3,\pm}_{j,M}(z_2) : . \quad (A.66)$$

$$: F^{1,+}_{M,j}(q^{2k+M+j}z)F^{3,\pm}_{j,M}(z) :=: F^{1,\pm}_{M,j+1}(q^{2k+M+j}z)F^{3}_{j,M+1}(z) :, \quad (A.67)$$

$$: F^{1,+}_{M,j+1}(q^{2k+M+j}z)F^{3,+}_{j,M}(z) :=: F^{1,-}_{M,j+1}(q^{2k+M+j}z)F^{3,-}_{j,M}(z) : . \quad (A.68)$$

- For $M+1 \le i \le M+N-1$ and $M+1 \le j \le i-2$ we have

$$[F_{i,j}^{1,+}(z_1), F_{j,i}^{3,\pm}(z_2)] = -q^{-1}(q-q^{-1})\delta(q^{2k+4M-i-j}z_2/z_1) : F_{i,j}^{1,+}(z_1)F_{j,i}^{3,\pm}(z_2) :, \quad (A.69)$$

$$[F_{i,j+1}^{1,\pm}(z_1), F_{j,i+1}^{3,-}(z_2)] = (q-q^{-1})\delta(q^{2k+4M-i-j}z_2/z_1) : F_{i,j+1}^{1,\pm}(z_1)F_{j,i+1}^{3,-}(z_2) :, \quad (A.70)$$

$$[F_{i,j+1}^{1,\pm}(z_1), F_{j,i}^{3,\pm}(z_2)] = \pm(q-q^{-1})\delta(q^{2k+4M-i-j}z_2/z_1) : F_{i,j+1}^{1,\pm}(z_1)F_{j,i}^{3,\pm}(z_2) : . \quad (A.71)$$

$$: F_{i,j}^{1,+}(q^{2k+4M-i-j}z)F_{j,i}^{3,\pm}(z) :=: F_{i,j+1}^{1,\pm}(q^{2k+4M-i-j}z)F_{j,i+1}^{3,-}(z) :, \quad (A.72)$$

$$: F_{i,j+1}^{1,+}(q^{2k+4M-i-j}z)F_{j,i}^{3,+}(z) :=: F_{i,j+1}^{1,-}(q^{2k+4M-i-j}z)F_{j,i}^{3,-}(z) : . \quad (A.73)$$

- For $M+2 \le i \le M+N-1$ and $j=M$ we have

$$[F_{i,M}^{1}(z_1), F_{M,i}^{3,\pm}(z_2)] = -\frac{q^{k+M-1}}{z_1}\delta(q^{2k+3M-i}z_2/z_1) : F_{i,M}^{1}(z_1)F_{M,i}^{3,\pm}(z_2) :, \quad (A.74)$$

$$[F_{i,M+1}^{1,\pm}(z_1), F_{M,i+1}^{3,-}(z_2)] = (q-q^{-1})\delta(q^{2k+3M-i}z_2/z_1) : F_{i,M+1}^{1,\pm}(z_1)F_{M,i+1}^{3,-}(z_2) :, \quad (A.75)$$

$$[F_{i,M+1}^{1,\pm}(z_1), F_{M,i}^{3,\pm}(z_2)] = \pm(q-q^{-1})\delta(q^{2k+3M-i}z_2/z_1) : F_{i,M+1}^{1,\pm}(z_1)F_{M,i}^{3,\pm}(z_2) : . \quad (A.76)$$

$$: F_{i,M}^{1,+}(q^{2k+3M-i}z)F_{M,i}^{3,\pm}(z) :=: F_{i,M+1}^{1,\pm}(q^{2k+3M-i}z)F_{M,i+1}^{3,-}(z) :, \quad (A.77)$$

$$: F_{i,M+1}^{1,+}(q^{2k+3M-i}z)F_{M,i}^{3,+}(z) :=: F_{i,M+1}^{1,-}(q^{2k+i+j}z)F_{M,i}^{3,-}(z) : . \quad (A.78)$$

- For $M+1 \le i \le M+N-1$ and $1 \le j \le M-1$ we have

$$[F_{i,j}^{1}(z_1), F_{j,i}^{3}(z_2)] = -\frac{q^{k+j-1}}{z_1}\delta(q^{2k+2M-i+j}z_2/z_1) : F_{i,j}^{1}(z_1)F_{j,i}^{3}(z_2) :, \quad (A.79)$$

$$[F_{i,j+1}^{1}(z_1), F_{j,i+1}^{3}(z_2)] = \frac{q^{k+j+1}}{z_1}\delta(q^{2k+2M-i+j}z_2/z_1) : F_{i,j+1}^{1}(z_1)F_{j,i+1}^{3}(z_2) :, \quad (A.80)$$

$$[F_{i,j+1}^{1}(z_1), F_{j,i}^{3}(z_2)] = 0. \quad (A.81)$$

$$: F_{i,j}^{1}(q^{2k+2M-i+j}z)F_{j,i}^{3}(z) :=: F_{i,j+1}^{1}(q^{2k+2M-i+j}z)F_{j,i+1}^{3}(z) : . \quad (A.82)$$

A.6. $[S_i(z_1), X^{+,j}(z_2)]$ $(1 \le i \le M+N-2, i+1 \le j \le M+N-1)$.

- For $1 \le i \le M-2$ and $i+1 \le j \le M-1$ we have

$$[S_{i,j}^{+}(z_1), E_{j,i}^{+}(z_2)] = q(q-q^{-1})\delta(q^{i+j-M+N}z_2/z_1) : S_{i,j}^{+}(z_1)E_{j,i}^{+}(z_2) :, \quad (A.83)$$

$$[S_{i,j+1}^{-}(z_1), E_{j,i+1}^{-}(z_2)]$$

$$= -q(q-q^{-1})\delta(q^{i+j-M+N}z_2/z_1) : S^-_{i,j+1}(z_1)E^-_{j,i+1}(z_2) :, \tag{A.84}$$

$$[S^\pm_{i,j+1}(z_1), E^\pm_{j,i}(z_2)]$$
$$= \mp(q-q^{-1})\delta(q^{i+j-M+N}z_2/z_1) : S^\pm_{i,j+1}(z_1)E^\pm_{j,i}(z_2) :, \tag{A.85}$$

$$[S^-_{i,j}(z_1), E^-_{j,i}(z_2)]$$
$$= q(q-q^{-1})\delta(q^{i+j-M+N}z_2/z_1) : S^-_{i,j}(z_1)E^-_{j,i}(z_2) :, \tag{A.86}$$

$$[S^+_{i,j+1}(z_1), E^-_{j,i+1}(z_2)]$$
$$= -q(q-q^{-1})\delta(q^{i+j-M+N}z_2/z_1) : S^+_{i,j+1}(z_1)E^-_{j,i+1}(z_2) : . \tag{A.87}$$

$$: S^+_{i,j}(q^{i+j-M+N}z)E^+_{j,i}(z) :=: S^-_{i,j+1}(q^{i+j-M+N}z)E^-_{j,i+1}(z) :, \tag{A.88}$$

$$: S^+_{i,j+1}(q^{i+j-M+N}z)E^+_{j,i}(z) :=: S^-_{i,j+1}(q^{i+j-M+N}z)E^-_{j,i}(z) :, \tag{A.89}$$

$$: S^+_{i,j}(q^{i+j-M+N}z)E^-_{j,i}(z) :=: S^+_{i,j+1}(q^{i+j-M+N}z)E^-_{j,i+1}(z) : . \tag{A.90}$$

- For $i = M-1$ and $M+1 \le j \le M+N-1$ we have

$$[S^+_{M-1,M}(z_1), E_{M,M-1}(z_2)] = q^{-1}(q-q^{-1})\delta(q^{M+N-1}z_2/z_1) : S^+_{M-1,M}(z_1)E_{M,M-1}(z_2) :, \tag{A.91}$$

$$[S_{M-1,j}(z_1), E_{j,M-1}(z_2)] = -\frac{q^{M+N-j-1}}{z_1}\delta(q^{2M+N-j-1}z_2/z_1) : S_{M-1,j}(z_1)E_{j,M-1}(z_2) : . \tag{A.92}$$

For $i = M-1$ and $M+1 \le j \le M+N-1$ we have

$$[S^\pm_{M-1,M}(z_1), E^\pm_{M-1,M-1}(z_2)] = \mp(q-q^{-1})\delta(q^{M+N-2}z_2/z_1) : S^\pm_{M-1,M}(z_1)E^\pm_{M-1,M-1}(z_2) :, \tag{A.93}$$

$$[S_{M-1,j}(z_1), E_{j,M-1}(z_2)] = -\frac{q^{M+N-j-1}}{z_1}\delta(q^{2M+N-j-1}z_2/z_1) : S_{M-1,j}(z_1)E_{j,M-1}(z_2) : . \tag{A.94}$$

For $i = M-1$ and $M \le j \le M+N-1$ we have

$$[S_{M-1,j+1}(z_1), E_{j,M}(z_2)]$$
$$= -\frac{q^{M+N-j-1}}{z_1}\delta(q^{2M+N-j-1}z_2/z_1) : S_{M-1,j+1}(z_1)E_{j,M}(z_2) : . \tag{A.95}$$

$$: S^+_{M-1,M}(q^{M+N-2}z)E^+_{M-1,M-1}(z)$$
$$:=: S^-_{M-1,M}(q^{M+N-2}z)E^-_{M-1,M-1}(z) :, \tag{A.96}$$

$$: S^+_{M-1,M}(q^{M+N-1}z)E_{M,M-1}(z) :=: S_{M-1,M+1}(q^{M+N-1}z)E_{M,M}(z) :, \tag{A.97}$$

$$: S_{M-1,j}(q^{2M+N-j-1}z)E_{j,M-1}(z) :=: S_{M-1,j+1}(q^{2M+N-j-1}z)E_{j,M}(z) :$$
$$(M+1 \le j \le M+N-1). \tag{A.98}$$

- For $i = M$ and $M+1 \le j \le M+N-1$ we have

$$[S_{M,j}(z_1), E_{j,M}(z_2)] = \frac{q^{M+N-j-1}}{z_1}\delta(q^{2M+N-j}z_2/z_1) : S_{M,j}(z_1)E_{j,M}(z_2) :, \quad \text{(A.99)}$$

$$[S_{M,j+1}(z_1), E^-_{j,M+1}(z_2)] = q^{-1}(q-q^{-1})\delta(q^{2M+N-j}z_2/z_1) : S_{M,j+1}(z_1)E^-_{j,M+1}(z_2) :, \quad \text{(A.100)}$$

$$[S_{M,j+1}(z_1), E_{j,M}(z_2)] = 0. \quad \text{(A.101)}$$

$$: S_{M,j}(q^{2M+N-j}z)E_{j,M}(z) :=: S_{M,j+1}(q^{2M+N-j}z)E^-_{j,M+1}(z) : . \quad \text{(A.102)}$$

- For $M+1 \le i \le M+N-1$ and $i+1 \le j \le M+N-1$ we have

$$[S^+_{i,j}(z_1), E^+_{j,i}(z_2)] = -q^{-1}(q-q^{-1})\delta(q^{3M+N-i-j}z_2/z_1) : S^+_{i,j}(z_1)E^+_{j,i}(z_2) :, \quad \text{(A.103)}$$

$$[S^-_{i,j+1}(z_1), E^-_{j,i+1}(z_2)] = q^{-1}(q-q^{-1})\delta(q^{3M+N-i-j}z_2/z_1) : S^-_{i,j+1}(z_1)E^-_{j,i+1}(z_2) :, \quad \text{(A.104)}$$

$$[S^\pm_{i,j+1}(z_1), E^\pm_{j,i}(z_2)] = \pm(q-q^{-1})\delta(q^{3M+N-i-j}z_2/z_1) : S^\pm_{i,j+1}(z_1)E^\pm_{j,i}(z_2) :, \quad \text{(A.105)}$$

$$[S^+_{i,j}(z_1), E^-_{j,i}(z_2)] \quad -q^{-1}(q-q^{-1})\delta(q^{3M+N-i-j}z_2/z_1) : S^+_{i,j}(z_1)E^-_{j,i}(z_2) :, \quad \text{(A.106)}$$

$$[S^+_{i,j+1}(z_1), E^-_{j,i+1}(z_2)] = q^{-1}(q-q^{-1})\delta(q^{3M+N-i-j}z_2/z_1) : S^+_{i,j+1}(z_1)E^-_{j,i+1}(z_2) : . \quad \text{(A.107)}$$

$$: S^+_{i,j}(q^{3M+N-i-j}z)E^+_{j,i}(z) :=: S^-_{i,j+1}(q^{3M+N-i-j}z)E^-_{j,i+1}(z) :, \quad \text{(A.108)}$$

$$: S^+_{i,j+1}(q^{3M+N-i-j}z)E^+_{j,i}(z) :=: S^-_{i,j+1}(q^{3M+N-i-j}z)E^-_{j,i}(z) :, \quad \text{(A.109)}$$

$$: S^+_{i,j}(q^{3M+N-i-j}z)E^-_{j,i}(z) :=: S^+_{i,j+1}(q^{3M+N-i-j}z)E^-_{j,i+1}(z) : . \quad \text{(A.110)}$$

A.7. $[S_i(z_1), X^{-,i-1}(z_2)]$ $(2 \le i \le M+N-1)$.

- For $2 \le i \le M-1$ and $i+1 \le j \le M$ we have

$$[S^\pm_{i,j}(z_1), F^{3,\pm}_{i-1,j}(z_2)] = \mp(q-q^{-1})\delta(q^{k+2j-M+N-1}z_2/z_1) : S^\pm_{i,j}(z_1)F^{3,\pm}_{i-1,j}(z_2) : . \quad \text{(A.111)}$$

$$: S^+_{i,j}(q^{k+2j-M+N-1}z)F^{3,+}_{i-1,j}(z) :=: S^-_{i,j}(q^{k+2j-M+N-1}z)F^{3,-}_{i-1,j}(z) : . \quad \text{(A.112)}$$

- For $i = M$ and $M+1 \le j \le M+N$ we have

$$[S_{M,j}(z_1), F^3_{M-1,j}(z_2)] = 0. \quad \text{(A.113)}$$

- For $i=M+1$ and $M+2\le j\le M+N$ we have

$$[S^{\pm}_{M+1,j}(z_1),F^{3,\pm}_{M,j}(z_2)] = \pm(q-q^{-1})\delta(q^{k+3M+N-2j+1}z_2/z_1):S^{\pm}_{M+1,j}(z_1)F^{3,\pm}_{M,j}(z_2):. \quad (A.114)$$

$$:S^{+}_{M+1,j}(q^{k+3M+N-2j+1}z)F^{3,+}_{M,j}(z) :=: S^{-}_{M+1,j}(q^{k+3M+N-2j+1}z)F^{3,-}_{M,j}(z):. \quad (A.115)$$

- For $M+2\le i\le M+N-1$ and $i+1\le j\le M+N$ we have

$$[S^{\pm}_{i,j}(z_1),F^{3,\pm}_{i-1,j}(z_2)] = \pm(q-q^{-1})\delta(q^{k+3M+N-2j+1}z_2/z_1):S^{\pm}_{i,j}(z_1)F^{3,\pm}_{i-1,j}(z_2):. \quad (A.116)$$

$$:S^{+}_{i,j}(q^{k+3M+N-2j+1}z)F^{3,+}_{i-1,j}(z) :=: S^{-}_{i,j}(q^{k+3M+N-2j+1}z)F^{3,-}_{i-1,j}(z):. \quad (A.117)$$

A.8. $[S_i(z_1),X^{-,j}(z_2)]$ $(1\le i\le M+N-2,\ i+1\le j\le M+N-1)$.

- For $1\le i\le M-1$ and $i+1\le j\le M$ we have

$$[S^{-}_{i,j+1}(z_1),F^{1,\pm}_{j,i}(z_2)] = -(q-q^{-1})\delta(q^{-k-M+N-i+j}z_2/z_1): S^{-}_{i,j+1}(z_1)F^{1,\pm}_{j,i}(z_2):, \quad (A.118)$$

$$[S^{\pm}_{i,j}(z_1),F^{1,-}_{j,i+1}(z_2)] = (q-q^{-1})\delta(q^{-k-M+N-i+j}z_2/z_1): S^{\pm}_{i,j}(z_1)F^{1,-}_{j,i+1}(z_2): \quad (1\le i\le M-2,\ i+2\le j\le M), \quad (A.119)$$

$$[S^{\pm}_{i,i+1}(z_1),F^{2,-}_{i+1,i+1}(z_2)] = (q-q^{-1})\delta(q^{-k-M+N+1}z_2/z_1): S^{\pm}_{i,i+1}(z_1)F^{2,-}_{i+1,i+1}(z_2):, \quad (A.120)$$

$$[S^{\pm}_{i,j}(z_1),F^{1,\mp}_{j,i}(z_2)] = \mp q^{-1}(q-q^{-1})\delta(q^{-k-M+N-i+j}z_2/z_1): S^{\pm}_{i,j}(z_1)F^{1,\mp}_{j,i}(z_2): \quad (1\le i\le M-2,\ i+1\le j\le M-1), \quad (A.121)$$

$$[S^{\pm}_{M-1,M}(z_1),F^{1,\mp}_{M,M-1}(z_2)] = \pm q^{-1}(q-q^{-1})\delta(q^{-k-M+N+1}z_2/z_1): S^{\pm}_{M-1,M}(z_1)F^{1,\mp}_{M,M-1}(z_2):. \quad (A.122)$$

$$:S^{\pm}_{i,i+1}(q^{-k-M+N+1}z)F^{2,-}_{i+1,i+1}(z) :=: S^{-}_{i,i+2}(q^{-k-M+N+1}z)F^{1,\pm}_{i+1,i}(z):, \quad (A.123)$$

$$:S^{\pm}_{i,j}(q^{-k-M+N-i+j}z)F^{1,-}_{j,i+1}(z) :=: S^{-}_{i,j+1}(q^{-k-M+N-i+j}z)F^{1,\pm}_{j,i}(z): \quad (1\le i\le M-2,\ i+2\le j\le M), \quad (A.124)$$

$$:S^{+}_{i,j}(q^{-k-M+N-i+j}z)F^{1,-}_{j,i}(z) :=: S^{-}_{i,j}(q^{-k-M+N-i+j}z)F^{1,+}_{j,i}(z): \quad (1\le i\le M-2,\ i+1\le j\le M-1), \quad (A.125)$$

$$: S^{+}_{M-1,M}(q^{-k-M+N+1}z)F^{1,-}_{M,M-1}(z) :=: S^{-}_{M-1,M}(q^{-k-M+N+1}z)F^{1,+}_{M,M-1}(z) : . \tag{A.126}$$

For $1 \le i \le M-1$ and $M+1 \le j \le M+N-1$ we have

$$[S_{i,j}(z_1), F^{1}_{j,i}(z_2)] = 0, \tag{A.127}$$

$$[S_{i,j+1}(z_1), F^{1}_{j,i}(z_2)] = -\frac{q^{M+N-j-1}}{z_1}\delta(q^{-k+M+N-i-j}z_2/z_1) : S_{i,j+1}(z_1)F^{1}_{j,i}(z_2) :, \tag{A.128}$$

$$[S_{i,j}(z_1), F^{1}_{j,i+1}(z_2)] = \frac{q^{M+N-j+1}}{z_1}\delta(q^{-k+M+M-i-j}z_2/z_1) : S_{i,j}(z_1)F^{1}_{j,i+1}(z_2) : . \tag{A.129}$$

$$: S_{i,j}(q^{-k+M+N-i-j}z)F^{1}_{j,i+1}(z) :=: S_{i,j+1}(q^{-k+M+N-i-j}z)F^{1}_{j,i}(z) : . \tag{A.130}$$

- For $i = M$ and $M+1 \le j \le M+N-1$ we have

$$[S_{M,j}(z_1), F^{1}_{j,M}(z_2)] = 0, \tag{A.131}$$

$$[S_{M,j+1}(z_1), F^{1}_{j,M}(z_2)] = -\frac{q^{M+N-j-1}}{z_1}\delta(q^{-k+N-j}z_2/z_1) : S_{M,j+1}(z_1)F^{1}_{j,M}(z_2) :, \tag{A.132}$$

$$[S_{M,j}(z_1), F^{1,-}_{j,M+1}(z_2)] = -(q-q^{-1})\delta(q^{-k+N-j}z_2/z_1) : S_{M,j}(z_1)F^{1,-}_{j,M+1}(z_2) : . \tag{A.133}$$

$$: S_{M,j+1}(q^{-k+N-j}z)F^{1}_{j,M}(z) :=: S_{M,j}(q^{-k+N-j}z)F^{1,-}_{j,M+1}(z) : . \tag{A.134}$$

- For $M+1 \le i \le M+N-1$ and $i+1 \le j \le M+N-1$ we have

$$[S^{-}_{i,j+1}(z_1), F^{1,\pm}_{j,i}(z_2)] = (q-q^{-1})\delta(q^{-k-M+N+i-j}z_2/z_1) : S^{-}_{i,j+1}(z_1)F^{1,\pm}_{j,i}(z_2) :, \tag{A.135}$$

$$[S^{\pm}_{i,j}(z_1), F^{1,-}_{j,i+1}(z_2)] = -(q-q^{-1})\delta(q^{-k-M+N+i-j}z_2/z_1) : S^{\pm}_{i,j}(z_1)F^{1,-}_{j,i+1}(z_2) : \quad (i+2 \le j \le M+N-1), \tag{A.136}$$

$$[S^{\pm}_{i,i+1}(z_1), F^{2,-}_{i+1,i+1}(z_2)] = -(q-q^{-1})\delta(q^{-k-M+N+1}z_2/z_1) : S^{\pm}_{i,i+1}(z_1)F^{2,-}_{i+1,i+1}(z_2) :, \tag{A.137}$$

$$[S^{\pm}_{i,j}(z_1), F^{1,\mp}_{j,i}(z_2)] = \pm q^{-1}(q-q^{-1})\delta(q^{-k-M+N+i-j}z_2/z_1) : S^{\pm}_{i,j}(z_1)F^{1,\mp}_{j,i}(z_2) : . \tag{A.138}$$

For $M+1 \le i \le M+N-1$ and $i+2 \le j \le M+N-1$ we have

$$: S^{\pm}_{i,i+1}(q^{-k-M+N-1}z)F^{2,-}_{i+1,i+1}(z) :=: S^{-}_{i,i+2}(q^{-k-M+N-1}z)F^{1,\pm}_{i+1,i}(z) :, \tag{A.139}$$

$$: S^{\pm}_{i,j}(q^{-k-M+N+i-j}z)F^{1,-}_{j,i+1}(z) :=: S^{-}_{i,j+1}(q^{-k-M+N+i-j}z)F^{1,\pm}_{j,i}(z) : \quad (i+2 \le j \le M+N-1), \tag{A.140}$$

$$: S^{+}_{i,j}(q^{-k-M+N+i-j}z)F^{1,-}_{j,i}(z) :=: S^{-}_{i,j}(q^{-k-M+N+i-j}z)F^{1,+}_{j,i}(z) : . \tag{A.141}$$

Appendix B: Difference Realization of $U_q(sl(M|N))$

In this Appendix we recall a q-difference realization of $U_q(sl(M|N))$ [19]. We introduce the coordinates $x_{i,j}$ $(1 \le i < j \le M+N)$ by

$$x_{i,j} = \begin{cases} z_{i,j} & (\nu_i \nu_j = +), \\ \theta_{i,j} & (\nu_i \nu_j = -), \end{cases} \tag{B.1}$$

where $z_{i,j}$ are complex variables and $\theta_{i,j}$ are the Grassmann odd variables that satisfy $\theta_{i,j}\theta_{i,j} = 0$ and $\theta_{i,j}\theta_{i',j'} = -\theta_{i',j'}\theta_{i,j}$. We set the differential operators $\vartheta_{i,j} = x_{i,j}\frac{\partial}{\partial x_{i,j}}$. We fix parameters $\lambda_i \in \mathbf{C}$, $(1 \le i \le M+N-1)$. We set q-difference operators h_i, e_i, f_i $(1 \le i \le M+N-1)$ as follows.

$$h_i = \lambda_i + \sum_{j=1}^{i-1}(\nu_i \vartheta_{j,i} - \nu_{i+1}\vartheta_{j,i+1}) - (\nu_i + \nu_{i+1})\vartheta_{i,i+1}$$
$$\sum_{j=i+1}^{M+N}(\nu_{i+1}\vartheta_{i+1,j} - \nu_i \vartheta_{i,j}), \tag{B.2}$$

$$e_i = e_{i,i} + \sum_{j=1}^{i-1} e_{i,j}, \quad f_i = \nu_i \sum_{j=1}^{i-1} f^1_{i,j} + f^2_{i,i} - \nu_{i+1}\sum_{j=i+2}^{M+N} f^3_{i,j}, \tag{B.3}$$

where we have set

$$e_{i,i} = \frac{1}{x_{i,i+1}}[\vartheta_{i,i+1}]_q \, q^{\sum_{l=1}^{i-1}(\nu_i \vartheta_{l,i} - \nu_{i+1}\vartheta_{l,i+1})}, \tag{B.4}$$

$$e_{i,j} = x_{j,i}\frac{1}{x_{j,i+1}}[\vartheta_{j,i+1}]_q \, q^{\sum_{l=1}^{j-1}(\nu_i \vartheta_{l,i} - \nu_{i+1}\vartheta_{l,i+1})}, \tag{B.5}$$

$$f^1_{i,j} = x_{j,i+1}\frac{1}{x_{j,i}}[\vartheta_{j,i}]_q$$
$$q^{\sum_{l=j+1}^{i-1}(\nu_{i+1}\vartheta_{l,i+1} - \nu_i \vartheta_{l,i}) - \lambda_i + (\nu_i + \nu_{i+1})\vartheta_{i,i+1} + \sum_{l=i+2}^{N+1}(\nu_i \vartheta_{i,l} - \nu_{i+1}\vartheta_{i+1,l})}, \tag{B.6}$$

$$f^2_{i,i} = x_{i,i+1}\left[\lambda_i - \nu_i \vartheta_{i,i+1} - \sum_{l=i+2}^{N+1}(\nu_i \vartheta_{i,l} - \nu_{i+1}\vartheta_{i+1,l})\right]_q, \tag{B.7}$$

$$f^3_{i,j} = x_{i,j+1}\frac{1}{x_{i+1,j}}[\vartheta_{i+1,j}]_q q^{\lambda_i + \sum_{l=j+1}^{N+1}(\nu_{i+1}\vartheta_{i+1,l} - \nu_i \vartheta_{i,l})}. \tag{B.8}$$

For Grassmann odd variables $x_{i,j} = \theta_{i,j}$, the expression $\frac{1}{x_{i,j}}$ stands for derivative $\frac{1}{x_{i,j}} = \frac{\partial}{\partial x_{i,j}}$. The q-difference operators h_i, e_i, f_i $(1 \le i \le M+N-1)$ satisfy the defining relations of $U_q(sl(M|N))$.

Appendix C: Limit $q \to 1$

In this Appendix we summarize useful formulae to take the limit $q \to 1$. Using the following relations

$$b^{i,j}_{\pm}(z) - (b+c)^{i,j}(q^{\pm 1}z) = -c^{i,j}(q^{\pm 1}z) - b^{i,j}(z; -1), \tag{C.1}$$

$$b_{\pm}^{i,j}(z)-(b+c)^{i,j}(q^{\mp1}z)=-c^{i,j}(q^{\mp1}z)-b^{i,j}(z;1), \tag{C.2}$$

$$b_{\pm}^{i,j}(q^{\pm\alpha}z)=\left(\frac{1}{M}b^{i,j}\right)(q^{\pm M}z;\alpha)-\left(\frac{1}{M}b^{i,j}\right)(z;\alpha-M), \tag{C.3}$$

$$b^{i,j}(q^{\pm\alpha}z)=\left(\frac{\alpha}{M}b^{i,j}\right)(q^{\pm M}z;0)+\left(\frac{M-\alpha}{M}b^{i,j}\right)(z;0), \tag{C.4}$$

we have

$$\frac{\pm1}{(q-q^{-1})z}:(e^{\pm b_{+}^{i,j}(z)-(b+c)^{i,j}(q^{\pm1}z)}-e^{\pm b_{-}^{i,j}(z)-(b+c)^{i,j}(q^{\mp1}z)})$$
$$=:{}_{1}\partial_z\left(e^{-c^{i,j}(z)}\right)e^{-b^{i,j}(z;\mp1)}:, \tag{C.5}$$

$$\frac{1}{(q-q^{-1})z}:(e^{a_{+}^{i}(q^{\frac{k+g}{2}}z)}-e^{a_{-}^{i}(q^{-\frac{k+g}{2}}z)})$$
$$=:{}_{k+g}\partial_z\left(e^{\left(\frac{1}{k+g}a^i\right)\left(z;\frac{k+g}{2}\right)}\right)e^{-\left(\frac{1}{k+g}a^i\right)\left(z;-\frac{k+g}{2}\right)}:, \tag{C.6}$$

$$\frac{1}{(q-q^{-1})z}:(e^{b^{i,j}(q^{k+l}z)}-e^{b^{i,j}(q^{-k-l}z)})$$
$$=:{}_{k+g}\partial_z\left(e^{\left(\frac{k+l}{k+g}b^{i,j}\right)(z;0)}\right)e^{\left(\frac{g-l}{k+g}b^{i,j}\right)(z;0)}:, \tag{C.7}$$

$$\frac{1}{(q-q^{-1})z}:(e^{b_{+}^{i,j}(q^{k+l}z)}-e^{b_{-}^{i,j}(q^{-k-l}z)})$$
$$=:{}_{k+g}\partial_z\left(e^{\left(\frac{1}{k+g}b^{i,j}\right)(z;k+l)}\right)e^{-\left(\frac{1}{k+g}b^{i,j}\right)(z;l-g)}:. \tag{C.8}$$

Hence we have the following formulae.

- For $1\le i\le M-1$ we have

$$H^i(z)={}_1\partial_z\left\{a^i\left(z;\frac{g}{2}\right)+\sum_{l=1}^{i}\left(b^{l,i+1}\left(z;\frac{k}{2}+l-1\right)-b^{l,i}\left(z;\frac{k}{2}+l\right)\right)\right.$$
$$-\sum_{l=i+1}^{M}\left(b^{i+1,l}\left(z;\frac{k}{2}+l-1\right)-b^{i,l}\left(z;\frac{k}{2}+l\right)\right)$$
$$\left.-\sum_{l=M+1}^{M+N}\left(b^{i+1,l}\left(z;\frac{k}{2}+2M-l\right)-b^{i,l}\left(z;\frac{k}{2}+2M+1-l\right)\right)\right\}. \tag{C.9}$$

- For $i=M$ we have

$$H^M(z)={}_1\partial_z\left\{a^M\left(z;\frac{g}{2}\right)-\sum_{l=1}^{M-1}\left(b^{l,M+1}\left(z;\frac{k}{2}+l\right)+b^{l,M}\left(z;\frac{k}{2}+l\right)\right)\right.$$
$$+\sum_{l=M+2}^{M+N}\left(b^{M+1,l}\left(z;\frac{k}{2}+2M+1-l\right)\right.$$
$$\left.\left.+b^{M,l}\left(z;\frac{k}{2}+2M+1-l\right)\right)\right\}. \tag{C.10}$$

- For $M+1 \le i \le M+N-1$ we have

$$
\begin{aligned}
H^i(z) = {}_1\partial_z \Bigg\{ & a^i\left(z; \frac{g}{2}\right) - \sum_{l=1}^{M} \left(b^{l,i+1}\left(z; \frac{k}{2}+l\right) - b^{l,i}\left(z; \frac{k}{2}+l-1\right)\right) \\
& - \sum_{l=M+1}^{i} \left(b^{l,i+1}\left(z; \frac{k}{2}+2M-l+1\right) - b^{l,i}\left(z; \frac{k}{2}+2M-l\right)\right) \\
& + \sum_{l=i+1}^{M+N} \left(b^{i+1,l}\left(z; \frac{k}{2}+2M-l+1\right) - b^{i,l}\left(z; \frac{k}{2}+2M-l\right)\right) \Bigg\}.
\end{aligned}
\tag{C.11}
$$

- For $1 \le i \le M-1$ we have

$$
\begin{aligned}
& \frac{1}{(q-q^{-1})z}(E^+_{i,j}(z) - E^-_{i,j}(z)) \\
& =: {}_1\partial_z \left(e^{-c^{j,i+1}(q^{j-1}z)}\right) e^{-b^{j,i+1}(q^{j-1}z;-1)+(b+c)^{j,i}(q^{j-1}z)+\sum_{l=1}^{j-1}(\Delta_R^- b_+^{l,i})(q^l z)} : \\
& (1 \le j \le i).
\end{aligned}
\tag{C.12}
$$

- For $M+1 \le i \le M+N-1$ we have

$$
\begin{aligned}
& \frac{1}{(q-q^{-1})z}(E^+_{i,j}(z) - E^-_{i,j}(z)) \\
& = - : {}_1\partial_z \left(e^{-c^{j,i+1}(q^{2M+1-j}z)}\right) e^{-b^{j,i+1}(q^{2M+1-j}z;1)+(b+c)^{j,i}(q^{2M+1-j}z)} \\
& \times e^{-\sum_{l=1}^{M}(\Delta_R^+ b_+^{l,i}(q^{l-1}z)) - \sum_{l=M+1}^{j-1}(\Delta_R^+ b_+^{l,i})(q^{2M-l}z)} : \quad (M+1 \le j \le i).
\end{aligned}
\tag{C.13}
$$

- For $1 \le i \le M-1$ we have

$$
\begin{aligned}
& \frac{-1}{(q-q^{-1})z}(F^{1,+}_{i,j}(z) - F^{1,-}_{i,j}(z)) \\
& =: {}_1\partial_z \left(e^{-c^{j,i}(q^{-k-j}z)}\right) e^{-b^{j,i}(q^{-k-j}z;1)+a_-^i(q^{-\frac{k+g}{2}}z)+(b+c)^{j,i+1}(q^{-k-j}z)} \\
& \times e^{\sum_{l=j+1}^{i}(\Delta_R^+ b_-^{l,i})(q^{-k-l}z) - \sum_{l=i+1}^{M}(\Delta_L^+ b_-^{i,l})(q^{-k-l}z) - \sum_{l=M+1}^{M+N}(\Delta_L^+ b_-^{i,l})(q^{-k-2M-1+l}z)} : \\
& (1 \le j \le i-1),
\end{aligned}
\tag{C.14}
$$

$$
\begin{aligned}
& \frac{1}{(q-q^{-1})z}(F^{2,+}_{i,i}(z) - F^{2,-}_{i,i}(z)) \\
& =: {}_{k+g}\partial_z \Bigg(e^{\left(\frac{1}{k+g}a^i\right)(z;\frac{k+g}{2})+\left(\frac{k+i}{k+g}(b+c)^{i,i+1}\right)(z;0)-\sum_{l=i+1}^{M}\left(\left(\frac{1}{k+g}b^{i+1,l}\right)(z;k+l-1)-\left(\frac{1}{k+g}b^{i,l}\right)(z;k+l)\right)} \\
& \times e^{-\sum_{l=M+1}^{M+N}\left(\left(\frac{1}{k+g}b^{i+1,l}\right)(z;k+2M-l)-\left(\frac{1}{k+g}b^{i,l}\right)(z;k+2M+1-l)\right)} \Bigg) \\
& \times e^{-\left(\frac{1}{k+g}a^i\right)(z;-\frac{k+g}{2})+\left(\frac{g-i}{k+g}(b+c)^{i,i+1}\right)(z;0)+\sum_{l=i+1}^{M}\left(\left(\frac{1}{k+g}b^{i+1,l}\right)(z;l-g-1)-\left(\frac{1}{k+g}b^{i,l}\right)(z;l-g)\right)} \\
& \times e^{\sum_{l=M+1}^{M+N}\left(\left(\frac{1}{k+g}b^{i+1,l}\right)(z;2M-l-g)-\left(\frac{1}{k+g}b^{i,l}\right)(z;2M-l-g+1)\right)} :,
\end{aligned}
\tag{C.15}
$$

$$
\begin{aligned}
&\frac{1}{(q-q^{-1})z}(F_{i,j}^{3,+}(z)-F_{i,j}^{3,-}(z))\\
&=:{}_1\partial_z\left(e^{-c^{i+1,j}(q^{k+j-1}z)}\right)e^{-b^{i+1,j}(q^{k+j-1}z;-1)+a_+^i(q^{\frac{k+g}{2}}z)+(b+c)^{i,j}(q^{k+j-1}z)}\\
&\quad\times e^{-\sum_{l=j}^{M}(\Delta_L^- b_+^{i,l})(q^{k+l}z)-\sum_{l=M+1}^{M+N}(\Delta_L^- b_+^{i,l})(q^{k+2M+1-i}z)}:\quad (1\le j\le i-1).
\end{aligned}
\tag{C.16}
$$

- For $i=M$ we have

$$
\begin{aligned}
&\frac{-1}{(q-q^{-1})z}(F_{M,j}^{1,+}(z)-F_{M,j}^{1,-}(z))\\
&=:{}_1\partial_z\left(e^{-c^{j,M}(q^{-k-j}z)}\right)e^{-b^{j,M}(q^{-k-j}z;1)+a_-^M(q^{-\frac{k+g}{2}}z)-b_-^{j,M+1}(q^{-k-j}z)-b^{j,M+1}(q^{-k-j+1}z)}
\end{aligned}
\tag{C.17}
$$

$$
\times e^{-\sum_{l=j+1}^{M-1}(\Delta_R^0 b_-^{l,M})(q^{-k-l}z)+\sum_{l=M+2}^{M+N}(\Delta_L^0 b_-^{M,l})(q^{-k-2M-1+l}z)}:\quad (1\le j\le M-1),
$$

$$
\begin{aligned}
&\frac{1}{(q-q^{-1})z}(F_{M,M}^{2,+}(z)-F_{M,M}^{2,-}(z))\\
&=:{}_{k+g}\partial_z\Bigg(e^{\left(\frac{1}{k+g}a^M\right)(z;\frac{k+g}{2})-\left(\frac{k+M-1}{k+g}b^{M,M+1}\right)(z;0)}\\
&\quad\times e^{\sum_{l=M+2}^{M+N}\left(\left(\frac{1}{k+g}b^{M+1,l}\right)(z;k+2M+1-l)+\left(\frac{1}{k+g}b^{M,l}\right)(z;k+2M+1-l)\right)}\\
&\quad\times e^{-\sum_{l=M+1}^{M+N}\left(\left(\frac{1}{k+g}b^{M+1,l}\right)(z;k+2M-l)-\left(\frac{1}{k+g}b^{M,l}\right)(z;k+M+1-l)\right)}\Bigg)
\end{aligned}
\tag{C.18}
$$

$$
\times e^{-\left(\frac{1}{k+g}a^M\right)(z;-\frac{k+g}{2})-\left(\frac{g-M+1}{k+g}b^{M,M+1}\right)(z;0)-\sum_{l=M+2}^{M+N}\left(\left(\frac{1}{k+g}b^{M+1,l}\right)(z;2M+1-l-g)+\left(\frac{1}{k+g}b^{M,l}\right)(z;2M+1-l-g)\right)}:,
$$

$$
\begin{aligned}
&\frac{-1}{(q-q^{-1})z}(F_{M,j}^{3,+}(z)-F_{M,j}^{3,-}(z))\\
&=:{}_1\partial_z\left(e^{-c^{M+1,j}(q^{k+2M+1-j}z)}\right)e^{-b^{M+1,j}(q^{k+2M+1-j}z;1)+a_+^M(q^{\frac{k+g}{2}}z)-b^{M,j}(q^{k+2M-j}z)}\\
&\quad\times e^{b_+^{M+1,j}(q^{k+2M+1-j}z)+\sum_{l=j+1}^{M+N}(\Delta_L^0 b_+^{M,l})(q^{k+2M+1-l}z)}:\quad (M+2\le j\le M+N).
\end{aligned}
\tag{C.19}
$$

- For $M+1\le i\le M+N-1$ we have

$$
\begin{aligned}
&\frac{1}{(q-q^{-1})z}(F_{i,j}^{1,+}(z)-F_{i,j}^{1,-}(z))\\
&=:{}_1\partial_z\left(e^{-c^{j,i}(q^{-k-2M+j}z)}\right)e^{-b^{j,i}(q^{-k-2M+j}z;-1)+a_-^i(q^{-\frac{k+g}{2}}z)+(b+c)^{j,i+1}(q^{-k-2M+j}z)}
\end{aligned}
\tag{C.20}
$$

$$
\times e^{-\sum_{l=j+1}^{i}(\Delta_R^- b_-^{l,i})(q^{-k-2M+l}z)+\sum_{l=i+1}^{M+N}(\Delta_L^- b_-^{i,l})(q^{-k-2M+l}z)}:\ (M+1\le j\le i-1),
$$

$$
\begin{aligned}
&\frac{1}{(q-q^{-1})z}(F_{i,i}^{2,+}(z)-F_{i,i}^{2,-}(z))\\
&=:{}_{k+g}\partial_z\left(e^{\left(\frac{1}{k+g}a^i\right)(z;\frac{k+g}{2})+\left(\frac{k+2M-i}{k+g}(b+c)^{i,i+1}\right)(z;0)+\sum_{l=i+1}^{M+N}\left(\left(\frac{1}{k+g}b^{i+1,l}\right)(z;k+2M-l+1)-\left(\frac{1}{k+g}b^{i,l}\right)(z;k+2M-l)\right)}\right)\\
&\quad\times e^{-\left(\frac{1}{k+g}a^i\right)(z;-\frac{k+g}{2})+\left(\frac{g-2M+i}{k+g}(b+c)^{i,i+1}\right)(z;0)-\sum_{l=i+1}^{M+N}\left(\left(\frac{1}{k+g}b^{i+1,l}\right)(z;2M-l-g+1)-\left(\frac{1}{k+g}b^{i,l}\right)(z;2M-l-g)\right)},
\end{aligned}
\tag{C.21}
$$

$$
\begin{aligned}
&\frac{-1}{(q-q^{-1})z}(F_{i,j}^{3,+}(z)-F_{i,j}^{3,-}(z))\\
&=:{}_1\partial_z\left(e^{-c^{i+1,j}(q^{k+2M-j+1}z)}\right)e^{-b^{i+1,j}(q^{k+2M-j+1}z;1)+a_+^i(q^{\frac{k+g}{2}}z)+(b+c)^{i,j}(q^{k+2M-j+1}z)}\\
&\quad\times e^{\sum_{l=j}^{M+N}(\Delta_L^+ b_+^{i,l})(q^{k+2M-l}z)}:\quad (i+2\le j\le M+N).
\end{aligned}
\tag{C.22}
$$

• For $1 \le i \le M-1$ we have

$$\frac{-1}{(q-q^{-1})z}(\tilde{S}^+_{i,j}(z)-\tilde{S}^-_{i,j}(z))$$
$$=:{}_1\partial_z\left(e^{-c^{i,j}(q^{M-N-j}z)}\right)e^{-b^{i,j}(q^{M-N-j}z;1)+(b+c)^{i+1,j}(q^{M-N-j}z)}$$
$$\times e^{\sum_{l=j+1}^{M}(\Delta_L^+ b_-^{i,l})(q^{M-N-l}z)+\sum_{l=M+1}^{M+N}(\Delta_L^+ b_-^{i,l})(q^{-M-N+l-1}z)}:$$
$$(i+1\le j\le M). \tag{C.23}$$

• For $M+1 \le i \le M+N-1$ we have

$$\frac{1}{(q-q^{-1})z}(\tilde{S}^+_{i,j}(z)-\tilde{S}^-_{i,j}(z))$$
$$=:{}_1\partial_z\left(e^{-c^{i,j}(q^{-M-N+j}z)}\right)e^{-b^{i,j}(q^{-M-N+j}z;-1)+(b+c)^{i+1,j}(q^{-M-N+j}z)}$$
$$\times e^{-\sum_{l=j+1}^{M+N}(\Delta_L^- b_-^{i,l})(q^{-M-N+l}z)}:$$
$$(i+1\le j\le M+N). \tag{C.24}$$

In the limit $q \to 1$ ${}_\alpha\partial_z$, $\left(\frac{L}{M}a^i\right)(z;\alpha)$ and $\left(\frac{L}{M}b^{i,j}\right)(z;\alpha)$ become $\alpha\partial_z$, $\frac{L}{M}a^i(z)$ and $\frac{L}{M}b^{i,j}(z)$ respectively. Because the operators $a^i_\pm(z)\, b^{i,j}_\pm(z)$, $(\Delta_L^\epsilon b^{i,j}_\pm)(z)$ and $(\Delta_R^\epsilon b^{i,j}_\pm)(z)$ disappear, our bosonization becomes simpler in the limit $q \to 1$.

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