Boundary state of a $U_q(\widehat{gl}(N|N))$ analog of the half-infinite $t-J$ model

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Takeo Kojima

Department of Mathematics and Physics, Faculty of Engineering, Yamagata University, Jonan 4-3-16, Yonezawa 992-8510, JAPAN

kojima@yz.yamagata-u.ac.jp

Abstract

A $U_q(\widehat{gl}(N|N))$-analog of the half-infinite $t-J$ model with a boundary is studied by using the vertex operator approach. We find an explicit bosonic formula of the boundary state in the integrable highest-weight module over the quantum superalgebra $U_q(\widehat{gl}(N|N))$.

1 Introduction

There have been many developments in exactly solvable models. Various methods were invented to solve models. The vertex operator approach (VOA) [1, 2] provides a powerful method to construct correlation functions of exactly solvable models. The VOA arose from an interplay between the corner transfer matrix method [3] and representation theory of infinite dimensional algebras, and it is completely different from the quantum inverse scattering method (QISM) [4]. In the QISM calculations are performed in two steps. First we perform calculations in a box of finite size, and then we take the infinite volume limit. In the VOA we perform calculations directly in the infinite volume limit, using representation theory of infinite dimensional algebras. Sometimes calculations in the infinite volume limit by the VOA are simpler than those in a box of finite size by the QISM. The VOA has chance of bringing something new. In this

article we consider systems defined on a half-infinite lattice with an open boundary. Exactly solvable lattice models with open boundary are defined by using the Yang-Baxter equation and the boundary Yang-Baxter equation [5, 6]

$$K_2(z_2)R_{2,1}(z_1z_2)K_1(z_1)R_{1,2}(z_1/z_2) = R_{2,1}(z_1/z_2)K_1(z_1)R_{1,2}(z_1z_2)K_2(z_2).$$

The VOA to half-infinite lattice models has been studied for the quantum affine algebras $U_q(\widehat{sl}(N))$, $U_q(A_2^{(2)})$, $U_q(\widehat{sl}(M|N))$ $(M \neq N)$ [2, 7, 8, 9, 10] and the elliptic deformed algebra $U_{q,p}(\widehat{sl}(N))$ [11, 12]. For $U_q(\widehat{sl}(N))$ and $U_{q,p}(\widehat{sl}(N))$ we have constructed bosonization of the boundary state ${}_B\langle i|$ and the dual boundary state $|i\rangle_B$. Using these bosonizations we calculated the correlation functions. The VOA program has been successfully completed. It is extraordinary complicated to calculate them by the QISM. For $U_q(A_2^{(2)})$ and $U_{q,p}(\widehat{sl}(M|N))$ we have constructed a bosonization of the boundary state ${}_B\langle i|$. A bosonization of the dual boundary state $|i\rangle_B$ has not yet been completed. It is necessary to obtain it for calculation of the correlation functions. In this paper we study a $U_q(\widehat{gl}(N|N))$-analog of the half-infinite $t-J$ model with a boundary. We describe the significance of the special case $M = N$. The physical significance is that the space of states of $M = N$ is completely different from those of $M \neq N$. The energy level degenerates violently for $M = N$. We study the algebra $U_q(\widehat{gl}(N|N))$ extended from $U_q(\widehat{sl}(N|N))$ because of this energy level degeneration. The mathematical significance is that the quantum superalgebra $U_q(\widehat{gl}(N|N))$ is the only untwisted superalgebra which has a nonstandard system where all simple roots are odd or fermionic. The boundary condition of our model is given by the most general diagonal solution of the boundary Yang-Baxter equation

$$\overline{K}(z) = \mathrm{diag}\left(z^2, \cdots, z^2, \frac{1-rz}{1-r/z}, \cdots, \frac{1-rz}{1-r/z}, 1, \cdots, 1\right).$$

In the VOA the transfer matrix $T_B(z)$ of solvable lattice model with a boundary is written by vertex operators $\Phi_j(z)$, $\Phi_j^*(z)$, and a solution of the boundary Yang-Baxter equation as follows.

$$T_B(z) = g\sum_{j,k=1}^{2N} \Phi_j^*(z^{-1})K(z)_j^k\Phi_k(z)(-1)^{[v_j]},$$

where $K(z) = \frac{\varphi(z)}{\varphi(z^{-1})}\overline{K}(z)$ and $\varphi(z)$ is given in (2.13). In this paper we give a bosonization of the boundary state ${}_B\langle i|$ that satisfies

$${}_B\langle i|T_B(z) = {}_B\langle i|.$$

The boundary state ${}_B\langle i|$ is realized by acting exponentially with a bosonic operator G (see (5.9)) on the highest-weight vector $\langle\Lambda_i|$ in the integrable highest-weight module $V^*(\Lambda_i)$

$${}_B\langle i| = \langle\Lambda_i|e^G \cdot Pr,$$

where Pr is a projection operator (see (4.36)). If we will find a bosonization of the dual boundary state $|i\rangle_B$ satisfying $T_B(z)|i\rangle_B = |i\rangle_B$ as the next step, we can calculate the correlation functions. A bosonization of the dual boundary state $|i\rangle_B$ for $M = N$ is more complicated than those for $M \neq N$,

because of the energy level degeneration for $M = N$. In this paper we report a bosonization of the boundary state ${}_B\langle i|$, based on the new relations in Propositions 6.4 and 6.5, as an important step of the VOA.

Over the last three decades, quantum supersymmetry has been developed in the literature. For the quantum super groups $GL_q(m|n)$, $U_q(\widehat{sl}(M|N))$, $U_q(\widehat{gl}(M|N))$, we refer to the literature [13, 14, 15, 16, 17, 30, 31, 32]. Supersymmetric models are discussed in [18, 19, 20, 21, 22, 23], and those with boundary are studied in [24, 25, 26, 27].

The text is organized as follows. In Section 2 we introduce the R-matrix and the boundary K-matrix. We introduce a $U_q(\widehat{gl}(N|N))$-analog of the half-infinite $t-J$ model with a boundary. In Section 3 we formulate the VOA to our problem. In Section 4 we review a bosonization of the quantum superalgebra $U_q(\widehat{gl}(N|N))$ and an integral representation of the vertex operator. In Section 5 we give a bosonization of the boundary state in the integrable highest-weight module $V^*(\Lambda_0)$. In Section 6 we give a proof of the formula for the boundary state by using an integral representation of the vertex operator. In Section 7 we discuss generalizations of the present paper. In Appendix A we review the quantum superalgebra $U_q(\widehat{gl}(N|N))$. In Appendix B we give a bosonization of the boundary state in the integrable highest-weight module $V^*(\Lambda_{2N-1})$. In Appendix C we summarize normal ordering rules of fundamental bosonic operators.

2 A $U_q(\widehat{gl}(N|N))$-analog of the half-infinite $t-J$ model

In this Section we introduce a $U_q(\widehat{gl}(N|N))$-analog of the half-infinite $t-J$ model with a boundary.

2.1 The R-matrix and the K-matrix

In this Section we introduce the R-matrix and the boundary K-matrix. Let $N \in \mathbf{N}_{\neq 0}$ and $q \in \mathbf{C}$ such that $0 < |q| < 1$. Let $L, M, R \in \mathbf{N}$ such that $L + M + R = 2N$. We set the vector space $V = \oplus_{j=1}^{2N} \mathbf{C} v_j$. The $\mathbf{Z}_2$-grading of the basis $\{v_j\}_{1 \leq j \leq 2N}$ is given to be $[v_j] = \begin{cases} 0 & (j = \text{odd}) \\ 1 & (j = \text{even}) \end{cases}$. The $\mathbf{Z}_2$-grading of matrix $A = (A_{j,k})_{1 \leq j,k \leq 2N} \in \mathrm{End}(V)$ is defined by $[A] = [v_j] + [v_k]$ $(mod.2)$ if RHS of the equation does not depend on j and k such that $A_{j,k} \neq 0$. We define action of operator $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ where $A_j \in \mathrm{End}(V)$ have $\mathbf{Z}_2$-grading. We set

$$\begin{aligned} & A_1 \otimes A_2 \otimes \cdots \otimes A_n \cdot v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_n} \\ = \quad & \exp\left(\pi\sqrt{-1} \sum_{k=1}^{n} [A_k] \sum_{s=1}^{k-1} [v_{j_s}]\right) A_1 v_{j_1} \otimes A_2 v_{j_2} \otimes \cdots \otimes A_n v_{j_n}. \end{aligned} \tag{2.1}$$

We have the following multiplication rule.

$$(A_1 \otimes A_2)(B_1 \otimes B_2) = (-1)^{[A_2][B_1]} (A_1 B_1 \otimes A_2 B_2). \tag{2.2}$$

We introduce super-trace "str" and super-transpose "st" of $A \in \mathrm{End}(V)$ by

$$\mathrm{str}_V(A) = \sum_{j=1}^{2N} (-1)^{[v_j]} A_{j,j}, \quad (A^{st})_{j,k} = A_{k,j}(-1)^{[v_j]([v_j]+[v_k])}. \tag{2.3}$$

Definition 2.1 *[28] Let $R(z) \in \mathrm{End}(V \otimes V)$ be the R-matrix of $U_q(\widehat{gl}(N|N))$ defined by*

$$R(z) = r(z)\overline{R}(z), \quad \overline{R}(z) v_i \otimes v_j = \sum_{k,l=1}^{2N} v_k \otimes v_l \overline{R}^{i,j}_{k,l}(z), \tag{2.4}$$

where

$$\overline{R}^{2j-1,2j-1}_{2j-1,2j-1}(z) = 1, \quad \overline{R}^{2j,2j}_{2j,2j}(z) = \frac{z-q^2}{q^2 z - 1} \quad (1 \le j \le N), \tag{2.5}$$

$$\overline{R}^{i,j}_{i,j}(z) = \frac{q(z-1)}{q^2 z - 1} \quad (1 \le i \ne j \le 2N), \tag{2.6}$$

$$\overline{R}^{j,i}_{i,j}(z) = \frac{q^2-1}{q^2 z - 1}(-1)^{[v_i][v_j]}, \quad \overline{R}^{i,j}_{j,i}(z) = \frac{(q^2-1)z}{q^2 z - 1}(-1)^{[v_i][v_j]} \quad (1 \le i < j \le 2N), \tag{2.7}$$

$$\overline{R}^{i,j}_{k,l}(z) = 0 \quad \text{(otherwise)}. \tag{2.8}$$

The scalar function $r(z)$ in (2.4) is

$$r(z) = -z^{\frac{1-2N}{N}} \left(\frac{1-q/z}{1-qz} \right)^{\frac{N-1}{N}}. \tag{2.9}$$

The R-matrix $R(z)$ of this paper is obtained from the R-matrix $R^{PS}(z)$ of the Part-Shultz model [28] as follows : $R^{j_1 j_2}_{k_1 k_2}(z) = (-1)^{[v_{k_1}][v_{k_2}]}(R^{PS})^{j_1 j_2}_{k_1 k_2}(z)$. The R-matrix $R(z)$ satisfies the graded Yang-Baxter equation in $V \otimes V \otimes V$.

$$R_{1,2}(z_1/z_2)R_{1,3}(z_1/z_3)R_{2,3}(z_2/z_3) = R_{2,3}(z_2/z_3)R_{1,3}(z_1/z_3)R_{1,2}(z_1/z_2). \tag{2.10}$$

The R-matrix $R(z)$ satisfies (i) initial condition $R(1) = P$ with P being the graded permutation operator $P^{k,l}_{i,j} = \delta_{i,k}\delta_{j,l}(-1)^{[v_i][v_j]}$, (ii) unitary condition $R_{1,2}(z)R_{2,1}(z^{-1}) = 1$, and (iii) crossing symmetry $(R(z)^{-1})^{st_1} R(z)^{st_1} = \frac{(1-z)^2}{(1-z/q^2)(1-q^2 z)}$.

Definition 2.2 *[10] Let $K(z) \in \mathrm{End}(V)$ be the boundary K-matrix of $U_q(\widehat{gl}(N|N))$ defined by*

$$K(z) = \frac{\varphi(z)}{\varphi(z^{-1})}\overline{K}(z), \quad \overline{K}(z) v_i = \sum_{j=1}^{2N} v_j \overline{K}^i_j(z), \tag{2.11}$$

where diagonal matrix $\overline{K}(z)$ is defined by

$$\overline{K}(z) = \mathrm{diag}\left(\underbrace{z^2, \cdots, z^2}_{L}, \underbrace{\frac{1-rz}{1-r/z}, \cdots, \frac{1-rz}{1-r/z}}_{M}, \underbrace{1, \cdots, 1}_{R} \right) \qquad (L+M+R = 2N). \tag{2.12}$$

The scalar function $\varphi(z)$ in (2.11) is

$$\varphi(z) = \begin{cases} \varphi^{[1]}(z) & (L = M = 0, R > 0) \\ \varphi^{[2]}(z) & (L = 0, M, R > 0) \\ \varphi^{[3]}(z) & (L, M, R > 0) \end{cases}, \tag{2.13}$$

where $L+M+R=2N$. Here we have set

$$\varphi^{[1]}(z)=(1-qz^2)^{\frac{1-N}{2N}}, \tag{2.14}$$

$$\varphi^{[2]}(z)=\varphi^{[1]}(z)\times\begin{cases}(1-rz/q)^{\frac{1-2N}{2N}} & (M=\text{odd})\\ 1 & (M=\text{even})\end{cases}, \tag{2.15}$$

$$\varphi^{[3]}(z)=\varphi^{[1]}(z)\times\begin{cases}(1-z/rq)^{\frac{1-2N}{2N}} & (L=\text{odd},M=\text{odd})\\ \{(1-z/rq)(1-rqz)\}^{\frac{1-2N}{2N}} & (L=\text{odd},M=\text{even})\\ (1-rz/q)^{\frac{1-2N}{2N}} & (L=\text{even},M=\text{odd})\\ 1 & (L=\text{even},M=\text{even})\end{cases}. \tag{2.16}$$

The boundary K-matrix $K(z)$ satisfies the graded boundary Yang-Baxter equation in $V\otimes V$

$$K_2(z_2)R_{2,1}(z_1z_2)K_1(z_1)R_{1,2}(z_1/z_2)=R_{2,1}(z_1/z_2)K_1(z_1)R_{1,2}(z_1z_2)K_2(z_2). \tag{2.17}$$

The boundary K-matrix $K(z)$ satisfies (i) initial condition $K(1)=1$, (ii) boundary unitary condition $K(z)K(z^{-1})=1$, and (iii) boundary crossing symmetry $K^*(z)K^*(z^{-1})=\dfrac{(1-rq^2z)(1-z/rq^2)}{(1-rz)(1-z/r)}$ with $K^{*i}_j(z)=\sum_{k,l=1}^{2N}R^{j,i}_{l,k}(z^2)K^l_k(z)(-1)^{[v_k]+[v_i][v_j]}$. The boundary K-matrix $K(z)$ given in (2.11) is general diagonal solution of the boundary Yang-Baxter equation.

2.2 A $U_q(\widehat{gl}(N|N))$-analog of the half-infinite $t-J$ model

We introduce the monodromy matrix $\mathcal{T}(z)$ by

$$\mathcal{T}(z)=R_{0,1}(z)R_{0,2}(z)\cdots R_{0,n}(z)\in\text{End}(V_n\otimes\cdots\otimes V_2\otimes V_1\otimes V_0), \tag{2.18}$$

where V_j are copies of V and $n\in\mathbf{N}$. We introduce the transfer matrix $T^{fin}_B(z)$ by

$$T^{fin}_B(z)=\text{str}_{V_0}(K(z^{-1})^{st}\mathcal{T}(z^{-1})^{-1}K(z)\mathcal{T}(z))\in\text{End}(V_n\otimes\cdots\otimes V_2\otimes V_1). \tag{2.19}$$

The Hamiltonian of a $U_q(\widehat{gl}(N|N))$-analog of the finite $t-J$ model is given by

$$H^{fin}_B=\frac{d}{dz}T^{fin}_B(z)|_{z=1}=\sum_{j=1}^{n-1}h_{j,j+1}+\frac{1}{2}\frac{d}{dz}K_1(z)|_{z=1}+\frac{\text{str}_{V_0}(K_0(1)^{st}h_{0,n})}{\text{str}_{V_0}(K_0(1)^{st})}, \tag{2.20}$$

where $h_{j,j+1}=P_{j,j+1}\frac{d}{dz}R_{j,j+1}(z)|_{z=1}$. We set the Hamiltonian H_B by taking the thermodynamic limit of H^{fin}_B in (2.20).

$$H_B=\lim_{n\to\infty}H^{fin}_B=\sum_{j=1}^{\infty}h_{j,j+1}+\frac{1}{2}\frac{d}{dz}K_1(z)|_{z=1}. \tag{2.21}$$

The Hamiltonian H_B acts on the half-infinite tensor product space $\cdots\otimes V_3\otimes V_2\otimes V_1$. We study a $U_q(\widehat{gl}(N|N))$-analog of the half-infinite $t-J$ model defined by the Hamiltonian H_B in (2.21).

3 The vertex operator approach

In this Section we give the formulation of the VOA to our model.

3.1 The transfer matrix

We diagonalize the Hamiltonian H_B in (2.21). It is convenient to study the transfer matrix

$$\widetilde{T}_B(z) = \lim_{n\to\infty} T_B^{fin}(z), \tag{3.1}$$

including spectral parameter z. The transfer matrix $\widetilde{T}_B(z)$ is given by infinite product of the R-matrix. Hence it isn't free from divergence. Later we would like to give mathematical formulation of our problem that is free from divergence. Following the strategy summarized in [1, 2, 29], we introduce the vertex operator $\widetilde{\Phi}_j(z)$ and the dual vertex operator $\widetilde{\Phi}_j^*(z)$, which act on half-infinite tensor product space $\cdots \otimes V_3 \otimes V_2 \otimes V_1$, as limit of the monodromy matrix $\mathcal{T}(z)$. The matrix elements of the vertex operator $\widetilde{\Phi}_j(z)$ and the dual vertex operator $\widetilde{\Phi}_j^*(z)$ $(j = 1, 2, \cdots, 2N)$ are given by

$$\left(\widetilde{\Phi}_j(z)\right)_{\cdots j_n,\cdots,j_2,j_1}^{\cdots k_n,\cdots,k_2,k_1} = \lim_{n\to\infty} \left(\mathcal{T}(z)\right)_{j_n,\cdots,j_2,j_1,j}^{k_n,\cdots,k_2,k_1,k}, \tag{3.2}$$

$$\left(\widetilde{\Phi}_j^*(z)\right)_{\cdots k_n,\cdots,k_2,k_1}^{\cdots j_n,\cdots,j_2,j_1} = \lim_{n\to\infty} \left(\mathcal{T}(z)^{-1}\right)_{k_n,\cdots,k_2,k_1,k}^{j_n,\cdots,j_2,j_1,j}, \tag{3.3}$$

where there exists k such that $\{k, k_1, k_2, \cdots, k_n\} = \{j, j_1, j_2, \cdots, j_n\}$ for $n >> 0$. Other matrix elements are given by $\left(\widetilde{\Phi}_j(z)\right)_{\cdots j_n,\cdots,j_2,j_1}^{\cdots k_n,\cdots,k_2,k_1} = \left(\widetilde{\Phi}_j^*(z)\right)_{\cdots k_n,\cdots,k_2,k_1}^{\cdots j_n,\cdots,j_2,j_1} = 0$. We expect that the vertex operator $\widetilde{\Phi}_j(z)$ and the dual vertex operator $\widetilde{\Phi}_j^*(z)$ give rise to well-defined operators. From heuristic argument by the Yang-Baxter equation [1, 2, 29], the vertex operator $\widetilde{\Phi}_j(z)$ is expected to satisfy the following commutation relation.

$$\widetilde{\Phi}_{j_2}(z_2)\widetilde{\Phi}_{j_1}(z_1) = \sum_{k_1,k_2=1}^{2N} R_{j_1,j_2}^{k_1,k_2}(z_1/z_2)\widetilde{\Phi}_{k_1}(z_1)\widetilde{\Phi}_{k_2}(z_2)(-1)^{[v_{j_1}][v_{j_2}]}. \tag{3.4}$$

Here $R_{j_1,j_2}^{k_1,k_2}(z)$ is the matrix element of the R-matrix given in (2.4). The transfer matrix $\widetilde{T}_B(z)$ is written by using the vertex operators $\widetilde{\Phi}_j(z)$ and $\widetilde{\Phi}_j^*(z)$ as follows.

$$\widetilde{T}_B(z) = \sum_{j=1}^{2N} \widetilde{\Phi}_j^*(z^{-1})K_j^j(z)\widetilde{\Phi}_j(z)(-1)^{[v_j]}. \tag{3.5}$$

The Hamiltonian H_B is given by $H_B = \frac{d}{dz}\widetilde{T}_B(z)|_{z=1}$. It is better to diagonalize $\widetilde{T}_B(z)$ instead of H_B. In order to diagonalize $\widetilde{T}_B(z)$, we follow the strategy called the VOA.

3.2 The vertex operator approach

Our useful tool is the vertex operator associated with the quantum superalgebra $U_q(\widehat{gl}(N|N))$. We introduce the evaluation representation V_z of the basic representation $V = \oplus_{j=1}^{2N}\mathbf{C}v_j$ for $U_q(\widehat{gl}(N|N))$. In what follows we use standard notation of q-integer

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{3.6}$$

Let $E_{i,j}$ be $2N \times 2N$ matrix satisfying $E_{i,j}v_k = \delta_{j,k}v_i$. The evaluation module V_z is given by the Drinfeld generators as follows.

$$H_m^i = (-1)^{i+1}\frac{[m]_q}{m}q^{(-1)^i m}(q^{[v_{i+1}]}z)^m(E_{i,i} + E_{i+1,i+1}) \quad (1 \le i \le 2N-1),$$

$$
\begin{aligned}
&H_m^{2N} = \frac{[2m]_q}{m} z^m \left\{ -q^m \sum_{l=1}^{N} E_{2l,2l} + \sum_{l=1}^{N} (1-N+(l-1)(1-q^m))(E_{2l-1,2l-1}+E_{2l,2l}) \right\}, \\
&H_0^i = (-1)^{i+1}(E_{i,i}+E_{i+1,i+1}) \quad (1 \le i \le 2N-1), \quad H_0^{2N} = \sum_{k=1}^{2N} (-1)^{k+1} E_{k,k}, \\
&X_m^{+,i} = (q^{[v_{i+1}]}z)^m E_{i,i+1}, \quad X_m^{-,i} = (-1)^{i+1}(q^{[v_{i+1}]}z)^m E_{i+1,i} \quad (1 \le i \le 2N-1).
\end{aligned} \tag{3.7}
$$

Denote by $V^{*S} = \oplus_{j=1}^{2N} \mathbf{C} v_j^*$ the left dual module of V defined by

$$
\langle a \cdot v^* | v \rangle = (-1)^{[a][v^*]} \langle v^* | S(a) \cdot v \rangle \quad (a \in U_q(\widehat{gl}(N|N)), v^* \in V^{*S}, v \in V). \tag{3.8}
$$

Here S is the antipode given in (A.7) and the $\mathbf{Z}_2$-grading of the basis is chosen to be $[v_j^*] = \frac{(-1)^j+1}{2}$. Namely the representation on V^{*S} are given by $\pi_{V^{*S}}(a) = \pi_V(S(a))^{st}$ where π_V denotes action of the module V. The evaluation module V_z^{*S} is given by the Drinfeld generators as follows.

$$
\begin{aligned}
&H_m^i = (-1)^i \frac{[m]_q}{m} q^{(-1)^{(i+1)}m} (q^{-[v_{i+1}]}z)^m (E_{i,i}+E_{i+1,i+1}) \quad (1 \le i \le 2N-1), \\
&H_m^{2N} = -\frac{[2m]_q}{m} z^m \left\{ -q^{-m} \sum_{l=1}^{N} E_{2l,2l} + \sum_{l=1}^{N} (1-N+(l-1)(1-q^{-m}))(E_{2l-1,2l-1}+E_{2l,2l}) \right\}, \\
&H_0^i = (-1)^i (E_{i,i}+E_{i+1,i+1}) \quad (1 \le i \le 2N-1), \quad H_0^{2N} = \sum_{k=1}^{2N} (-1)^k E_{k,k}, \\
&X_m^{+,i} = -(-1)^i q^{(-1)^i} (q^{-[v_{i+1}]}z)^m E_{i+1,i}, \quad X_m^{-,i} = -q^{(-1)^{i+1}} (q^{-[v_{i+1}]}z)^m E_{i,i+1} \quad (1 \le i \le 2N-1).
\end{aligned} \tag{3.9}
$$

Definition 3.1 *Let $V(\lambda)$ be the highest-weight $U_q(\widehat{gl}(N|N))$-module with the highest-weight λ. We define the vertex operator $\Phi(z)$ and the dual vertex operator $\Phi^*(z)$ as the intertwiner of $U_q(\widehat{gl}(N|N))$-module as follows.*

$$
\Phi(z) : V(\lambda) \longrightarrow V(\mu) \otimes V_z, \qquad \Phi(z) \cdot x = \Delta(x) \cdot \Phi(z), \tag{3.10}
$$

$$
\Phi^*(z) : V(\lambda) \longrightarrow V(\mu) \otimes V_z^{*S}, \qquad \Phi^*(z) \cdot x = \Delta(x) \cdot \Phi^*(z), . \tag{3.11}
$$

We expand the vertex operators as follows.

$$
\Phi(z) = \sum_{j=1}^{2N} \Phi_j(z) \otimes v_j, \quad \Phi^*(z) = \sum_{j=1}^{2N} \Phi_j^*(z) \otimes v_j^*. \tag{3.12}
$$

We set the $\mathbf{Z}_2$-grading of the vertex operators by $[\Phi(z)] = [\Phi^*(z)] = 0$. Hence we have the $\mathbf{Z}_2$-grading $[\Phi_j(z)] = [\Phi_j^*(z)] = [v_j] = [v_j^*]$ $(j = 1, 2, \cdots, 2N)$. The vertex operators are expected to satisfy the following relations [32].

$$
\Phi_{j_2}(z_2)\Phi_{j_1}(z_1) = \sum_{k_1,k_2=1}^{2N} R_{j_1,j_2}^{k_1,k_2}(z_1/z_2)\Phi_{k_1}(z_1)\Phi_{k_2}(z_2)(-1)^{[v_{j_1}][v_{j_2}]}, \tag{3.13}
$$

$$
g\Phi_i(z)\Phi_j^*(z) = (-1)^{[v_i]}\delta_{i,j}, \quad g\sum_{j=1}^{2N}(-1)^{[v_j]}\Phi_j^*(z)\Phi_j(z) = 1. \tag{3.14}
$$

Here $R_{j_1,j_2}^{k_1,k_2}(z)$ is matrix element of the R-matrix given in (2.4). Here g is a constant. We introduce the transfer matrix $T_B(z)$ as follows.

$$
T_B(z) = g \sum_{i,j=1}^{2N} \Phi_i^*(z^{-1}) K_i^j(z) \Phi_j(z) (-1)^{[v_j]}. \tag{3.15}
$$

Following the strategy proposed in [1, 2], we consider our problem upon the following identification.

$$T_B(z) = \widetilde{T}_B(z), \quad \Phi_j(z) = \widetilde{\Phi}_j(z), \quad \Phi_j^*(z) = \widetilde{\Phi}_j^*(z). \tag{3.16}$$

We call studies based on this identification "vertex operator approach". The point of using the vertex operators $\Phi_j(z), \Phi_j^*(z)$ is that they are well-defined objects and are free from divergence.

4 A bosonization of the vertex operator

In this Section we review a bosonization of the vertex operator associated with the quantum superalgebra $U_q(\widehat{gl}(N|N))$. This bosonization scheme is given by Yang and Zhang [32]. We give an integral representation of the dual vertex operator.

4.1 The quantum superalgebra $U_q(\widehat{gl}(N|N))$

We introduce bosons $\{a_m^i, c_m^j, Q_{a^i}, Q_{c^j} | i = 1, 2, \cdots, 2N, j = 1, 2, \cdots, N, m \in \mathbf{Z}\}$ satisfying the following commutation relations.

$$[a_m^i, a_n^j] = (-1)^{i+1}\delta_{i,j}\delta_{m+n,0}\frac{[m]_q^2}{m}, \quad [a_0^i, Q_{a^j}] = \delta_{i,j} \quad (1 \le i, j \le 2N), \tag{4.1}$$

$$[c_m^i, c_n^j] = \delta_{i,j}\delta_{m+n,0}\frac{[m]_q^2}{m}, \quad [c_0^i, Q_{c^j}] = \delta_{i,j} \quad (1 \le i, j \le N). \tag{4.2}$$

The remaining commutators vanish. We use the standard normal ordering :: given by

$$: a_m^i a_n^i := \begin{cases} a_m^i a_n^i & (m \le 0) \\ a_n^i a_m^i & (m > 0) \end{cases} \quad (1 \le i \le 2N), \tag{4.3}$$

$$: a_0^i Q_{a^i} :=: Q_{a_i} a_0^i := Q_{a^i} a_0^i \ (1 \le i \le 2N),$$

$$: c_m^j c_n^j := \begin{cases} c_m^j c_n^j & (m \le 0) \\ c_n^j c_m^j & (m > 0) \end{cases} \quad (1 \le j \le N), \tag{4.4}$$

$$: c_0^i Q_{c^i} :=: Q_{c_i} c_0^i := Q_{c^i} c_0^i \ (1 \le i \le N).$$

We set auxiliary bosonic operators B_m^j, Q_B^j $(j = 1, 2, \cdots, 2N)$ by

$$B_m^j = (-1)^{j+1}(a_m^j + a_m^{j+1}), \quad Q_{B^j} = Q_{a^j} - Q_{a^{j+1}} \quad (1 \le j \le 2N-1), \tag{4.5}$$

$$B_m^{2N} = \frac{q^m + q^{-m}}{2}\sum_{l=1}^{2N}(-1)^{l+1}a_m^l, \quad Q_{B^{2N}} = \sum_{l=1}^{2N} Q_{a^l}. \tag{4.6}$$

They satisfy the following commutation relations.

$$[B_m^i, B_n^j] = \frac{[A_{i,j}m]_q[m]_q}{m}\delta_{m+n,0}, \quad [B_0^i, Q_{B^j}] = A_{i,j}\delta_{m+n,0} \quad (1 \le i, j \le 2N). \tag{4.7}$$

Here $A_{i,j}$ is a matrix element of the Cartan matrix given in Appendix A. We set the notation

$$B^j(z;\kappa) = Q_{B^j} + B_0^j \log z - \sum_{m \ne 0}\frac{B_m^j}{[m]_q}q^{\kappa|m|}z^{-n} \quad (j = 1, 2, \cdots, 2N), \tag{4.8}$$

$$c^j(z) = Q_{c^j} + c_0^j \log z - \sum_{m\neq 0} \frac{c_m^j}{[m]_q} z^{-m} \quad (j = 1, 2, \cdots, N), \tag{4.9}$$

$$B_\pm^j(z) = \pm(q - q^{-1}) \sum_{m>0} B_{\pm m}^j z^{\mp m} \pm B_0^j \log q \quad (j = 1, 2, \cdots, 2N). \tag{4.10}$$

We set the auxiliary bosonic operators $B_m^{*j}, Q_{B^{*j}}$ $(j = 1, 2, \cdots, 2N)$ by

$$\begin{aligned}
B_m^{*1} &= \frac{1}{2N}\{(2N-1)a_m^1 - \sum_{l=2}^{2N} a_m^l\}, \\
B_m^{*j} &= \frac{1}{N}\{(N-1)\sum_{l=1}^{j} a_m^l - \sum_{l=j+1}^{2N} a_m^l\} \quad (1 \le j \le 2N-2), \\
B_m^{*2N-1} &= \frac{1}{2N}\{\sum_{l=1}^{2N-1} a_m^l - (2N-1)a_m^{2N}\}, \quad B_m^{*2N} = \frac{1}{2N}\sum_{l=1}^{2N} a_m^l,
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
Q_{B^{*1}} &= \frac{1}{2N}\{(2N-1)Q_{a^1} - \sum_{l=2}^{2N} Q_{a^l}\} \\
Q_{B^{*j}} &= \frac{1}{N}\{(N-1)\sum_{l=1}^{j} Q_{a^l} - \sum_{l=j+1}^{2N} Q_{a^l}\} \quad (1 \le j \le 2N-2), \\
Q_{B^{*2N-1}} &= \frac{1}{2N}\{\sum_{l=1}^{2N-1} Q_{a^l} - (2N-1)Q_{a^{2N}}\}, \quad Q_{B^{*2N}} = \frac{1}{2N}\sum_{l=1}^{2N} Q_{a^l}.
\end{aligned} \tag{4.12}$$

They satisfy the following commutation relations.

$$[B_m^{*i}, B_n^{*j}] = (\bar{A}^{-1})_{i,j} \frac{[m]_q^2}{m} \delta_{m+n,0} \quad (1 \le i, j \le 2N), \tag{4.13}$$

where $\bar{A} = (A_{i,j})_{1\le i,j\le 2N}$ is the Cartan matrix given in Appendix A. For instance, we have

$$\begin{aligned}
&[B_m^{*1}, B_n^{*1}] = \frac{N-1}{N}\frac{[m]_q^2}{m}\delta_{m+n,0}, \quad [B_m^{*2N}, B_n^{*2N}] = 0, \\
&[B_m^{*1}, B_n^{*2N}] = \frac{1}{2N}\frac{[m]_q^2}{m}\delta_{m+n,0}, \quad [B_m^{*2N-1}, B_n^{*2N-1}] = -\frac{N-1}{N}\frac{[m]_q^2}{m}\delta_{m+n,0}, \\
&[B_m^{*2N-1}, B_n^{*2N}] = \frac{1}{2N}\frac{[m]_q^2}{m}\delta_{m+n,0}.
\end{aligned} \tag{4.14}$$

We introduce the q-difference operator ∂_z given by $\partial_z f(z) = \frac{f(qz)-f(q^{-1}z)}{(q-q^{-1})z}$.

Theorem 4.1 *[30] The Drinfeld generators of $U_q(\widehat{gl}(N|N))$ at Level-1 are realized as follows.*

$$\begin{aligned}
&c = 1, \quad H_m^i = B_m^i \quad (m \in \mathbf{Z}, i = 1, 2, \cdots, 2N), \quad H_m^0 = c - \sum_{j=1}^{2N-1} B_m^j, \\
&X^{\pm,i}(z) =: e^{\pm B^i(z;-\frac{1}{2})} Y^{\pm,i}(z) F^{\pm,i} : \quad (i = 1, 2, \cdots, 2N-1),
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
d = -\sum_{m>0} \frac{m^2}{[m]_q^2} &\left\{ \sum_{i=1}^{2N-1} B_{-m}^i B_m^{*i} + \frac{2}{q^m + q^{-m}} B_{-m}^{2N} B_m^{*2N} + \sum_{j=1}^{N} c_{-m}^j c_m^j \right\} \\
&- \frac{1}{2}\left\{ \sum_{i=1}^{2N} B_0^i B_0^{*i} + \sum_{j=1}^{N} c_0^j (c_0^j + 1) \right\},
\end{aligned}$$

where we have set

$$
\begin{aligned}
&Y^{+,2j}(z) = Y^{-,2j-1}(z) =: \partial_z e^{c^j(z)} : \quad (1 \le j \le N),\\
&Y^{-,2j}(z) = -Y^{+,2j-1}(z) = - : e^{c^j(z)} : \quad (1 \le j \le N),\\
&F^{\pm,2j-1} = \prod_{l=1}^{j-1} e^{\pm\sqrt{-1}a_0^{2l-1}}, \quad F^{\pm,2j} = \prod_{l=1}^{j} e^{\mp\sqrt{-1}\pi a_0^{2l}} \quad (1 \le j \le N).
\end{aligned} \tag{4.16}
$$

We introduce the Fock module. The vacuum vector $|0\rangle \neq 0$ is defined by

$$
a_m^i|0\rangle = c_m^j|0\rangle = 0 \quad (1 \le i \le 2N, 1 \le j \le N, m \ge 0). \tag{4.17}
$$

For $\lambda^a = (\lambda_1^a, \lambda_2^a, \cdots, \lambda_{2N}^a) \in \mathbf{C}^{2N}$ and $\lambda^c = (\lambda_1^c, \lambda_2^c, \cdots, \lambda_N^c) \in \mathbf{C}^N$, we set

$$
|\lambda^a; \lambda^c\rangle = |\lambda_1^a, \cdots, \lambda_{2N}^a; \lambda_1^c, \cdots, \lambda_N^c\rangle = e^{\sum_{i=1}^{2N} \lambda_i Q_{a^i} + \sum_{j=1}^{N} \lambda_j^c Q_{c^j}}|0\rangle. \tag{4.18}
$$

Denote by $F_{\lambda_1^a,\cdots,\lambda_{2N}^a;\lambda_1^c,\cdots,\lambda_N^c}$ the Fock space generated by $\{a_{-m}^i, c_{-m}^j | m > 0, i = 1, 2, \cdots, 2N, j = 1, 2, \cdots, N\}$ over the vector $|\lambda^a; \lambda^c\rangle$. Action of the bosonization of $U_q(\widehat{gl}(N|N))$ on $F_{\lambda_1^a,\cdots,\lambda_{2N}^a;\lambda_1^c,\cdots,\lambda_N^c}$ is not closed. We introduce the space $\mathcal{F}_{\lambda^a;\lambda^c}$ by

$$
\mathcal{F}_{\lambda^a;\lambda^c} = \bigoplus_{j_1,j_2,\cdots,j_{2N-1}\in\mathbf{Z}} F_{\lambda_1^a+j_1,\lambda_2^a-j_1+j_2,\cdots,\lambda_{2N}^a-j_{2N-1};\lambda_1^c+j_1-j_2,\lambda_2^c+j_3-j_4,\cdots,\lambda_N^c+j_{2N-1}}. \tag{4.19}
$$

Action of the bosonization for $U_q(\widehat{gl}(N|N))$ on $\mathcal{F}_{\lambda^a;\lambda^c}$ is closed.

$$
U_q(\widehat{gl}(N|N))\mathcal{F}_{\lambda^a;\lambda^c} = \mathcal{F}_{\lambda^a;\lambda^c}. \tag{4.20}
$$

To obtain highest-weight vector, we impose the condition

$$
e_i|\lambda^a; \lambda^c\rangle = 0 \quad (i = 0, 1, 2, \cdots, 2N-1). \tag{4.21}
$$

We have the following necessary and sufficient condition.

$$
\begin{aligned}
&\lambda_1^a + \lambda_2^a + \lambda_1^c = 0, 1, \quad \lambda_{2j-1}^a + \lambda_{2j}^a + \lambda_j^c = 0 \quad (2 \le j \le N),\\
&\lambda_{2j}^a + \lambda_{2j+1}^a + \lambda_j^c = 0 \quad (1 \le j \le N-1).
\end{aligned} \tag{4.22}
$$

For $\lambda_1^a + \lambda_2^a + \lambda_1^c = 0$, we have $\lambda_{2j-1}^a = \lambda_1^a$ $(2 \le j \le N)$, $\lambda_j^c = -(\lambda_1^a + \lambda_{2j}^a)$ $(1 \le j \le N)$. For free parameter $\lambda_1^a = \beta_1, \lambda_{2j}^a = \beta_{2j} \in \mathbf{C}$ $(1 \le j \le N)$, we have the following highest-weight vector.

$$
|(1 - (\beta_1 + \beta_{2N}))\Lambda_0 + \sum_{j=1}^{N-1}(\beta_1 + \beta_{2j})(\Lambda_{2j-1} - \Lambda_{2j}) + (\beta_1 + \beta_{2N})\Lambda_{2N-1} + (N\beta_1 - \sum_{j=1}^{N}\beta_{2j})\Lambda_{2N}\rangle. \tag{4.23}
$$

As the special cases we have

$$
|\Lambda_0\rangle = |0, \cdots, 0; 0, \cdots, 0\rangle = |0\rangle, \tag{4.24}
$$

$$
|\Lambda_{2N-1}\rangle = \left|\frac{1}{2N}, -\frac{1}{2N}, \frac{1}{2N}, -\frac{1}{2N}, \cdots, \frac{1}{2N}, -\frac{1}{2N}\ ; 0, 0, \cdots, 0, 0, -1\right\rangle. \tag{4.25}
$$

For $\lambda_1^a+\lambda_2^a+\lambda_1^c=1$, we have $\lambda_{2j-1}^a=\lambda_1^a$ $(2\le j\le N)$, $\lambda_j^c=-(\lambda_1^a+\lambda_{2j}^a)+1$ $(1\le j\le N)$. For free parameter $\lambda_1^a=\beta_1,\lambda_{2j}^a=\beta_{2j}\in\mathbf{C}$ $(1\le j\le N)$, we have

$$\begin{aligned}&|(1-(\beta_1+\beta_{2N}))\Lambda_0+(\beta_1+\beta_2)\Lambda_1+(1-(\beta_1+\beta_2))\Lambda_{2N-1}\\&+\sum_{j=2}^{N-1}((\beta_1+\beta_{2j})-1)(\Lambda_{2j-1}-\Lambda_{2j})+(\beta_1+\beta_{2N})\Lambda_{2N-1}+(N\beta_1-\sum_{j=1}^{N}\beta_{2j}-N+1)\Lambda_{2N}\rangle.\end{aligned}\tag{4.26}$$

As the special cases we have

$$|\Lambda_1\rangle=\left|\frac{2N-1}{2N},\frac{1}{2N},-\frac{1}{2N},\frac{1}{2N}\cdots,-\frac{1}{2N},\frac{1}{2N};0,0,0,\cdots,0\right\rangle,\tag{4.27}$$

$$|\Lambda_2\rangle=\left|\frac{N-1}{N},\frac{1-N}{N},-\frac{1}{N},\frac{1}{N},\cdots,-\frac{1}{N},\frac{1}{N};1,0,0,\cdots,0\right\rangle.\tag{4.28}$$

We are interested in the *irreducible* highest-weight module. We have obtained bosonizations in the Fock space $\mathcal{F}_{\lambda^a;\lambda^c}$. The module $\mathcal{F}_{\lambda^a;\lambda^c}$ is not irreducible in general. To obtain an irreducible representation, we introduce a pair of fermionic operators $\eta(z),\xi(z)$.

$$\eta^i(z)=\sum_{n\in\mathbf{Z}}\eta_n^iz^{-n-1}=:e^{c^i(z)}:,\quad \xi^i(z)=\sum_{n\in\mathbf{Z}}\xi_n^iz^{-n}=:e^{-c^i(z)}:\quad (i=1,2,\cdots,N).\tag{4.29}$$

The Fourier components $\xi_m^j=\oint\frac{dw}{2\pi\sqrt{-1}}w^{m-1}\xi^j(w)$ and $\eta_m^j=\oint\frac{dw}{2\pi\sqrt{-1}}w^{m-1}\eta^j(w)$ are well-defined on the Fock space for $\lambda_c^j\in\mathbf{Z}$. Hence we assume $\lambda_1^c,\lambda_2^c,\cdots,\lambda_N^c\in\mathbf{Z}$ in what follows. They satisfy anti-commutation relations

$$\xi_m^i\xi_n^i+\xi_n^i\xi_m^i=\eta_m^i\eta_n^i+\eta_n^i\eta_m^i=0,\quad \xi_m^i\eta_n^i+\eta_n^i\xi_m^i=\delta_{m+n,0},\quad (i=1,2,\cdots,N).\tag{4.30}$$

The products $\eta_0^j\xi_0^j$ and $\xi_0^j\eta_0^j$ are orthogonal projection operators.

$$\eta_0^j\xi_0^j+\xi_0^j\eta_0^j=1,\quad \eta_0^j\xi_0^j\cdot\xi_0^j\eta_0^j=\xi_0^j\eta_0^j\cdot\eta_0^j\xi_0^j=0\quad (j=1,2,\cdots,N).\tag{4.31}$$

Hence we have direct sum decomposition

$$\mathcal{F}_{\lambda^a;\lambda^c}=\eta_0^j\xi_0^j\cdot\mathcal{F}_{\lambda^a;\lambda^c}\oplus\xi_0^j\eta_0^j\cdot\mathcal{F}_{\lambda^a;\lambda^c}\quad (j=1,2,\cdots,N).\tag{4.32}$$

They commute with each other.

$$[\xi_m^i,\xi_n^j]=[\eta_m^i,\eta_n^j]=[\xi_m^i,\eta_n^j]=0\quad (1\le i\ne j\le N).\tag{4.33}$$

We note that

$$\begin{aligned}&\eta_0^j|\lambda^a;\lambda^c\rangle=0,\quad \xi_0^j|\lambda^a;\lambda^c\rangle\ne0\quad (\lambda_j^c=0,1,2,\cdots),\\&\eta_0^j|\lambda^a;\lambda^c\rangle\ne0,\quad \xi_0^j|\lambda^a;\lambda^c\rangle=0\quad (\lambda_j^c=-1,-2,-3,\cdots).\end{aligned}\tag{4.34}$$

We set the abbreviation ζ_ϵ^j by

$$\zeta_\epsilon^j=\begin{cases}\xi_0^j & (\epsilon=+)\\ \eta_0^j & (\epsilon=-)\end{cases}\qquad (j=1,2,\cdots,N).\tag{4.35}$$

We introduce the projection operator Pr by

$$Pr = \prod_{j=1}^{N} \zeta^j_{\epsilon_j} \zeta^j_{-\epsilon_j} \qquad ((\epsilon_1, \epsilon_2, \cdots, \epsilon_N) \in J(\lambda^c)), \tag{4.36}$$

where we have used

$$J(\lambda^c) = \left\{ (\epsilon_1, \epsilon_2, \cdots, \epsilon_N) \middle| \epsilon_j = \left\{ \begin{array}{ll} + & (\lambda^c_j = -1, -2, \cdots) \\ - & (\lambda^c_j = 0, 1, 2, \cdots) \end{array} \right. \quad (j = 1, 2, \cdots, N) \right\}. \tag{4.37}$$

The projection operator Pr commutes with every element of $U_q(\widehat{gl}(N|N))$, and satisfies $Pr^2 = Pr$. For general $N = 1, 2, 3, \cdots$, we have the following direct sum of decomposition.

$$\mathcal{F}_{\lambda^a;\lambda^c} = \bigoplus_{\epsilon_1, \epsilon_2, \cdots, \epsilon_N = \pm} \prod_{j=1}^{N} \zeta^j_{\epsilon_j} \zeta^j_{-\epsilon_j} \mathcal{F}_{\lambda^a;\lambda^c}. \tag{4.38}$$

From calculation of character for $N = 1, 2$ in [31, 32], we expect that the projection operator Pr is a map onto the irreducible highest-weight module $V(\lambda)$ with the highest-weight λ given by (4.23) and (4.26).

$$V(\lambda) = Pr \cdot \mathcal{F}_{\lambda^a;\lambda^c}. \tag{4.39}$$

The module $V(\lambda) = Pr \cdot \mathcal{F}_{\lambda^a;\lambda^c}$ is one of the sub-modules of the decomposition (4.38).

4.2 The vertex operator

In this Section we give bosonizations of the vertex operators that intertwine irreducible highest-weight modules, and derive an integral representation of the dual vertex operator. First, we give the vertex operators $\phi(z), \phi^*(z)$ between the Fock spaces $\mathcal{F}_{\lambda^a;\lambda^c}$. We set the vertex operator $\phi(z)$ and the dual vertex operator $\phi^*(z)$ as the intertwiner of $U_q(\widehat{gl}(N|N))$-module as follows.

$$\phi(z) : \mathcal{F}_{\lambda^a;\lambda^c} \longrightarrow \mathcal{F}_{\mu^a;\mu^c} \otimes V_z, \qquad \phi(z) \cdot x = \Delta(x) \cdot \phi(z), \tag{4.40}$$

$$\phi^*(z) : \mathcal{F}_{\lambda^a;\lambda^c} \longrightarrow \mathcal{F}_{\mu^a;\mu^c} \otimes V^{*S}_z, \qquad \phi^*(z) \cdot x = \Delta(x) \cdot \phi^*(z). \tag{4.41}$$

We expand the vertex operators as the same way as (3.12). We introduce the notation

$$\begin{aligned} B^{*i}(z;\kappa) &= Q_{B^{*i}} + B^{*i}_0 \log z - \sum_{m \neq 0} \frac{B^{*i}_m}{[m]_q} q^{\kappa|m|} z^{-m} \quad (i = 1, 2, \cdots, 2N-1), \\ B^1(z;\kappa) &= Q_{B^{*2N}} + B^{*2N}_0 \log z + 2(N-1) \sum_{m \neq 0} \frac{B^{*2N}_m}{[m]_q} q^{\kappa|m|} z^{-m}, \\ B^{2N}(z;\kappa) &= Q_{B^{*2N}} + B^{*2N}_0 \log z - 2N \sum_{m \neq 0} \frac{B^{*2N}_m}{[m]_q} q^{\kappa|m|} z^{-m}, \end{aligned} \tag{4.42}$$

and the Fermi number operator $N_f = \sum_{j=1}^{N} a^{2j}_0$.

Theorem 4.2 *[30, 31, 32] The vertex operators $\phi_j(z)$ and $\phi^*_j(z)$ $(j = 1, 2, \cdots, 2N)$ in (4.40) and (4.41) are realized as follows.*

$$\phi_{2N}(z) =: e^{-B^{*2N-1}(qz;\frac{1}{2}) + B^{2N}(q^2 z;\frac{1}{2}) + c^N(qz)} : e^{-\sqrt{-1}\pi N_f},$$

$$(-1)^j\phi_j(z) = [\phi_{j+1}(z), f_j]_{q^{(-1)^j}} \quad (j = 1, 2, \cdots, 2N-1), \tag{4.43}$$

$$\phi_1^*(z) =: e^{B^{*,1}(qz;\frac{1}{2})+B^1(qz;\frac{1}{2})} : e^{\sqrt{-1}\pi N_f}(qz)^{1-\lambda_1^a},$$

$$q^{(-1)^{j+1}}\phi_{j+1}^*(z) = [\phi_j^*(z), f_j]_{q^{(-1)^{j+1}}} \quad (j = 1, 2, \cdots, 2N-1). \tag{4.44}$$

Here the Chevalley generator f_j is realized as follows.

$$f_j = X_0^{-,j} = \oint \frac{dw}{2\pi\sqrt{-1}} X_j^-(w) \quad (j = 1, 2, \cdots, 2N-1). \tag{4.45}$$

Next, we focus our attention to the vertex operators $\Phi_j(z)$ between the irreducible highest-weight module $V(\lambda)$, given in Definition 3.1 : $\Phi(z) : V(\lambda) \longrightarrow V(\mu) \otimes V_z$, $\Phi^*(z) : V(\lambda) \longrightarrow V(\mu) \otimes V_z^{*S}$. We expect

$$\Phi(z) = Pr \cdot \phi(z) \cdot Pr, \quad \Phi^*(z) = Pr \cdot \phi^*(z) \cdot Pr, \tag{4.46}$$

where Pr is the projection operator in (4.36). The vertex operators $\phi(z)$ and $\phi^*(z)$ commute with Pr. Hence we have $\Phi(z) = Pr \cdot \phi(z) = \phi(z) \cdot Pr$ and $\Phi^*(z) = Pr \cdot \phi^*(z) = \phi^*(z) \cdot Pr$.

In what follows we give integral representations of $\phi_j^*(z)$ which are convenient for construction of the boundary state. We introduce the notation

$$X_\epsilon^{-,2j-1}(z) =: e^{-B^{2j-1}(z;\frac{1}{2})-c^j(q^\epsilon z)} : F^{-,2j-1}. \tag{4.47}$$

Then we have $X^{-,2j-1}(z) = \frac{1}{(q-q^{-1})z}(X_+^{-,2j-1}(z) - X_-^{-,2j-1}(z))$. Using the formulae of the normal orderings in Appendix C, we have the following integral representations.

Theorem 4.3 *The dual vertex operators $\phi_i^*(z)$ $(i = 1, 2, \cdots, 2N)$ have the following integral representations.*

$$\begin{aligned}
\phi_{2j+1}^*(z) &= (q-q^{-1})^j \sum_{\epsilon_1,\epsilon_3,\epsilon_5,\cdots,\epsilon_{2j-1}=\pm} \prod_{s=1}^{j} \epsilon_{2s-1} \prod_{s=1}^{2j} \oint_{C_s} \frac{dw_s}{2\pi\sqrt{-1}w_s} \frac{qw_{2j}}{qz(1-qw_1/z)(1-qz/w_1)} \\
&\times \prod_{s=1}^{j-1} \frac{1}{(1-qw_{2s+1}/w_{2s})(1-qw_{2s}/w_{2s+1})} \prod_{s=1}^{j} \frac{1}{(1-q^{\epsilon_{2s-1}}w_{2s-1}/w_{2s})} \\
&\times : \phi_1^*(z) \prod_{s=1}^{j} X_{\epsilon_{2s-1}}^{-,2s-1}(qw_{2s-1}) \prod_{s=1}^{j} X^{-,2s}(qw_{2s}) : \quad (j = 1, 2, \cdots, N-1),
\end{aligned} \tag{4.48}$$

$$\begin{aligned}
\phi_{2j+2}^*(z) &= (q-q^{-1})^j \sum_{\epsilon_1,\epsilon_3,\epsilon_5,\cdots,\epsilon_{2j+1}=\pm} \prod_{s=1}^{j+1} \epsilon_{2s-1} \prod_{s=1}^{2j+1} \oint_{C_s} \frac{dw_s}{2\pi\sqrt{-1}w_s} \frac{1}{qz(1-qw_1/z)(1-qz/w_1)} \\
&\times \prod_{s=1}^{j} \frac{1}{(1-qw_{2s+1}/w_{2s})(1-qw_{2s}/w_{2s+1})} \prod_{s=1}^{j} \frac{1}{(1-q^{\epsilon_{2s-1}}w_{2s-1}/w_{2s})} \\
&\times : \phi_1^*(z) \prod_{s=1}^{j+1} X_{\epsilon_{2s-1}}^{-,2s-1}(qw_{2s-1}) \prod_{s=1}^{j} X^{-,2s}(qw_{2s}) : \quad (j = 1, 2, \cdots, N-1).
\end{aligned} \tag{4.49}$$

We take the integration contour C_s $(s = 1, 2, \cdots, 2N-1)$ to be a simple closed curve encircles $w_s = 0, qw_{s-1}$ but not $w_s = q^{-1}w_{s-1}$. Here we set $w_0 = z$.

5 The boundary state

In this Section we give a bosonization of the boundary state ${}_B\langle 0| \neq 0$ satisfying

$$ {}_B\langle 0|T_B(z) = {}_B\langle 0|. $$

The construction of the boundary state is main result of this paper.

5.1 A bosonization

For $\lambda^a = (\lambda^a_1, \lambda^a_2, \cdots, \lambda^a_{2N}) \in \mathbf{C}^{2N}$ and $\lambda^c = (\lambda^c_1, \lambda^c_2, \cdots, \lambda^c_N) \in \mathbf{C}^N$, we set the vector $\langle \lambda^a; \lambda^c|$ by

$$ \langle \lambda^a; \lambda^c| = \langle 0| e^{-\sum_{i=1}^{2N} \lambda^a_i Q_{a^i} - \sum_{j=1}^{N} \lambda^c_j Q_{c^j}}, \tag{5.1} $$

where the vector $\langle 0| \neq 0$ satisfies

$$ \langle 0| a^i_{-m} = \langle 0| c^j_{-m} = 0 \quad (m \geq 0, i = 1, 2, \cdots, 2N, j = 1, 2, \cdots, N). \tag{5.2} $$

We introduce the space $\mathcal{F}^*_{\lambda^a;\lambda^c}$ by

$$ \mathcal{F}^*_{\lambda^a;\lambda^c} = \bigoplus_{j_1, j_2, \cdots, j_{2N-1} \in \mathbf{Z}} F^*_{\lambda^a_1 + j_1, \lambda^a_2 - j_1 + j_2, \cdots, \lambda^a_{2N} - j_{2N-1}; \lambda^c_1 + j_1 - j_2, \lambda^c_2 + j_3 - j_4, \cdots, \lambda^c_N + j_{2N-1}}. \tag{5.3} $$

Denote by $F^*_{\lambda^a_1, \cdots, \lambda^a_{2N}; \lambda^c_1, \cdots, \lambda^c_N}$ the dual Fock space generated by $\{a^i_m, c^j_m | m > 0, i = 1, 2, \cdots, 2N, j = 1, 2, \cdots, N\}$ over the vector $\langle \lambda^a; \lambda^c|$. Following arguments in the previous section, we expect the following identification between the restricted dual module $V^*(\lambda)$ and the Fock module.

$$ V^*(\lambda) = \mathcal{F}^*_{\lambda^a;\lambda^c} \cdot Pr. \tag{5.4} $$

For instance we have the highest-weight vectors $\langle \Lambda_i| \in V^*(\Lambda_i)$ as follows.

$$ \langle \Lambda_0| = \langle 0, \cdots, 0; 0, \cdots, 0|, \tag{5.5} $$

$$ \langle \Lambda_{2N-1}| = \langle \frac{1}{2N}, -\frac{1}{2N}, \frac{1}{2N}, -\frac{1}{2N}, \cdots, \frac{1}{2N}, -\frac{1}{2N}; 0, 0, \cdots, 0, 0, -1|, \tag{5.6} $$

$$ \langle \Lambda_1| = \langle \frac{2N-1}{2N}, \frac{1}{2N}, -\frac{1}{2N}, \frac{1}{2N} \cdots, -\frac{1}{2N}, \frac{1}{2N}; 0, 0, 0, \cdots, 0|, \tag{5.7} $$

$$ \langle \Lambda_2| = \langle \frac{N-1}{N}, \frac{1-N}{N}, -\frac{1}{N}, \frac{1}{N}, \cdots, -\frac{1}{N}, \frac{1}{N}; 1, 0, 0, \cdots, 0|. \tag{5.8} $$

In this Section we focus our attention to the integrable highest-weight module $V^*(\Lambda_0)$ for simplicity. Later we summarize results associated with the integrable modules $V^*(\Lambda_{2N-1})$ in Appendix B. For $V^*(\Lambda_0)$ the projection operator Pr is given by $Pr = \prod_{j=1}^{N} \eta^j_0 \prod_{j=1}^{N} \xi^j_0$.

Definition 5.1 *We define the bosonic operator G by*

$$ G = -\frac{1}{2} \sum_{m>0} \frac{m q^{-2m}}{[m]_q^2} \left(\sum_{i=1}^{2N} (-1)^{i+1} (a^i_m)^2 + \sum_{j=1}^{N} (c^j_m)^2 \right) + \sum_{m>0} \left(\sum_{i=1}^{2N} \beta^i_m a^i_m + \sum_{j=1}^{N} \delta^j_m c^j_m \right). \tag{5.9} $$

Here we have set

$$\delta_m^j = \frac{q^{-m}}{[m]_q}(1-2(-1)^{\frac{m}{2}})\theta_m \quad (j=1,2,\cdots,N), \tag{5.10}$$

$$\beta_m^i = \begin{cases} \beta_m^{[1],i} & (L=M=0, R>0) \\ \beta_m^{[2],i} & (L=0, M, R>0) \\ \beta_m^{[3],i} & (L, M, R>0) \end{cases} \qquad (i=1,2,\cdots,2N), \tag{5.11}$$

where we have set

$$\beta_m^{[1],2s-1} = \beta_m^{[1],2s} = \frac{2(1-s)}{[m]_q} q^{-\frac{3}{2}m}\theta_m \quad (1 \le s \le N), \tag{5.12}$$

$$\beta_m^{[2],i} = \beta_m^{[1],i} - \frac{r^m q^{-\frac{3}{2}m}}{[m]_q} + \begin{cases} 0 & (1 \le j \le M) \\ \frac{r^m q^{-\frac{5}{2}m}}{[m]_q} & (M < i \le 2N, M = \text{odd}) \\ \frac{r^m q^{-\frac{3}{2}m}}{[m]_q} & (M < i \le 2N, M = \text{even}) \end{cases}, \tag{5.13}$$

$$\beta_m^{[3],i} = \beta_m^{[1],i} - \frac{(-1)^{\frac{m}{2}} 2q^{-\frac{3}{2}m}}{[m]_q}\theta_m + \begin{cases} 0 & (1 \le i \le L) \\ \frac{r^{-m} q^{-\frac{5}{2}m}}{[m]_q} & (L < i \le L+M, L = \text{odd}) \\ \frac{r^{-m} q^{-\frac{3}{2}m}}{[m]_q} & (L < i \le L+M, L = \text{even}) \\ \frac{r^m q^{\frac{1}{2}m}}{[m]_q} & (L+M < i \le 2N, L = \text{odd}, M = \text{odd}) \\ \frac{r^m q^{-\frac{1}{2}m}}{[m]_q} & (L+M < i \le 2N, L = \text{odd}, M = \text{even}) \\ \frac{r^m q^{-\frac{5}{2}m}}{[m]_q} & (L+M < i \le 2N, L = \text{even}, M = \text{odd}) \\ \frac{r^m q^{-\frac{3}{2}m}}{[m]_q} & (L+M < i \le 2N, L = \text{even}, M = \text{even}) \end{cases}. \tag{5.14}$$

Here we have used

$$\theta_m = \begin{cases} 1 & (m = \text{even}) \\ 0 & (m = \text{odd}) \end{cases}. \tag{5.15}$$

The following theorem is **main result** of this paper.

Theorem 5.2 *A bosonization of the boundary state* ${}_B\langle 0| \in V^*(\Lambda_0)$ *is realized as follows.*

$${}_B\langle 0| = \langle \Lambda_0 | e^G \cdot Pr. \tag{5.16}$$

Here G is given in (5.9), and Pr is the projection operator.

We set the type-II vertex operator $\psi^*(z)$ as the intertwiner of the $U_q(\widehat{gl}(N|N))$-module as follows.

$$\psi^*(z) : \mathcal{F}_{\lambda^a;\lambda^c} \longrightarrow V_z^{*S} \otimes \mathcal{F}_{\mu^a;\mu^c}, \qquad \psi^*(z) \cdot x = \Delta(x) \cdot \psi^*(z). \tag{5.17}$$

We expand the type-II vertex operators as the same way as (3.12). The type-II vertex operators $\psi_i^*(z)$ $(i = 1, 2, \cdots, 2N)$ satisfy the following commutation relation [32].

$$\psi_i^*(z_1)\phi_j(z_2) = (qz_2/z_1)^{2-\frac{1}{N}} \phi_j(z_2)\psi_i^*(z_1)(-1)^{[v_i][v_j]}. \tag{5.18}$$

We set

$$ {}_B\langle k_1, k_2, \cdots, k_n| = {}_B\langle 0|\psi^*_{k_1}(\xi_1)\psi^*_{k_2}(\xi_2)\cdots\psi^*_{k_n}(\xi_n)\cdot Pr. \tag{5.19} $$

These vectors are eigen-vectors of $T_B(z)$.

$$ {}_B\langle k_1, k_2, \cdots, k_n|T_B(z) = {}_B\langle k_1, k_2, \cdots, k_n|\prod_{s=1}^{n}(q/\xi_s)^{2(1-\frac{2}{N})}. \tag{5.20} $$

We expect that the eigen-vectors ${}_B\langle k_1, k_2, \cdots, k_n|$ are basis of the space of physical state. The energy level degenerates violently.

6 Proof of Theorem 5.2

In this Section we give a proof of the boundary state. The following proposition gives sufficient condition of Theorem 5.2.

Proposition 6.1 *A sufficient condition of ${}_B\langle 0|T_B(z) = {}_B\langle 0|$ is given by*

$$ \langle 0|e^G\phi^*_j(z^{-1})K^j_j(z) = \langle 0|e^G\phi^*_j(z) \quad (j = 1, 2, \cdots, 2N). \tag{6.1} $$

Proof. Multiplying $\Phi^*_j(z)$ to ${}_B\langle 0|T_B(z) = {}_B\langle 0|$ from the right and using the inversion relation of the vertex operators (3.14), we have ${}_B\langle 0|\Phi^*_j(z^{-1})K^j_j(z) = {}_B\langle 0|\Phi^*_j(z)$. The projection operator Pr commutes with the vertex operators, we have $\langle 0|e^G\phi^*_j(z^{-1})K^j_j(z)\cdot Pr = \langle 0|e^G\phi^*_j(z)\cdot Pr$. Removing the projection operator Pr, we have the above sufficient condition. **Q.E.D.**

6.1 Action of the vertex operator

In this Section we study the action of the vertex operator on the boundary state.

Proposition 6.2

$$ e^G a^j_{-m}e^{-G} = a^j_{-m} - q^{-2m}a^j_m + (-1)^{j+1}\beta^j_m\frac{[m]^2_q}{m} \quad (m > 0, 1 \le i \le 2N), \tag{6.2} $$

$$ e^G c^j_{-m}e^{-G} = c^j_{-m} - q^{-2m}c^j_m + \delta^j_m\frac{[m]^2_q}{m} \quad (m > 0, 1 \le j \le N), \tag{6.3} $$

$$ e^G B^j_{-m}e^{-G} = B^j_{-m} - q^{-2m}B^j_m + \frac{[m]^2_q}{m}(\beta^j_m - \beta^{j+1}_m) \quad (m > 0, 1 \le i \le 2N-1). \tag{6.4} $$

Proof. We note that

$$ e^G X e^{-G} = e^{ad(G)}(X) = \sum_{n=0}^{\infty}\frac{1}{n!}ad(G)^n(X). $$

Using $ad^2(G)(X) = [G, [G, X]] = 0$ for $X = a^i_m, c^j_m$, we have

$$ e^G X e^{-G} = X + [G, X]. $$

Using the following relations, we get proposition.

$$[G, a^i_{-m}] = -q^{-2m} a^i_m + \frac{[m]_q^2}{m}(-1)^{i+1}\beta^i_m, \tag{6.5}$$

$$[G, c^j_{-m}] = -q^{-2m} c^j_m + \frac{[m]_q^2}{m}\delta^j_m, \tag{6.6}$$

$$[G, B^i_{-m}] = -q^{-2m} B^i_m + \frac{[m]_q^2}{m}(\beta^i_m - \beta^{i+1}_m). \tag{6.7}$$

Q.E.D.

We set notations.

$$B^i_\pm(z;\kappa) = \mp \sum_{m=1}^{\infty} \frac{B^i_{\pm m}}{[m]_q} q^{\kappa m} z^{\mp m} \quad (i = 1, 2, \cdots, 2N-1), \tag{6.8}$$

$$B_\pm(z;\kappa) = \mp \sum_{m=1}^{\infty} \frac{1}{[m]_q} \left(B^{*1}_{\pm m} - 2(N-1)B^{*2N}_{\pm m}\right) q^{\kappa m} z^{\mp m}, \tag{6.9}$$

$$c^j_\pm(z) = \mp \sum_{m=1}^{\infty} \frac{c^j_{\pm m}}{[m]_q} z^{\mp m} \quad (j = 1, 2, \cdots, N). \tag{6.10}$$

We have $B^{*1}(z;\kappa) + B^1(z;\kappa) = Q_{a^1} + a^1_0 \log z + B_+(z;\kappa) + B_-(z;\kappa)$.

Proposition 6.3 *Action of the basic operators is given as follows.*

$$\langle 0| e^G e^{-B^i_-(qw;\frac{1}{2})} = g_i(w) \langle 0| e^G e^{-B^i_+(q/w;\frac{1}{2})} \quad (i = 1, 2, \cdots, 2N-1), \tag{6.11}$$

$$\langle 0| e^G e^{B_-(qz;\frac{1}{2})} = f_1(z) \langle 0| e^G e^{B_+(q/z;\frac{1}{2})}, \tag{6.12}$$

$$\langle 0| e^G e^{c^j_-(qw)} = c_j(w) \langle 0| e^G e^{c^j_+(q/w)} \quad (j = 1, 2, \cdots, N), \tag{6.13}$$

where $c_j(w)$, $f_1(z)$, and $g_i(w)$ are given as follows.

$$c_j(w) = 1 + w^2 \quad (j = 1, 2, \cdots, N), \tag{6.14}$$

$$f_1(z) = \begin{cases} \varphi^{[1]}(z) & (L = M = 0, R > 0) \\ (1 - rz)\varphi^{[2]}(z) & (L = 0, M, R > 0) \\ (1 + z^2)\varphi^{[3]}(z) & (L, M, R > 0) \end{cases}, \tag{6.15}$$

$$g_i(w) = \begin{cases} g^{[1]}_i(w) & (L = M = 0, R > 0) \\ g^{[2]}_i(w) & (L = 0, M, R > 0) \\ g^{[3]}_i(w) & (L, M, R > 0) \end{cases} \qquad (i = 1, 2, \cdots, 2N-1). \tag{6.16}$$

Here we have set

$$g^{[1]}_i(w) = \begin{cases} 1 & (i = \text{odd}) \\ (1 - w^2) & (i = \text{even}) \end{cases}, \tag{6.17}$$

$$g^{[2]}_i(w) = g^{[1]}_i(w) \times \begin{cases} \frac{1}{(1-rw/q)} & (i = M, M = \text{odd}) \\ \frac{1}{(1-rw)} & (i = M, M = \text{even}) \\ 1 & (\text{otherwise}) \end{cases}, \tag{6.18}$$

$$
g_i^{[3]}(w) \quad = \quad g_i^{[1]}(w) \times \begin{cases} \frac{1}{(1-w/qr)} & (i=L, L=\text{odd}) \\ \frac{1}{(1-w/r)} & (i=L, L=\text{even}) \\ \frac{1}{(1-q^2rw)} & (i=L+M, L=\text{odd}, M=\text{odd}) \\ \frac{1}{(1-qrw)} & (i=L+M, L=\text{odd}, M=\text{even}) \\ \frac{1}{(1-rw/q)} & (i=L+M, L=\text{even}, M=\text{odd}) \\ \frac{1}{(1-rw)} & (i=L+M, L=\text{even}, M=\text{even}) \\ 1 & (\text{otherwise}) \end{cases} . \tag{6.19}
$$

Proof. Using Proposition 6.2, we have

$$
e^G e^{-B_-^j(qw;\frac{1}{2})} e^{-G} = g_j(w) e^{-B_-^j(qw;\frac{1}{2})} e^{-B_+^j(q/w;\frac{1}{2})}, \tag{6.20}
$$

$$
e^G e^{B_-(qz;\frac{1}{2})} e^{-G} = f_1(z) e^{B_-(qz;\frac{1}{2})} e^{B_+(q/z;\frac{1}{2})}, \tag{6.21}
$$

$$
e^G e^{-c_-^j(qw)} e^{-G} = c_j(w) e^{-c_-^j(qw)} e^{-c_+^j(q/w)}, \tag{6.22}
$$

where

$$
g_j(w) \quad = \quad \exp\left(-\sum_{m>0} \frac{[m]_q}{m} q^{\frac{3}{2}m} (\beta_m^j - \beta_m^{j+1}) w^m \right) \quad (1 \le j \le 2N-1), \tag{6.23}
$$

$$
\begin{aligned}
f_1(z) \quad &= \quad \exp\left(\frac{N-1}{N} \sum_{m>0} \frac{1}{2m} q^m z^{2m} + \sum_{m>0} \frac{[m]_q}{m} q^{\frac{3}{2}m} \beta_m^1 z^m \right. \\
&\quad \left. + \left(1 - \frac{1}{2N}\right) \sum_{m>0} \frac{[m]_q}{m} q^{\frac{3}{2}m} \left(\sum_{j=1}^{2N} (-1)^j \beta_m^j \right) z^m \right),
\end{aligned} \tag{6.24}
$$

$$
c_j(w) \quad = \quad \exp\left(-\sum_{m>0} \frac{1}{2m} w^{2m} + \sum_{m>0} \frac{[m]_q}{m} q^m \delta_m^j w^m \right) \quad (1 \le j \le N). \tag{6.25}
$$

Using (5.10) and (5.11), we have the explicit formulae of $g_j(w)$, $f_1(z)$, and $c_j(w)$. Acting these operators to the vacuum vector $\langle 0|$, we obtain this proposition. **Q.E.D.**

We set

$$
D(z,w) = (1-qzw)(1-qw/z)(1-qz/w)(1-q/zw), \tag{6.26}
$$

$$
H_\epsilon^{2j-1}(w_1,w_2) = \frac{e^{-c_+^j(q^{1+\epsilon}w_1) - c_+^j(q^{1-\epsilon}/w_1) - B_+^{2j-1}(qw_1) - B_+^{2j-1}(q/w_1)}}{(1-q^\epsilon w_1 w_2)(1-q^\epsilon w_1/w_2)}, \tag{6.27}
$$

$$
H^{2j}(w_1,w_2) = \frac{e^{-B_+^{2j}(qw_1) - B_+^{2j}(q/w_1) + c_+^j(qw_1) + c_+^j(q/w_1)}}{D(w_1,w_2)}, \tag{6.28}
$$

$$
J_\epsilon^{2j-1}(w) = e^{-c_+^j(q^{1+\epsilon}w) - c_+^j(q^{1-\epsilon}/w) - B_+^{2j-1}(qw) - B_+^{2j-1}(q/w)}, \tag{6.29}
$$

$$
J^{2j}(w) = e^{-B_+^{2j}(qw) - B_+^{2j}(q/w) + c_+^j(qw) + c_+^j(q/w)}. \tag{6.30}
$$

Proposition 6.4 *The following relations for $H_\epsilon^{2j-1}(w_1,w_2)$ $(j=1,2,\cdots,N, \epsilon=\pm)$ hold.*

$$
\begin{aligned}
&\oint_C \frac{dw_1}{w_1} \sum_{\epsilon=\pm} \epsilon q^\epsilon \left(q^\epsilon w_1 + \frac{1}{q^\epsilon w_1} \right) (1-qw_1w_2) H_\epsilon^{2j-1}(w_1,w_2) \\
&= \oint_C \frac{dw_1}{w_1} q \left(qw_1 + \frac{1}{qw_1} \right) (1-w_1^2) H_+^{2j-1}(w_1,w_2),
\end{aligned} \tag{6.31}
$$

$$\oint_C \frac{dw_1}{w_1} \sum_{\epsilon=\pm} \epsilon q^\epsilon \left(q^\epsilon w_1 + \frac{1}{q^\epsilon w_1} \right) (1-sw_1)(1-qw_1w_2) H_\epsilon^{2j-1}(w_1,w_2)$$
$$= (1-qsw_2) \oint_C \frac{dw_1}{w_1} q \left(qw_1 + \frac{1}{qw_1} \right) (1-w_1^2) H_+^{2j-1}(w_1,w_2), \tag{6.32}$$

$$\oint_C \frac{dw_1}{w_1} \sum_{\epsilon=\pm} \epsilon q^\epsilon \left(q^\epsilon w_1 + \frac{1}{q^\epsilon w_1} \right) \frac{(1-qw_1w_2)}{(1-s/w_1)} H_\epsilon^{2j-1}(w_1,w_2)$$
$$= (1-qsw_2) \oint_C \frac{dw_1}{w_1} q \left(qw_1 + \frac{1}{qw_1} \right) \frac{(1-w_1^2)}{(1-s/w_1)(1-sw_1)} H_+^{2j-1}(w_1,w_2). \tag{6.33}$$

$$\oint_C \frac{dw_1}{w_1} \sum_{\epsilon=\pm} \epsilon q^\epsilon \left(q^\epsilon w_1 + \frac{1}{q^\epsilon w_1} \right) w_1(1-qw_1w_2) H_\epsilon^{2j-1}(w_1,w_2)$$
$$= q^2 w_2 \oint_C \frac{dw_1}{w_1} \left(qw_1 + \frac{1}{qw_1} \right) (1-w_1^2) H_+^{2j-1}(w_1,w_2). \tag{6.34}$$

Here the integration contour C encircles $w_1 = 0$, s, $qw_2^{\pm 1}$ but not s^{-1}, $q^{-1}w_2^{\pm 1}$. Note that the integral $\oint_C \frac{dw_1}{w_1} \left(qw_1 + \frac{1}{qw_1} \right) \frac{(1-w_1^2)}{(1-s/w_1)(1-sw_1)} H_+^{2j-1}(w_1,w_2)$ is invariant under $w_2 \to w_2^{-1}$.

Proof. We show the second relation (6.32). We start from LHS :

$$\oint_C \frac{dw_1}{w_1} q \left(qw_1 + \frac{1}{qw_1} \right) (1-sw_1)(1-qw_1w_2) H_+^{2j-1}(w_1,w_2)$$
$$- \oint_C \frac{dw_1}{w_1} q^{-1} \left(\frac{w_1}{q} + \frac{q}{w_1} \right) (1-sw_1)(1-qw_1w_2) H_-^{2j-1}(w_1,w_2).$$

Changing the variable $w_1 \to w_1^{-1}$ in the second term and using $H_-^{2j-1}(w_1^{-1},w_2) = q^2 w_1^2 H_+^{2j-1}(w_1,w_2)$, we have RHS after summing up the first and the second terms. The relations (6.33) and (6.34) are obtained in the same way. Upon the specialization $s = 0$, (6.31) is obtained from both (6.32) and (6.33). **Q.E.D.**

Proposition 6.5 *The following relations for $H^{2j}(w_1,w_2)$ $(j = 1, 2, \cdots, N)$ hold.*

$$\oint_C \frac{dw_1}{w_1} (1-sw_1)(1-q/w_1w_2) \frac{(1-w_1^2)}{(1+w_1^2)} H^{2j}(w_1,w_2)$$
$$= -\frac{q}{2w_2}(1-sw_2/q) \times \oint_C \frac{dw_1}{w_1} \frac{(1-w_1^2)^2}{w_1(1+w_1^2)} H^{2j}(w_1,w_2), \tag{6.35}$$

$$\oint_C \frac{dw_1}{w_1} w_1(1-q/w_1w_2) \frac{(1-w_1^2)}{(1+w_1^2)} H^{2j}(w_1,w_2) = -\frac{1}{2} \times \oint_C \frac{dw_1}{w_1} \frac{(1-w_1^2)^2}{w_1(1+w_1^2)} H^{2j}(w_1,w_2), \tag{6.36}$$

$$\oint_C \frac{dw_1}{w_1} \frac{(1-q/w_1w_2)}{(1-s/w_1)} \frac{(1-w_1^2)}{1+w_1^2} H^{2j}(w_1,w_2)$$
$$= -\frac{q}{2w_2}(1-sw_2/q) \times \oint_C \frac{dw_1}{w_1} \frac{(1-w_1^2)^2}{w_1(1+w_1^2)} \frac{H^{2j}(w_1,w_2)}{(1-sw_1)(1-s/w_1)}. \tag{6.37}$$

Here the integration contour C encircles $w_1 = 0$, s, $\sqrt{-1}$, $qw_2^{\pm 1}$ but not $s^{-1}, -\sqrt{-1}$, $q^{-1}w_2^{\pm 1}$. Note that the integral $\oint_C \frac{dw_1}{w_1} \frac{(1-w_1^2)^2}{w_1(1+w_1^2)} H^{2j}(w_1,w_2)$ is invariant under $w_2 \to w_2^{-1}$.

Proof. We start from LHS. Taking into account the relation $\oint \frac{dw}{w} f(w) = \frac{1}{2} \oint \frac{dw}{w} (f(w) + f(w^{-1}))$ and using $H^{2j}(w_1^{-1}, w_2) = H^{2j}(w_1, w_2)$, we have

$$\frac{1}{2} \oint_C \frac{dw_1}{w_1} \left\{ (1 - sw_1)(1 - q/w_1 w_2) - (1 - s/w_1)(1 - qw_1/w_2) \right\} \frac{1 - w_1^2}{1 + w_1^2} H^{2j}(w_1, w_2).$$

From $(1 - sw_1)(1 - q/w_1 w_2) - (w_1 \leftrightarrow w_1^{-1}) = \frac{q}{w_1 w_2}(w_1^2 - 1)(1 - sw_2/q)$, the relation (6.35) is obtained. The relations (6.36) and (6.37) are obtained in the same way. **Q.E.D.**

We have action of the vertex operators as follows.

Proposition 6.6 *Action of the vertex operators $\langle \lambda^a; \lambda^c | e^G \phi_i^*(z)$ $(i = 1, 2, \cdots, 2N)$ is given as follows.*

$$\begin{aligned}
\langle \lambda^a; \lambda^c | e^G \phi_{2j}^*(z) &= (-1)(q - q^{-1})^{j-1} \prod_{s=1}^{N} e^{\sqrt{-1}\pi\lambda_{2s}^a} \prod_{s=1}^{j} \prod_{t=1}^{s} e^{-\sqrt{-1}\pi\lambda_{2t-1}^a} \prod_{s=1}^{j-1} \prod_{t=1}^{s} e^{-\sqrt{-1}\pi\lambda_{2t}^a} f_1(z) \\
&\times \sum_{\epsilon_1, \epsilon_3, \cdots, \epsilon_{2j-1} = \pm} \prod_{s=1}^{j} \epsilon_{2s-1} \cdot q^{-\epsilon_{2j-1}(\lambda_j^c + 1)} \prod_{s=1}^{j-1} q^{-\lambda_s^c \epsilon_{2s-1}} \prod_{s=1}^{2j-1} \oint_{\widetilde{C}_s} \frac{dw_s}{2\pi\sqrt{-1} w_s} \\
&\times \prod_{s=1}^{j} (qw_{2s-1})^{-\lambda_{2s-1}^a - \lambda_{2s}^a - \lambda_s^c} \prod_{s=1}^{j-1} (qw_{2s})^{\lambda_{2s}^a + \lambda_{2s+1}^a + \lambda_s^c} \prod_{s=1}^{2j-1} g_s(w_s) \frac{\prod_{s=1}^{j} c_s(q^{\epsilon_{2s-1}} w_{2s-1})}{\prod_{s=1}^{j-1} c_s(w_{2s})} \\
&\times \frac{(1 - q/zw_1)}{D(z, w_1)} \prod_{s=1}^{j-1} (1 - q/w_{2s} w_{2s+1}) \prod_{s=1}^{j-1} (1 - qw_{2s-1} w_{2s}) \\
&\times \langle \lambda^a; \lambda^c | e^G e^{Q_{a2j} - Q_{cj}} e^{B_+(qz; \frac{1}{2}) + B_+(q/z; \frac{1}{2})} \qquad (6.38) \\
&\times \prod_{s=1}^{j-1} H_{\epsilon_{2s-1}}^{2s-1}(w_{2s-1}, w_{2s}) \prod_{s=1}^{j-1} H^{2s}(w_{2s}, w_{2s+1}) J_{\epsilon_{2j-1}}^{2j-1}(w_{2j-1}) \qquad (j = 1, 2, \cdots, N),
\end{aligned}$$

$$\begin{aligned}
\langle \lambda^a; \lambda^c | e^G \phi_{2j+1}^*(z) &= (q - q^{-1})^{j} \prod_{s=1}^{N} e^{\sqrt{-1}\pi\lambda_{2s}^a} \prod_{s=1}^{j} \prod_{t=1}^{s} e^{-\sqrt{-1}\pi\lambda_{2t-1}^a} \prod_{s=1}^{j} \prod_{t=1}^{s} e^{-\sqrt{-1}\pi\lambda_{2t}^a} f_1(z) \\
&\times \sum_{\epsilon_1, \epsilon_3, \cdots, \epsilon_{2j-1} = \pm} \prod_{s=1}^{j} \epsilon_{2s-1} \prod_{s=1}^{j} q^{-\lambda_s^c \epsilon_{2s-1}} \prod_{s=1}^{2j} \oint_{\widetilde{C}_s} \frac{dw_s}{2\pi\sqrt{-1} w_s} \\
&\times \prod_{s=1}^{j} (qw_{2s-1})^{-\lambda_{2s-1}^a - \lambda_{2s}^a - \lambda_s^c} \prod_{s=1}^{j} (qw_{2s})^{\lambda_{2s}^a + \lambda_{2s+1}^a + \lambda_s^c} \prod_{s=1}^{2j} g_s(w_s) \prod_{s=1}^{j} \frac{c_s(q^{\epsilon_{2s-1}} w_{2s-1})}{c_s(w_{2s})} \\
&\times \frac{(1 - q/zw_1)}{D(z, w_1)} \prod_{s=1}^{j-1} (1 - q/w_{2s} w_{2s+1}) \prod_{s=1}^{j} (1 - qw_{2s-1} w_{2s}) \\
&\times \langle \lambda^a; \lambda^c | e^G e^{Q_{a2j+1}} e^{B_+(qz; \frac{1}{2}) + B_+(q/z; \frac{1}{2})} \qquad (6.39) \\
&\times \prod_{s=1}^{j} H_{\epsilon_{2s-1}}^{2s-1}(w_{2s-1}, w_{2s}) \prod_{s=1}^{j-1} H^{2s}(w_{2s}, w_{2s+1}) J^{2j}(w_{2j}) \qquad (j = 0, 1, 2, \cdots, N-1).
\end{aligned}$$

Here we take the integration contour $\widetilde{C}_s$ $(s = 1, 2, \cdots, 2N-1)$ to be a simple closed curve that encircles $w_s = 0$, $\sqrt{-1}$, $qw_{s-1}^{\pm 1}$ but not $-\sqrt{-1}$, $q^{-1}w_{s-1}^{\pm 1}$. Here we set $w_0 = z$. Here $c_j(w), f_1(z)$, and $g_j(w)$ are given by (6.14), (6.15), and (6.16), respectively.

6.2 Proof for $U_q(\widehat{gl}(1|1))$

First we show the case $U_q(\widehat{gl}(1|1))$ for simplicity. It is enough to study the case $K(z)=\frac{\varphi^{[2]}(z)}{\varphi^{[2]}(z^{-1})}\begin{pmatrix}\frac{1-rz}{1-r/z} & 0\\ 0 & 1\end{pmatrix}$.

Proof. We show the following two relations that are equivalent to the condition (6.1).

$$(1-rz)\varphi^{[2]}(z)\langle 0|e^G\phi_1^*(z^{-1}) = (1-r/z)\varphi^{[2]}(z^{-1})\langle 0|e^G\phi_1^*(z), \tag{6.40}$$

$$\varphi^{[2]}(z)\langle 0|e^G\phi_2^*(z^{-1}) = \varphi^{[2]}(z^{-1})\langle 0|e^G\phi_2^*(z). \tag{6.41}$$

• The relation (6.40) : From (6.39), we have

$$\begin{aligned}&(1-rz)\varphi^{[2]}(z)\langle 0|e^G\phi_1^*(z^{-1})\\ =\ &(1-rz)(1-r/z)\varphi^{[2]}(z)\varphi^{[2]}(z^{-1})e^{\sqrt{-1}\pi\lambda_2^a}\langle 0|e^Ge^{Q_{a^1}}e^{B_+(qz;\frac{1}{2})+B_+(q/z;\frac{1}{2})}\\ =\ &(1-r/z)\varphi^{[2]}(z^{-1})\langle 0|e^G\phi_1^*(z).\end{aligned} \tag{6.42}$$

We have shown the first relation (6.40).

• The relation (6.41) : From (6.38), we have LHS of (6.41) as follows.

$$\begin{aligned}\mathrm{LHS} &= -e^{\sqrt{-1}\pi(-\lambda_1^a+\lambda_2^a)}\varphi(z)\varphi(z^{-1})\langle 0|e^Ge^{Q_{a^2}-Q_{c^1}}e^{B_+(qz;\frac{1}{2})+B_+(q/z;\frac{1}{2})}\\ &\times \sum_{\epsilon=\pm}\epsilon\oint\frac{dw}{2\pi\sqrt{-1}w}\frac{(1-qz/w)(1-r/z)}{D(z,w)}g_1(w)q^{-\epsilon}c_1(q^\epsilon w)J_\epsilon^1(w).\end{aligned} \tag{6.43}$$

Because $\mathrm{RHS}=\mathrm{LHS}|_{z\to 1/z}$, we have

$$\begin{aligned}\mathrm{LHS}-\mathrm{RHS} &= qe^{\sqrt{-1}\pi(-\lambda_1^a+\lambda_2^a)}(z-z^{-1})\varphi(z)\varphi(z^{-1})\langle 0|e^Ge^{Q_{a^2}-Q_{c^1}}e^{B_+(qz;\frac{1}{2})+B_+(q/z;\frac{1}{2})}\\ &\times \sum_{\epsilon=\pm}\epsilon\oint\frac{dw}{2\pi\sqrt{-1}w}\frac{1}{D(z,w)}\left(q^\epsilon w+\frac{1}{q^\epsilon w}\right)J_\epsilon^1(w).\end{aligned} \tag{6.44}$$

Here we have used $D(z,w)=D(z^{-1},w)$ and $g_1(w)=\frac{1}{1-rw/q}$, $c_1(w)=1+w^2$. Taking into account the relation $\oint\frac{dw}{w}f(w)=\frac{1}{2}\oint\frac{dw}{w}(f(w)+f(w^{-1}))$, we have

$$\sum_{\epsilon=\pm}\epsilon\oint\frac{dw}{w}\frac{q^\epsilon w+(q^\epsilon w)^{-1}}{D(z,w)}J_\epsilon^1(w)=\sum_{\epsilon=\pm}\epsilon\oint\frac{dw}{2w}\frac{(q^\epsilon w)+(q^\epsilon w)^{-1}}{D(z,w)}(J_\epsilon^1(w)+J_\epsilon(w^{-1}))=0.$$

Here we have used $D(z,w)=D(z,w^{-1})$ and $J_\epsilon^1(w^{-1})=J_{-\epsilon}^1(w)$. Hence we have LHS = RHS. We have shown (6.41). **Q.E.D.**

6.3 Proof for $U_q(\widehat{gl}(N|N))$

In this Section we show the case of higher-rank $U_q(\widehat{gl}(N|N))$ $(N\geq 2)$. Upon the specialization $r=0$ for the case $L=0,M,R>0$ such that $M+R=2N$, we have $\mathrm{diag}\left(\frac{1-rz}{1-r/z},\cdots,\frac{1-rz}{1-r/z},1,\cdots,1\right)\Big|_{r=0}=id$. Hence, it is enough to study two cases $K(z)=\frac{\varphi^{[s]}(z)}{\varphi^{[s]}(z^{-1})}\overline{K}(z)$ $(s=2,3)$.

Proof. First we show

$$(1-rz)\varphi^{[2]}(z)\langle 0|e^G\phi_i^*(z^{-1}) = (1-r/z)\varphi^{[2]}(z^{-1})\langle 0|e^G\phi_i^*(z) \quad (1\leq i\leq M), \tag{6.45}$$

$$\varphi^{[2]}(z)\langle 0|e^G\phi_i^*(z^{-1}) = \varphi^{[2]}(z^{-1})\langle 0|e^G\phi_i^*(z) \qquad (M<i\leq 2N), \tag{6.46}$$

where $L=0, M, R>0$ such that $M+R=2N$.

• The relation (6.45) for $i=1$: From (6.39), we have LHS and RHS of (6.45) as follows.

$$\text{LHS} = (1-rz)(1-r/z)\varphi^{[2]}(z)\varphi^{[2]}(z^{-1})\langle 0|e^{G}e^{Q_{a^1}}e^{B_+(qz;\frac{1}{2})+B_+(q/z;\frac{1}{2})} = \text{RHS}. \tag{6.47}$$

We have shown (6.45) for $i=1$.

• The relation (6.45) for $i=2j$ $(j\geq 1)$: From (6.38), we have LHS $-$ RHS of (6.45) for $i=2j$ as follows.

$$\begin{aligned}
& \text{LHS} - \text{RHS} \\
= \;& q(q-q^{-1})^{j-1}(z-z^{-1})(1-rz)(1-r/z)\varphi^{[2]}(z)\varphi^{[2]}(z^{-1}) \\
\times \;& \sum_{\epsilon_1,\cdots,\epsilon_{2j-1}=\pm}\prod_{s=1}^{j}\epsilon_{2s-1}q^{-\epsilon_{2j-1}}\prod_{s=1}^{2j-1}\oint_{\widetilde{C}_s}\frac{dw_s}{2\pi\sqrt{-1}w_s}\prod_{s=1}^{2j-1}g_s(w_s)\frac{\prod_{s=1}^{j}c_s(q^{\epsilon_{2s-1}}w_{2s-1})}{\prod_{s=1}^{j-1}c_s(w_{2s})} \\
\times \;& \frac{\prod_{s=1}^{j-1}(1-q/w_{2s}w_{2s+1})(1-qw_{2s-1}w_{2s})}{w_1D(z,w_1)}\langle 0|e^{G}e^{Q_{a^{2j}-c^j}}e^{B_+(qz;\frac{1}{2})+B_+(q/z;\frac{1}{2})} \\
\times \;& \prod_{s=1}^{j-1}H^{2s-1}_{\epsilon_{2s-1}}(w_{2s-1},w_{2s})\prod_{s=1}^{j-1}H^{2s}(w_{2s},w_{2s+1})J^{2j-1}_{\epsilon_{2j-1}}(w_{2j-1}).
\end{aligned} \tag{6.48}$$

We focus our attention to the following integral relating to the variables $w_1, w_2, \cdots, w_{2j-1}$ in (6.48).

$$\begin{aligned}
& \sum_{\epsilon_1,\cdots,\epsilon_{2j-1}=\pm}\prod_{s=1}^{j}\epsilon_{2s-1}q^{-\epsilon_{2j-1}}\prod_{s=1}^{2j-1}\oint_{\widetilde{C}_s}\frac{dw_s}{2\pi\sqrt{-1}w_s}\prod_{s=1}^{2j-1}g_s(w_s)\frac{\prod_{s=1}^{j}c_s(q^{\epsilon_{2s-1}}w_{2s-1})}{\prod_{s=1}^{j-1}c_s(w_{2s})} \\
\times \;& \prod_{s=1}^{j-1}(1-q/w_{2s}w_{2s+1})(1-qw_{2s-1}w_{2s})\prod_{s=1}^{j-1}H^{2s-1}_{\epsilon_{2s-1}}(w_{2s-1},w_{2s})\prod_{s=1}^{j-1}H^{2s}(w_{2s},w_{2s+1}) \\
\times \;& J^{2j-1}_{\epsilon_{2j-1}}(w_{2j-1})\frac{1}{w_1D(z,w_1)}.
\end{aligned} \tag{6.49}$$

First we study (6.49) for $j=1$. We have

$$\sum_{\epsilon_1=\pm}\epsilon_1\oint\frac{dw_1}{w_1}\frac{q^{-\epsilon_1}c_1(q^{\epsilon_1}w_1)}{w_1}\frac{1}{D(z,w_1)}J^1_{\epsilon_1}(w_1)=0.$$

Now we have shown (6.49) for $j=1$.

Next we study (6.49) for $j\geq 2$. We focus our attention to the variable w_1. Using (6.31) and $D(z,w_1)=D(z,w_1^{-1})$, we have

$$\begin{aligned}
& \sum_{\epsilon_1=\pm}\epsilon_1\oint\frac{dw_1}{w_1}\frac{c_1(q^{\epsilon_1}w_1)}{w_1}\frac{(1-qw_1w_2)}{D(z,w_1)}H^1_{\epsilon_1}(w_1,w_2) \\
= \;& \oint\frac{dw_1}{w_1}q\left(qw_1+\frac{1}{qw_1}\right)\frac{(1-w_1^2)}{D(z,w_1)}H^1_+(w_1,w_2).
\end{aligned}$$

After calculation for w_1 we focus our attention to the variable w_2. Using (6.35) and $H^1_+(w_1^{-1},w_2)=H^1_+(w_1,w_2)$ we have

$$\begin{aligned}
& \oint\frac{dw_2}{w_2}(1-q/w_2w_3)\frac{g_2(w_2)}{c_2(w_2)}H^2(w_2,w_3)H^1_+(w_1,w_2) \\
= \;& -\frac{q}{2w_3}\oint\frac{dw_2}{w_2}(1-q/w_2w_3)\frac{(1-w_2^2)^2}{w_2(1+w_2^2)}H^2(w_2,w_3)H^1_+(w_1,w_2).
\end{aligned}$$

We get the following integral relating to the variables $w_3, w_4, \cdots, w_{2j-1}$.

$$\begin{aligned}
&\sum_{\epsilon_3,\cdots,\epsilon_{2j-1}=\pm}\prod_{s=2}^{j}\epsilon_{2s-1}q^{-\epsilon_{2j-1}}\prod_{s=3}^{2j-1}\oint_{\widetilde{C}_s}\frac{dw_s}{2\pi\sqrt{-1}w_s}\prod_{s=3}^{2j-1}g_s(w_s)\frac{\prod_{s=2}^{j}c_s(q^{\epsilon_{2s-1}}w_{2s-1})}{\prod_{s=2}^{j-1}c_s(w_{2s})}\\
\times\ &\prod_{s=2}^{j-1}(1-q/w_{2s}w_{2s+1})(1-qw_{2s-1}w_{2s})\prod_{s=2}^{j-1}H^{2s-1}_{\epsilon_{2s-1}}(w_{2s-1},w_{2s})H^{2s}(w_{2s},w_{2s+1})\\
\times\ &J^{2j-1}_{\epsilon_{2j-1}}(w_{2j-1})\frac{1}{w_3}H^2(w_2,w_3). \qquad (6.50)
\end{aligned}$$

The structure of (6.49) and (6.50) are the same except for their sizes and minor difference between $D(z,w_1)$ and $H^2(w_2,w_3)$. We note that $D(z,w)$ and $H^2(z,w)$ are invariant under $w\to w^{-1}$. Using (6.31) and (6.36) we calculate the variables $w_3, w_4, \cdots, w_{2j-2}$ iteratively. Then we get the same relation as (6.49) for $j=1$, that we have already shown.

• The relation (6.45) for $i=2j+1$ $(j\geq 1)$: From (6.39), we have LHS − RHS of (6.45) for $i=2j+1$ as follows.

$$\begin{aligned}
&\text{LHS}-\text{RHS}\\
=\ &-q(q-q^{-1})^j(z-z^{-1})(1-rz)(1-r/z)\varphi^{[2]}(z)\varphi^{[2]}(z^{-1})\\
\times\ &\sum_{\epsilon_1,\cdots,\epsilon_{2j-1}=\pm}\prod_{s=1}^{j}\epsilon_{2s-1}\prod_{s=1}^{2j}\oint_{\widetilde{C}_s}\frac{dw_s}{2\pi\sqrt{-1}w_s}\prod_{s=1}^{2j}g_s(w_s)\prod_{s=1}^{j}\frac{c_s(q^{\epsilon_{2s-1}}w_{2s-1})}{c_s(w_{2s})}\\
\times\ &\frac{\prod_{s=1}^{j-1}(1-q/w_{2s}w_{2s+1})\prod_{s=1}^{j}(1-qw_{2s-1}w_{2s})}{w_1D(z,w_1)}\langle 0|e^{G}e^{Q_{a^{2j}}}e^{B_+(qz;\frac{1}{2})+B_+(q/z;\frac{1}{2})}\\
\times\ &\prod_{s=1}^{j}H^{2s-1}_{\epsilon_{2s-1}}(w_{2s-1},w_{2s})\prod_{s=1}^{j-1}H^{2s}(w_{2s},w_{2s+1})J^{2j}(w_{2j}). \qquad (6.51)
\end{aligned}$$

We focus our attention to the following integral relating to the variables $w_1, w_2, \cdots, w_{2j-1}$ in (6.51).

$$\begin{aligned}
&\sum_{\epsilon_1,\cdots,\epsilon_{2j-1}=\pm}\prod_{s=1}^{j}\epsilon_{2s-1}\prod_{s=1}^{2j}\oint_{\widetilde{C}_s}\frac{dw_s}{2\pi\sqrt{-1}w_s}\prod_{s=1}^{2j}g_s(w_s)\prod_{s=1}^{j}\frac{c_s(q^{\epsilon_{2s-1}}w_{2s-1})}{c_s(w_{2s})}\\
\times\ &\prod_{s=1}^{j-1}(1-q/w_{2s}w_{2s+1})\prod_{s=1}^{j}(1-qw_{2s-1}w_{2s})\prod_{s=1}^{j}H^{2s-1}_{\epsilon_{2s-1}}(w_{2s-1},w_{2s})\prod_{s=1}^{j-1}H^{2s}(w_{2s},w_{2s+1})\\
\times\ &J^{2j}(w_{2j})\frac{1}{w_1D(z,w_1)}. \qquad (6.52)
\end{aligned}$$

First we study (6.52) for $j=1$. We calculate the variable w_1 using (6.31). Then we have the following integral relating to w_2.

$$\oint\frac{dw_2}{w_2}\frac{g_2(w_2)}{c_2(w_2)}J^2(w_2)=0,$$

where we have used $J^2(w_2^{-1})=J^2(w_2)$. Now we have shown (6.52) for $j=1$.

Next we study (6.52) for $j\geq 2$. We focus our attention to two variables w_1, w_2. Using relations (6.31) and (6.36) in the same way as the above, we have the following integral relating to $w_3, w_4, \cdots, w_{2j}$.

$$\sum_{\epsilon_3,\cdots,\epsilon_{2j-1}=\pm}\prod_{s=2}^{j}\epsilon_{2s-1}\prod_{s=3}^{2j}\oint_{\widetilde{C}_s}\frac{dw_s}{2\pi\sqrt{-1}w_s}\prod_{s=3}^{2j}g_s(w_s)\prod_{s=2}^{j}\frac{c_s(q^{\epsilon_{2s-1}}w_{2s-1})}{c_s(w_{2s})}$$

$$\times \quad \prod_{s=2}^{j-1}(1-q/w_{2s}w_{2s+1})\prod_{s=2}^{j}(1-qw_{2s-1}w_{2s})\prod_{s=2}^{j}H^{2s-1}_{\epsilon_{2s-1}}(w_{2s-1},w_{2s})\prod_{s=2}^{j-1}H^{2s}(w_{2s},w_{2s+1})$$
$$\times \quad J^{2j}(w_{2j})\frac{1}{w_3}H^2(w_2,w_3). \tag{6.53}$$

The structure of (6.52) and (6.53) are the same except for their sizes and minor difference between $D(z,w_1)$ and $H^2(w_2,w_3)$. Using (6.31) and (6.36) we calculate the variables $w_3,w_4,\cdots,w_{2j-2}$ iteratively. Then we get the same relation as (6.52) for $j=1$, that we have already shown.

• The relation (6.46) for $i=2j$: From (6.38), the relation (6.46) is reduced to the following relation.

$$\sum_{\epsilon_1,\cdots,\epsilon_{2j-1}=\pm}\prod_{s=1}^{j}\epsilon_{2s-1}q^{-\epsilon_{2j-1}}\prod_{s=1}^{2j-1}\oint_{\widetilde{C}_s}\frac{dw_s}{2\pi\sqrt{-1}w_s}\prod_{s=1}^{2j-1}g_s(w_s)\frac{\prod_{s=1}^{j}c_s(q^{\epsilon_{2s-1}}w_{2s-1})}{\prod_{s=1}^{j-1}c_s(w_{2s})}$$
$$\times \quad \prod_{s=1}^{j-1}(1-q/w_{2s}w_{2s+1})(1-qw_{2s-1}w_{2s})\prod_{s=1}^{j-1}H^{2s-1}_{\epsilon_{2s-1}}(w_{2s-1},w_{2s})\prod_{s=1}^{j-1}H^{2s}(w_{2s},w_{2s+1})$$
$$\times \quad (1-rw_1/q)\times J^{2j-1}_{\epsilon_{2j-1}}(w_{2j-1})\frac{1}{w_1D(z,w_1)}=0. \tag{6.54}$$

Using (6.31), (6.32) and (6.36), the relation (6.54) is shown in the same way as the above.

• The relation (6.46) for $i=2j+1$: From (6.39) the relation (6.46) is reduced to the following relation.

$$\sum_{\epsilon_1,\cdots,\epsilon_{2j-1}=\pm}\prod_{s=1}^{j}\epsilon_{2s-1}\prod_{s=1}^{2j}\oint_{\widetilde{C}_s}\frac{dw_s}{2\pi\sqrt{-1}w_s}\prod_{s=1}^{2j}g_s(w_s)\prod_{s=1}^{j}\frac{c_s(q^{\epsilon_{2s-1}}w_{2s-1})}{c_s(w_{2s})}$$
$$\times \quad \prod_{s=1}^{j-1}(1-q/w_{2s}w_{2s+1})\prod_{s=1}^{j}(1-qw_{2s-1}w_{2s})\prod_{s=1}^{j}H^{2s-1}_{\epsilon_{2s-1}}(w_{2s-1},w_{2s})\prod_{s=1}^{j-1}H^{2s}(w_{2s},w_{2s+1})$$
$$\times \quad (1-rw_1/q)\times J^{2j}(w_{2j})\frac{1}{w_1D(z,w_1)}=0. \tag{6.55}$$

Using (6.32), (6.35) and (6.36), the relation (6.55) is shown in the same way as the above.

Next we show For $L,M,R>0$ such that $L+M+R=2N$, we have

$$z\varphi^{[3]}(z)\langle 0|e^G\phi_i^*(z^{-1}) = z^{-1}\varphi^{[3]}(z^{-1})\langle 0|e^G\phi_i^*(z) \qquad (1\le i\le L), \tag{6.56}$$
$$(1-rz)\varphi^{[3]}(z)\langle 0|e^G\phi_i^*(z^{-1}) = (1-r/z)\varphi^{[3]}(z^{-1})\langle 0|e^G\phi_i^*(z) \ (L<i\le L+M), \tag{6.57}$$
$$\varphi^{[3]}(z)\langle 0|e^G\phi_i^*(z^{-1}) = \varphi^{[3]}(z^{-1})\langle 0|e^G\phi_i^*(z) \qquad (L+M<i\le 2N), \tag{6.58}$$

where $L,M,R>0$ such that $L+M+R=2N$.

• The relation (6.56) for $j=2i$: From (6.38) the relation (6.56) is reduced to the following relation.

$$\sum_{\epsilon_1,\cdots,\epsilon_{2j-1}=\pm}\prod_{s=1}^{j}\epsilon_{2s-1}q^{-\epsilon_{2j-1}}\prod_{s=1}^{2j-1}\oint_{\widetilde{C}_s}\frac{dw_s}{2\pi\sqrt{-1}w_s}\prod_{s=1}^{2j-1}g_s(w_s)\frac{\prod_{s=1}^{j}c_s(q^{\epsilon_{2s-1}}w_{2s-1})}{\prod_{s=1}^{j-1}c_s(w_{2s})}$$
$$\times \quad \prod_{s=1}^{j-1}(1-q/w_{2s}w_{2s+1})(1-qw_{2s-1}w_{2s})\prod_{s=1}^{j-1}H^{2s-1}_{\epsilon_{2s-1}}(w_{2s-1},w_{2s})\prod_{s=1}^{j-1}H^{2s}(w_{2s},w_{2s+1})$$
$$\times \quad J^{2j-1}_{\epsilon_{2j-1}}(w_{2j-1})\frac{1}{w_1D(z,w_1)}=0. \tag{6.59}$$

Using (6.31) and (6.36), the relation (6.59) is shown in the same way as the above.

• The relation (6.56) for $j = 2i+1$: From (6.39) the relation (6.56) is reduced to the following relation.

$$\begin{aligned}
&\sum_{\epsilon_1,\cdots,\epsilon_{2j-1}=\pm}\prod_{s=1}^{j}\epsilon_{2s-1}\prod_{s=1}^{2j}\oint_{\widetilde{C}_s}\frac{dw_s}{2\pi\sqrt{-1}w_s}\prod_{s=1}^{2j}g_s(w_s)\prod_{s=1}^{j}\frac{c_s(q^{\epsilon_{2s-1}}w_{2s-1})}{c_s(w_{2s})}\\
\times\ &\prod_{s=1}^{j-1}(1-q/w_{2s}w_{2s+1})\prod_{s=1}^{j}(1-qw_{2s-1}w_{2s})\prod_{s=1}^{j}H^{2s-1}_{\epsilon_{2s-1}}(w_{2s-1},w_{2s})\prod_{s=1}^{j-1}H^{2s}(w_{2s},w_{2s+1})\\
\times\ &J^{2j}(w_{2j})\frac{1}{w_1D(z,w_1)}=0. \qquad (6.60)
\end{aligned}$$

Using (6.31) and (6.36), the relation (6.60) is shown in the same way as the above.

• The relation (6.57) for $j = 2i$: From (6.38) the relation (6.57) is reduced to the following relation.

$$\begin{aligned}
&\sum_{\epsilon_1,\cdots,\epsilon_{2j-1}=\pm}\prod_{s=1}^{j}\epsilon_{2s-1}q^{-\epsilon_{2j-1}}\prod_{s=1}^{2j-1}\oint_{\widetilde{C}_s}\frac{dw_s}{2\pi\sqrt{-1}w_s}\prod_{s=1}^{2j-1}g_s(w_s)\frac{\prod_{s=1}^{j}c_s(q^{\epsilon_{2s-1}}w_{2s-1})}{\prod_{s=1}^{j-1}c_s(w_{2s})}\\
\times\ &\prod_{s=1}^{j-1}(1-q/w_{2s}w_{2s+1})(1-qw_{2s-1}w_{2s})\prod_{s=1}^{j-1}H^{2s-1}_{\epsilon_{2s-1}}(w_{2s-1},w_{2s})\prod_{s=1}^{j-1}H^{2s}(w_{2s},w_{2s+1})\\
\times\ &(1-w_1/qr)\times J^{2j-1}_{\epsilon_{2j-1}}(w_{2j-1})\frac{1}{w_1D(z,w_1)}=0. \qquad (6.61)
\end{aligned}$$

Using (6.31), (6.32), (6.35), (6.36), the relation (6.61) is shown in the same way as the above.

• The relation (6.57) for $j = 2i+1$: From (6.39) the relation (6.57) is reduced to the following relation.

$$\begin{aligned}
&\sum_{\epsilon_1,\cdots,\epsilon_{2j-1}=\pm}\prod_{s=1}^{j}\epsilon_{2s-1}\prod_{s=1}^{2j}\oint_{\widetilde{C}_s}\frac{dw_s}{2\pi\sqrt{-1}w_s}\prod_{s=1}^{2j}g_s(w_s)\prod_{s=1}^{j}\frac{c_s(q^{\epsilon_{2s-1}}w_{2s-1})}{c_s(w_{2s})}\\
\times\ &\prod_{s=1}^{j-1}(1-q/w_{2s}w_{2s+1})\prod_{s=1}^{j}(1-qw_{2s-1}w_{2s})\prod_{s=1}^{j}H^{2s-1}_{\epsilon_{2s-1}}(w_{2s-1},w_{2s})\prod_{s=1}^{j-1}H^{2s}(w_{2s},w_{2s+1})\\
\times\ &(1-w_1/qr)\times J^{2j}(w_{2j})\frac{1}{w_1D(z,w_1)}=0. \qquad (6.62)
\end{aligned}$$

Using (6.31), (6.32), (6.35), (6.36), the relation (6.62) is shown in the same way as the above.

• The relation (6.58) for $j = 2i$: From (6.38) the relation (6.58) is reduced to the following relation.

$$\begin{aligned}
&\sum_{\epsilon_1,\cdots,\epsilon_{2j-1}=\pm}\prod_{s=1}^{j}\epsilon_{2s-1}q^{-\epsilon_{2j-1}}\prod_{s=1}^{2j-1}\oint_{\widetilde{C}_s}\frac{dw_s}{2\pi\sqrt{-1}w_s}\prod_{s=1}^{2j-1}g_s(w_s)\frac{\prod_{s=1}^{j}c_s(q^{\epsilon_{2s-1}}w_{2s-1})}{\prod_{s=1}^{j-1}c_s(w_{2s})}\\
\times\ &\prod_{s=1}^{j-1}(1-q/w_{2s}w_{2s+1})(1-qw_{2s-1}w_{2s})\prod_{s=1}^{j-1}H^{2s-1}_{\epsilon_{2s-1}}(w_{2s-1},w_{2s})\prod_{s=1}^{j-1}H^{2s}(w_{2s},w_{2s+1})\\
\times\ &J^{2j-1}_{\epsilon_{2j-1}}(w_{2j-1})\frac{1}{D(z,w_1)}=0. \qquad (6.63)
\end{aligned}$$

Using (6.31), (6.32), (6.33), (6.34), (6.35), (6.36), (6.37), the relation (6.63) is shown in the same way as the above.

• The relation (6.58) for $j = 2i+1$: From (6.39) the relation (6.58) is reduced to the following relation.

$$\sum_{\epsilon_1,\cdots,\epsilon_{2j-1}=\pm}\prod_{s=1}^{j}\epsilon_{2s-1}\prod_{s=1}^{2j}\oint_{\widetilde{C}_s}\frac{dw_s}{2\pi\sqrt{-1}w_s}\prod_{s=1}^{2j}g_s(w_s)\prod_{s=1}^{j}\frac{c_s(q^{\epsilon_{2s-1}}w_{2s-1})}{c_s(w_{2s})}$$

$$\times \quad \prod_{s=1}^{j-1}(1-q/w_{2s}w_{2s+1})\prod_{s=1}^{j}(1-qw_{2s-1}w_{2s})\prod_{s=1}^{j}H^{2s-1}_{\epsilon_{2s-1}}(w_{2s-1},w_{2s})\prod_{s=1}^{j-1}H^{2s}(w_{2s},w_{2s+1})$$
$$\times \quad J^{2j}(w_{2j})\frac{1}{D(z,w_1)}=0. \tag{6.64}$$

Using (6.32), (6.32), (6.33), (6.34), (6.35), (6.36), (6.37), the relation (6.64) is shown in the same way as the above. **Q.E.D.**

7 Concluding remarks

In the present paper, the $U_q(\widehat{gl}(N|N))$-analog of the half-infinite t-J model with a diagonal boundary is considered by the VOA. A bosonization of the boundary state ${}_B\langle i|$ satisfying ${}_B\langle i|T_B(z) = {}_B\langle i|$ is constructed by acting exponentially with the bosonic operator G on the highest-weight vector $\langle\Lambda_i|$ in the integrable highest-weight module $V^*(\Lambda_i)$:

$${}_B\langle i| = \langle\Lambda_i|e^G\cdot Pr,$$

where Pr is the projection operator. In the present paper we focus our attention to $V^*(\Lambda_i)$ for $i=0$ and $i=2N-1$. The boundary states in $V^*(\Lambda_i)$ for $i=1,2$ can be constructed in the same way.

For more general integrable boundary conditions, bosonization of the boundary states is an open problem. Here we study non-diagonal solutions of the boundary Yang-Baxter equation associated with the quantum superalgebra $U_q(\widehat{gl}(N|N))$. Let us set $V^{(+)}=\oplus_{j=1}^N\mathbf{C}v_{2j-1}$ and $V^{(-)}=\oplus_{j=1}^N\mathbf{C}v_{2j}$. We have $V=V^{(+)}\oplus V^{(-)}$. Let us study the boundary Yang-Baxter equation in $V\otimes V$ with the R-matrix (2.4).

$$\overline{K}_2(z_2)\overline{R}_{2,1}(z_1z_2)\overline{K}_1(z_1)\overline{R}_{1,2}(z_1/z_2)=\overline{R}_{2,1}(z_1/z_2)\overline{K}_1(z_1)\overline{R}_{1,2}(z_1z_2)\overline{K}_2(z_2).$$

Let us set $\overline{K}(z)v_i=\sum_{j=1}^{2N}v_j\overline{K}^i_j(z)$. For $a\neq b$ we have

$$\left(\overline{K}_2(z_2)\overline{R}_{2,1}(z_1z_2)\overline{K}_1(z_1)\overline{R}_{1,2}(z_1/z_2)\right)^{b,b}_{a,a}=(-1)^{([v_a]+[v_b])[v_a]}K^b_a(z_2)R^{b,a}_{b,a}(z_1z_2)K^b_a(z_1)R^{b,b}_{b,b}(z_1/z_2),$$
$$\left(\overline{R}_{2,1}(z_1/z_2)\overline{K}_1(z_1)\overline{R}_{1,2}(z_1z_2)\overline{K}_2(z_2)\right)^{b,b}_{a,a}=(-1)^{([v_a]+[v_b])[v_a]}R^{a,a}_{a,a}(z_1/z_2)K^b_a(z_1)R^{b,a}_{b,a}(z_1z_2)K^b_a(z_2).$$

Hence we have the following necessary and sufficient condition upon the assumption $\overline{K}^b_a(z)\neq 0$.

$$(\mathrm{LHS})^{b,b}_{a,a}=(\mathrm{RHS})^{b,b}_{a,a}\Longleftrightarrow\overline{R}^{a,a}_{a,a}(z)=\overline{R}^{b,b}_{b,b}(z). \tag{7.1}$$

Because $\overline{R}^{a,a}_{a,a}(z)\neq\overline{R}^{b,b}_{b,b}(z)$ for $a\neq b$ $(mod.2)$, we have

$$a\neq b\ (mod.2)\Longrightarrow\overline{K}^b_a(z)=0. \tag{7.2}$$

Hence the boundary Yang-Baxter equation in $V\otimes V$ associated with $U_q(\widehat{gl}(N|N))$ splits into two boundary Yang-Baxter equations in $V^{(\pm)}\otimes V^{(\pm)}$ associated with $U_q(\widehat{sl}(N))$. Hence we get the following procedure to construct non-diagonal solution of the boundary Yang-Baxter equation associated with $U_q(\widehat{gl}(N|N))$. *(i) First, we give a diagonal solution in* $\mathrm{End}(V)$ *associated with* $U_q(\widehat{gl}(N|N))$. *(ii) Next, we extend it to non-diagonal by using two boundary Yang-Baxter equation in* $V^{(\pm)}\otimes V^{(\pm)}$ *associated with* $U_q(\widehat{sl}(N))$. The same argument holds for $U_q(\widehat{sl}(M|N))$ $(M\neq N)$.

We study the $U_q(\widehat{gl}(N|N))$-analog of the half-infinite $t-J$ model with a non-diagonal boundary.

• $U_q(\widehat{gl}(1|1))$: From the argument above, there is no non-diagonal solution of the boundary Yang-Baxter equation.

• $U_q(\widehat{gl}(2|2))$: Let $D(z) \in \mathrm{End}(V)$ be a diagonal solution. Let $O^{(\pm)}(z)$ in $\mathrm{End}(V^{(\pm)})$ be two off-diagonal parts of non-diagonal solutions associated with $U_q(\widehat{sl}(2))$. Then the following $K(z)$ gives a non-diagonal solution associated with $U_q(\widehat{gl}(2|2))$.

$$K(z) = D(z) + O^{(+)}(z) + O^{(-)}(z). \tag{7.3}$$

Here $O^{(\pm)}(z) \in \mathrm{End}(V^{(\pm)})$ are understood as operators in $\mathrm{End}(V)$. For *triangular boundary condition*, we have progress on the boundary state associated with $U_q(\widehat{sl}(2))$ [33, 34, 35]. We will report on the boundary state of $U_q(\widehat{gl}(2|2))$ spin chain with a non-diagonal boundary in another paper.

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A The quantum superalgebra $U_q(\widehat{gl}(N|N))$

In this Appendix we give the definition of the quantum superalgebra $U_q(\widehat{gl}(N|N))$ [16, 30]. We introduce the enlarged Cartan matrix $A = (A_{i,j})_{0\le i,j\le 2N}$ for $\widehat{gl}(N|N)$ as follows. Let $\{\alpha_i | i = 0, 1, 2, \cdots, 2N-1\}$ a set of simple roots of the quantum superalgebra $\widehat{sl}(N|N)$ given by

$$\alpha_0 = \delta - \epsilon_1 + \epsilon_{2N}, \quad \alpha_j = \epsilon_j - \epsilon_{j+1} \quad (j = 1, 2, \cdots, 2N-1), \tag{A.1}$$

where we have set $\{\epsilon_j\}_{1\le j\le 2N}$, δ satisfying $(\delta|\delta) = 0, (\delta|e_j) = 0$, and $(\epsilon_i|\epsilon_j) = (-1)^{j+1}\delta_{i,j}$. The enlarged Cartan matrix $(A_{i,j})_{0\le i,j\le 2N-1}$ for $\widehat{sl}(N|N)$ is given by $A_{i,j} = (\alpha_i|\alpha_j)$ $(0 \le i, j \le 2N-1)$. We extend $\widehat{sl}(N|N)$ by adding the element $\alpha_{2N} = \sum_{j=1}^{2N} \epsilon_j$. The enlarged Cartan matrix A for $\widehat{gl}(N|N)$ is given by $A_{i,j} = (\alpha_i|\alpha_j)$ $(0 \le i, j \le 2N)$. For $N = 1$ we have

$$A = (A)_{0\le i,j\le 2} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 2 \\ -2 & 2 & 0 \end{pmatrix}. \tag{A.2}$$

For $N = 2, 3, 4, \cdots$, we have

$$A = (A_{i,j})_{0\le i,j\le 2N} = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 1 & -2 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 2 \\ 0 & 1 & 0 & -1 & \cdots & \cdots & -2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 1 & 0 & -1 & -2 \\ 1 & 0 & \cdots & 0 & -1 & 0 & 2 \\ -2 & 2 & -2 & \cdots & -2 & 2 & 0 \end{pmatrix}. \tag{A.3}$$

Note that the Cartan matrix $\bar{A} = (A_{i,j})_{1\le i,j\le 2N}$ is invertible.

Definition A.1 *[16, 30] The quantum superalgebra $U_q(\widehat{gl}(N|N))$ $(N = 1, 2, 3, \cdots)$ is generated by the Chevalley generators $\{e_i, f_i, h_j, d | i = 0, 1, \cdots, 2N-1, j = 0, 1, 2, \cdots, 2N\}$. The $\mathbf{Z}_2$-grading of the Chevalley generators is $[e_i] = [f_i] = 1$ $(i = 0, 1, 2, \cdots, 2N-1)$ and zero otherwise. The defining relations are*

$$
\begin{aligned}
&[h_i, h_j] = 0 \quad (0 \le i, j \le 2N),\\
&[h_i, e_j] = A_{i,j} e_j, \quad [d, e_j] = \delta_{j,0} e_j \quad (0 \le i \le 2N, 0 \le j \le 2N-1),\\
&[h_i, f_j] = -A_{i,j} f_j, \quad [d, f_j] = -\delta_{j,0} f_j \quad (0 \le i \le 2N, 0 \le j \le 2N-1),\\
&[e_i, f_j] = \delta_{i,j} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} \quad (1 \le i, j \le 2N-1),\\
&[e_i, e_j] = [f_i, f_j] = 0 \quad \text{for } A_{i,j} = 0,\\
&[[e_0, e_1]_{q^{-1}}, [e_0, e_{2N-1}]_q] = 0,\\
&[[e_j, e_{j-1}]_{q^{(-1)^j}}, [e_j, e_{j+1}]_{q^{(-1)^{j+1}}}] = 0 \quad (1 \le j \le 2N-2),\\
&[[e_{2N-1}, e_{2N-2}]_{q^{-1}}, [e_{2N-1}, e_0]_q] = 0,\\
&[[f_0, f_1]_{q^{-1}}, [f_0, f_{2N-1}]_q] = 0,\\
&[[f_j, f_{j-1}]_{q^{(-1)^j}}, [f_j, f_{j+1}]_{q^{(-1)^{j+1}}}] = 0 \quad (1 \le j \le 2N-2),\\
&[[f_{2N-1}, f_{2N-2}]_{q^{-1}}, [f_{2N-1}, f_0]_q] = 0.
\end{aligned} \tag{A.4}
$$

Here and throughout this paper, we use

$$
[a, b]_x = ab - (-1)^{[a][b]} xba. \tag{A.5}
$$

The multiplication rule on the tensor products is $\mathbf{Z}_2$-graded.

$$
(a \otimes b)(a' \otimes b') = (-1)^{[b][a']} (aa' \otimes bb'). \tag{A.6}
$$

The quantum superalgebra $U_q(\widehat{gl}(N|N))$ has the $\mathbf{Z}_2$-graded Hopf algebra structure with the following coproduct Δ, counit ϵ, and antipode S.

$$
\begin{aligned}
&\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i,\\
&\Delta(e_j) = e_j \otimes 1 + q^{h_j} \otimes e_j, \quad \Delta(f_j) = f_j \otimes q^{-h_j} + 1 \otimes f_j,\\
&\epsilon(e_j) = \epsilon(f_j) = \epsilon(h_i) = 0,\\
&S(e_i) = -q^{-h_i} e_i, \quad S(f_j) = -f_j q^{h_j}, \quad S(h_i) = -h_i.
\end{aligned} \tag{A.7}
$$

where $i = 0, 1, 2, \cdots, 2N$ and $j = 0, 1, 2, \cdots, 2N-1$. The coproduct Δ satisfies algebra automorphism $\Delta(ab) = \Delta(a)\Delta(b)$ and the antipode satisfies $\mathbf{Z}_2$-graded algebra anti-automorphism $S(ab) = (-1)^{[a][b]} S(b) S(a)$.

We denote by $\mathcal{H} = \oplus_{j=0}^{2N} \mathbf{C} h_j \oplus \mathbf{C} d$ the extended Cartan subalgebra. Let $\{\Lambda_0, \Lambda_1, \cdots, \Lambda_{2N}, \delta\}$ be the dual basis with Λ_i being fundamental weight. Explicitly

$$
(\Lambda_i | h_j) = \delta_{i,j}, \quad (\Lambda_i | \delta) = 0, \quad (d | h_j) = 0, \quad (\delta | d) = 1 \quad (0 \le i, j \le 2N). \tag{A.8}
$$

The quantum superalgebra $U_q(\widehat{gl}(N|N))$ has another realization that we call the Drinfeld realization.

Theorem A.2 *[16, 30] The quantum superalgebra $U_q(\widehat{gl}(N|N))$ $(N=1,2,3,\cdots)$ is generated by the Drinfeld generators $\{X_m^{\pm,i}, H_m^j, c, d | m\in\mathbf{Z}, i=1,2,\cdots,2N-1, j=1,2,\cdots,2N\}$. The $\mathbf{Z}_2$-grading of the Drinfeld generators is $[X_m^{\pm,i}]=1$ $(i=1,2,\cdots,2N-1)$ and zero otherwise. The defining relations are*

$$
\begin{aligned}
&[c,a]=[d,H_0^j]=[H_0^i,H_n^j]=0 \quad (a\in U_q(\widehat{gl}(N|N)), 1\le i,j\le 2N, n\in\mathbf{Z}),\\
&[H_m^i,H_n^j]=\delta_{m+n,0}\frac{[A_{i,j}m]_q[cm]_q}{m} \qquad (m,n\in\mathbf{Z}_{\neq 0}, 1\le i,j\le 2N),\\
&[H_0^i,X_j^{\pm}(z)]=\pm A_{i,j}X_j^{\pm}(z) \quad (1\le i\le 2N, 1\le j\le 2N-1),\\
&[H_m^i,X_j^{\pm}(z)]=\pm\frac{[A_{i,j}m]_q}{m}q^{\mp c|m|}z^m X_j^{\pm}(z) \quad (1\le i\le 2N, 1\le j\le 2N-1, m\in\mathbf{Z}_{\neq 0}),\\
&[X_i^+(z_1),X_j^-(z_2)]\\
=\ &\frac{\delta_{i,j}}{(q-q^{-1})z_1z_2}\left(\delta(q^{-c}z_1/z_2)\psi_i^+(q^{\frac{c}{2}}z_2)-\delta(q^c z_1/z_2)\psi_i^-(q^{-\frac{c}{2}}z_2)\right) \quad (1\le i,j\le 2N-1),\\
&[X_i^{\pm}(z_1),X_j^{\pm}(z_2)]=0 \quad (A_{i,j}=0),\\
&(z_1-q^{\pm A_{i,j}}z_2)X_i^{\pm}(z_1)X_j^{\pm}(z_2)=(q^{\pm A_{j,i}}z_1-z_2)X_j^{\pm}(z_2)X_i^{\pm}(z_1) \quad (A_{i,j}\neq 0),\\
&\left[[X_j^{\pm}(z_1),X_{j-1}^{\pm}(z_2)]_{q^{(-1)^j}},[X_j^{\pm}(z_3),X_{j+1}^{\pm}(z_4)]_{q^{(-1)^{j+1}}}\right]\\
&+\left[[X_j^{\pm}(z_3),X_{j-1}^{\pm}(z_2)]_{q^{(-1)^j}},[X_j^{\pm}(z_1),X_{j+1}^{\pm}(z_4)]_{q^{(-1)^{j+1}}}\right]=0 \quad (2\le j\le 2N-2).
\end{aligned}
\tag{A.9}
$$

Here we have used the generating functions

$$
\begin{aligned}
&X_j^{\pm}(z)=\sum_{m\in\mathbf{Z}}X_m^{\pm,j}z^{-m-1} \quad (1\le j\le 2N-1),\\
&\psi_j^{\pm}(z)=q^{\pm H_0^j}\exp\left(\pm(q-q^{-1})\sum_{m=1}^{\infty}H_m^j z^{\mp m}\right) \quad (1\le j\le 2N-1).
\end{aligned}
\tag{A.10}
$$

The Chevalley generators are related to the Drinfeld generators as follows.

$$
\begin{aligned}
&h_0=c-\sum_{j=1}^{2N-1}H_0^j, \quad h_i=H_0^i, \quad e_j=X_0^{+,j}, \quad f_j=X_0^{-,j} \quad (1\le i\le 2N, 1\le j\le 2N-1),\\
&e_0=[X_0^{-,2N-1},[X_0^{-,2N-2},\cdots,[X_0^{-,3},[X_0^{-,2},X_1^{-,1}]_q]_{q^{-1}}\cdots]_q]_{q^{-1}}q^{h_0-c},\\
&f_0=(-1)^N q^{c-h_0}[[\cdots[[X_{-1}^{+,1},X_0^{+,2}]_{q^{-1}},X_0^{+,3}]_q,\cdots,X_0^{+,2N-2}]_{q^{-1}},X_0^{+,2N-1}]_q.
\end{aligned}
\tag{A.11}
$$

B The boundary state in $V(\Lambda_{2N-1})$

In this Appendix we give a bosonization of the boundary state in the integrable highest-weight module $V(\Lambda_{2N-1})$. Let $L>0$ and $M,R\ge 0$ $(L+M+R=2N)$. The boundary K-matrix $K(z)$ is given by $K(z)=\frac{\varphi(z)}{\varphi(z^{-1})}\overline{K}(z)$. Here the matrix $\overline{K}(z)$ is given by

$$
\overline{K}(z)=\mathrm{diag}\left(\underbrace{1,\cdots,1}_{L},\underbrace{\frac{1-r/z}{1-rz},\cdots,\frac{1-r/z}{1-rz}}_{M},\underbrace{z^{-2},\cdots,z^{-2}}_{R}\right).
\tag{B.1}
$$

Here the function $\varphi(z)$ is given by

$$\varphi(z) = \begin{cases} \varphi^{[1]}(z) & (L>0, M=R=0) \\ \varphi^{[2]}(z) & (L,M>0, R=0) \\ \varphi^{[3]}(z) & (L,M,R>0) \end{cases}, \tag{B.2}$$

where we have set

$$\varphi^{[1]}(z) = (1-qz^2)^{\frac{1-N}{2N}}(1+z^2)^{\frac{1-2N}{2N}}, \tag{B.3}$$

$$\varphi^{[2]}(z) = \varphi^{[1]}(z) \times \begin{cases} (1-rz/q)^{\frac{1-2N}{2N}} & (L=\text{odd}) \\ 1 & (L=\text{even}) \end{cases}, \tag{B.4}$$

$$\varphi^{[3]}(z) = \varphi^{[2]}(z) \times \begin{cases} (1-qz/r)^{\frac{1-2N}{2N}} & (L=\text{odd}, M=\text{even}) \\ (1-z/rq)^{\frac{1-2N}{2N}} & (L=\text{even}, M=\text{odd}) \\ 1 & (\text{otherwise}) \end{cases}. \tag{B.5}$$

Theorem B.1 *The boundary state ${}_B\langle 2N-1| \in V^*(\Lambda_{2N-1})$ is realized as follows.*

$${}_B\langle 2N-1| = \langle \Lambda_{2N-1}|e^G \cdot Pr, \tag{B.6}$$

where the highest-weight vector $\langle \Lambda_{2N-1}|$ is given in (5.6). Here the projection operator Pr is given by $Pr = \prod_{j=1}^{N-1} \eta_0^j \prod_{j=1}^{N-1} \xi_0^j \cdot \xi_0^N \eta_0^N$. Here the bosonic operator G is given in (5.9), where δ_m^j and β_m^i are given as follows.

$$\delta_m^j = \begin{cases} \frac{q^{-m}}{[m]_q}(1-2(-1)^{\frac{m}{2}})\theta_m & (1 \le j \le N-1) \\ \frac{q^{-m}}{[m]_q}\theta_m & (j=N) \end{cases}, \tag{B.7}$$

$$\beta_m^j = \begin{cases} \beta_m^{[1],j} & (L>0, M=R=0) \\ \beta_m^{[2],j} & (L,M>0, R=0) \\ \beta_m^{[3],j} & (L,M,R>0) \end{cases} \quad (i=1,2,\cdots,2N), \tag{B.8}$$

where we have set

$$\beta_m^{[1],2s-1} = \frac{2(1-s)}{[m]_q} q^{-\frac{3}{2}m}\theta_m \quad (1 \le s \le N), \tag{B.9}$$

$$\beta_m^{[1],2s} = \frac{2(1-s)}{[m]_q} q^{-\frac{3}{2}m}\theta_m - \begin{cases} 0 & (1 \le s \le N-1) \\ \frac{2(-1)^{\frac{m}{2}}}{[m]_q} q^{-\frac{3}{2}m}\theta_m & (s=N) \end{cases}, \tag{B.10}$$

$$\beta_m^{[2],i} = \beta_m^{[1],i} + \begin{cases} 0 & (1 \le j \le M) \\ \frac{r^m q^{-\frac{5}{2}m}}{[m]_q} & (M < i \le 2N, M=\text{odd}) \\ \frac{r^m q^{-\frac{3}{2}m}}{[m]_q} & (M < i \le 2N, M=\text{even}) \end{cases}, \tag{B.11}$$

$$\beta_m^{[3],i} = \beta_m^{[2],i} + \begin{cases} 0 & (1 \le i \le L+M) \\ \dfrac{r^{-m} q^{\frac{1}{2}m}}{[m]_q} & (L+M < i \le 2N, L = \text{odd}, M = \text{odd}) \\ \dfrac{r^{-m} q^{-\frac{1}{2}m}}{[m]_q} & (L+M < i \le 2N, L = \text{odd}, M = \text{even}) \\ \dfrac{r^{-m} q^{-\frac{5}{2}m}}{[m]_q} & (L+M < i \le 2N, L = \text{even}, M = \text{odd}) \\ \dfrac{r^{-m} q^{-\frac{3}{2}m}}{[m]_q} & (L+M < i \le 2N, L = \text{even}, M = \text{even}) \end{cases} . \tag{B.12}$$

Proposition B.2 *We have*

$$c_j(w) = \begin{cases} 1+w^2 & (1 \le j \le N-1) \\ 1 & (j = N) \end{cases} , \quad f_1(z) = \begin{cases} \varphi^{[1]}(z) & (L > 0, M = R = 0) \\ \varphi^{[2]}(z) & (L, M > 0, R = 0) \\ \varphi^{[3]}(z) & (L, M, R > 0) \end{cases} , \tag{B.13}$$

and

$$g_i(w) = \begin{cases} g_i^{[1]}(w) & (L > 0, M = R = 0) \\ g_i^{[2]}(w) & (L, M > 0, R = 0) \\ g_i^{[3]}(w) & (L, M, R > 0) \end{cases} \quad (i = 1, 2, \cdots, 2N-1), \tag{B.14}$$

where we have set

$$g_i^{[1]}(w) = \begin{cases} 1 & (i = \text{odd}, 1 \le i \le 2N-3) \\ (1-w^2) & (i = \text{even}, 1 \le i \le 2N-2) \\ (1+w^2) & (i = 2N-1) \end{cases} , \tag{B.15}$$

$$g_i^{[2]}(w) = g_i^{[1]}(w) \times \begin{cases} \frac{1}{(1-rw/q)} & (i = L, L = \text{odd}) \\ \frac{1}{(1-rw)} & (i = L, L = \text{even}) \\ 1 & (\text{otherwise}) \end{cases} , \tag{B.16}$$

$$g_i^{[3]}(w) = g_i^{[2]}(w) \times \begin{cases} \frac{1}{(1-q^2w/r)} & (i = L+M, L = \text{odd}, M = \text{odd}) \\ \frac{1}{(1-qw/r)} & (i = L+M, L = \text{odd}, M = \text{even}) \\ \frac{1}{(1-w/rq)} & (i = L+M, L = \text{even}, M = \text{odd}) \\ \frac{1}{(1-w/r)} & (i = L+M, L = \text{even}, M = \text{even}) \\ 1 & (\text{otherwise}) \end{cases} . \tag{B.17}$$

C Normal ordering rules

In this Appendix we summarize normal orderings.

$$\begin{aligned} \phi_1^*(z) X_\epsilon^{-,1}(qw) &= :\phi_1^*(z) X_\epsilon^{-,1}(qw): \frac{-1}{qz(1-qw/z)}, \\ X_\epsilon^{-,1}(qw) \Phi_1^*(z) &= :X_\epsilon^{-,1}(qw) \Phi_1^*(z): \frac{1}{qw(1-qz/w)}, \\ X_\epsilon^{-,2j-1}(qw_1) X^{-,2j}(qw_2) &= :X_\epsilon^{-,2j-1}(qw_1) X^{-,2j}(qw_2): q^{-\epsilon} \frac{(1-qw_2/w_1)}{(1-w_2/q^\epsilon w_1)} \quad (1 \le j \le N-1), \\ X^{-,2j}(qw_2) X_\epsilon^{-,2j-1}(qw_1) &= -:X^{-,2j}(qw_2) X_\epsilon^{-,2j-1}(qw_1): \frac{(1-qw_1/w_2)}{(1-q^\epsilon w_1/w_2)} \quad (1 \le j \le N-1), \end{aligned}$$

$$
\begin{aligned}
X^{-,2j}(qw_1)X_\epsilon^{-,2j+1}(qw_2) &= :X^{-,2j}(qw_1)X_\epsilon^{-,2j+1}(qw_2): \frac{1}{qw_1(1-qw_2/w_1)} \quad (1\le j\le N-1),\\
X_\epsilon^{-,2j+1}(qw_2)X^{-,2j}(qw_1) &= :X_\epsilon^{-,2j+1}(qw_2)X^{-,2j}(qw_1): \frac{1}{qw_2(1-qw_1/w_2)} \quad (1\le j\le N-1),\\
X^{-,2j-1}(qw_1)X^{-,2j+1}(qw_2) &= :X^{-,2j-1}(qw_1)X^{-,2j+1}(qw_2): \quad (1\le j\le N-1),\\
X^{-,2j+1}(qw_2)X^{-,2j-1}(qw_1) &= -:X^{-,2j+1}(qw_2)X^{-,2j-1}(qw_1): \quad (1\le j\le N-1).
\end{aligned} \tag{C.1}
$$

$$
\begin{aligned}
e^{B_+(z;\frac{1}{2})}e^{-B_-^1(w;\frac{1}{2})} &= \frac{1}{(1-qw/z)} :e^{-B_-^1(w;\frac{1}{2})}e^{B_+(z;\frac{1}{2})}:,\\
e^{-B_+^{2j-1}(w_1;\frac{1}{2})}e^{-B_-^{2j}(w_2;\frac{1}{2})} &= (1-qw_2/w_1):e^{-B_-^{2j}(w_2;\frac{1}{2})}e^{-B_+^{2j-1}(w_1;\frac{1}{2})}: \quad (1\le j\le N-1),\\
e^{-B_+^{2j}(w_1;\frac{1}{2})}e^{-B_-^{2j+1}(w_2;\frac{1}{2})} &= \frac{1}{(1-qw_2/w_1)} :e^{-B_-^{2j+1}(w_2;\frac{1}{2})}e^{-B_+^{2j}(w_1;\frac{1}{2})}: \quad (1\le j\le N-1),\\
e^{-c_+^j(q^{1-\epsilon}/w_1)}e^{c_-^j(qw_2)} &= \frac{1}{(1-q^\epsilon w_1w_2)} :e^{c_-^j(qw_2)}e^{-c_+^j(q^{1-\epsilon}/w_1)}: \quad (1\le j\le N).
\end{aligned} \tag{C.2}
$$

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