BOSONIZATION OF SUPERALGEBRA $U_q(\widehat{sl}(N|1))$ FOR AN ARBITRARY LEVEL *

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Abstract

We give a bosonization of the quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \in \mathbf{C}$. The bosonization of level $k \in \mathbf{C}$ is completely different from those of level $k = 1$. From this bosonization, we induce the Wakimoto realization whose character coincides with those of the Verma module. We give the screening that commute with $U_q(\widehat{sl}(N|1))$. Using this screening, we propose the vertex operator that is the intertwiner among the Wakimoto realization and typical realization. We study non-vanishing property of the correlation function defined by a trace of the vertex operators.

* Work supported by Grant-in-Aid for Scientific Research **C** from Japan Society for Promotion of Science.

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1. Introduction

Bosonizations provide a powerful method to construct correlation function of exactly solvable models. We construct a bosonization of the quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ $(N \geq 2)$ for an arbitrary level $k \in \mathbf{C}$ [1, 2]. For the special level $k = 1$, bosonizations have been constructed for the quantum affine algebra $U_q(g)$ in many cases $g = (ADE)^{(r)}$, $(BC)^{(1)}$, $G_2^{(1)}$, $\widehat{sl}(M|N)$, $osp(2|2)^{(2)}$ [3, 4, 5, 6, 7, 8, 9, 10]. Bosonizations of level $k \in \mathbf{C}$ are completely different from those of level $k = 1$. For an arbitrary level $k \in \mathbf{C}$ bosonizations have been studied only for $U_q(\widehat{sl}_N)$ [11, 12] and $U_q(\widehat{sl}(N|1))$ [1, 2]. Our construction is based on the ghost-boson system. We need more consideration to get the Wakimoto realization whose character coincides with those of the Verma module. Using ξ-η system we construct the Wakimoto realization [13, 14] from our level k bosonization. For an arbitrary level $k \neq -N + 1$ we construct the screening current that commutes with $U_q(\widehat{sl}(N|1))$ modulo total difference. By using Jackson integral and the screening current, we construct the screening that commute with $U_q(\widehat{sl}(N|1))$ [13, 15]. We propose the vertex operator that is the intertwiner among the Wakimoto realization and typical realization. By using the Gelfand-Zetlin basis, we have checked the intertwining property of the vertex operator for rank $N = 2, 3, 4$ [15]. We balance the background charge of the vertex operator by using the screening and propose the correlation function by a trace of them, which gives quantum and super generalization of Dotsenko-Fateev theory [16].

The paper is organized as follows. In section 2 we review bosonizations of $U_q(\widehat{sl}_2)$. In section 3 we construct a bosonization of $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \in \mathbf{C}$. We induce the Wakimoto realization by ξ-η system. In section 4 we construct the screening that commute with $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \neq -N + 1$. We propose the vertex operator and the correlation function.

2. Bosonization : Level $k = 1$ vs. Level $k \in \mathbf{C}$

In this section we review the bosonization of the quantum affine algebra $U_q(\widehat{sl}_2)$. The purpose of this section is to make readers understand that the bosonization of level $k \in \mathbf{C}$ is complete different from those of level $k = 1$. In what follows let q be a generic complex number $0 < |q| < 1$. We use the standard q-integer notation :

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

First we recall the definition of $U_q(\widehat{sl}_2)$. We recall the Drinfeld realization of the quantum affine algebra $U_q(\widehat{sl}_2)$.

Definition 2.1 *[17] The generators of the quantum affine algebra $U_q(\widehat{sl}_2)$*

are $x_{i,n}^{\pm}$, h_m, h, c $(n \in \mathbf{Z}, m \in \mathbf{Z}_{\neq 0})$. *Defining relations are*

$$\begin{aligned}
& c : \text{central}, \quad [h, h_m] = 0, \\
& [h_m, h_n] = \delta_{m+n,0} \frac{[2m]_q [cm]_q}{m}, \\
& [h, x^{\pm}(z)] = \pm 2 x^{\pm}(z), \\
& [h_m, x^{\pm}(z)] = \pm \frac{[2m]_q}{m} q^{\mp \frac{c|m|}{2}} z^m x^{\pm}(z), \\
& (z_1 - q^{\pm 2} z_2) x^{\pm}(z_1) x^{\pm}(z_2) = (q^{\pm 2} z_1 - z_2) x^{\pm}(z_2) x^{\pm}(z_1), \\
& [x^+(z_1), x^-(z_2)] = \frac{1}{(q - q^{-1}) z_1 z_2} \\
\times \quad & \left(\delta(q^{-c} z_1/z_2) \psi^+(q^{\frac{c}{2}} z_2) - \delta(q^c z_1/z_2) \psi^-(q^{-\frac{c}{2}} z_2) \right).
\end{aligned}$$

where we have used $\delta(z) = \sum_{n \in \mathbf{Z}} z^n$. *We have set the generating function*

$$\begin{aligned}
x^{\pm}(z) &= \sum_{n \in \mathbf{Z}} x_n^{\pm} z^{-n-1}, \\
\psi^{\pm}(q^{\pm \frac{c}{2}} z) &= q^{\pm h} e^{\pm (q - q^{-1}) \sum_{m>0} h_{\pm m} z^{\mp m}}.
\end{aligned}$$

When the center c takes the complex number $c = k \in \mathbf{C}$, we call it the level k representation. We call the realization by the differential operators the bosonization. Frenkel-Jing [3] constructed the level $k = 1$ bosonization of the quantum affine algebra $U_q(g)$ for simply-laced $g = (ADE)^{(1)}$. Here we recall the level $k = 1$ bosonization of $U_q(\widehat{sl}_2)$. We introduce the boson a_n $(n \in \mathbf{Z}_{\neq 0})$ and the zero-mode operator ∂, α by

$$[a_m, a_n] = \frac{[2m]_q [m]_q}{m} \delta_{m+n,0}, \quad [\partial, \alpha] = 2.$$

In what follows, in order to avoid divergences, we restrict ourselves to the Fock space of the bosons.

Theorem 2.2 *[3] A bosonization of the quantum affine algebra* $U_q(\widehat{sl}_2)$ *for the level* $k = 1$ *is given as follows.*

$$\begin{aligned}
& c = 1, \quad h = \partial, \quad h_n = a_n, \\
& x^{\pm}(z) =: e^{\mp \sum_{n \neq 0} \frac{a_n}{[n]_q} q^{\mp \frac{n}{2}} z^{-n} \pm (\alpha + \partial)} : .
\end{aligned}$$

We have used the normal ordering symbol ::

$$: a_k a_l := \begin{cases} a_k a_l & (k < 0), \\ a_l a_k & (k > 0), \end{cases} \qquad : \alpha \partial :=: \partial \alpha := \alpha \partial.$$

Next we recall the level k bosonization of the quantum affine algebra $U_q(\widehat{sl}_2)$ [11]. We introduce the bosons and the zero-mode operator $a_n, b_n, c_n, Q_a, Q_b, Q_c$ ($n \in \mathbf{Z}$) as follows.

$$[a_m, a_n] = \delta_{m+n,0}\frac{[2m]_q[(k+2)m]_q}{m}, \ [\tilde{a}_0, Q_a] = 2(k+2),$$
$$[b_m, b_n] = -\delta_{m+n,0}\frac{[2m]_q[2m]_q}{m}, \ [\tilde{b}_0, Q_b] = -4,$$
$$[c_m, c_n] = \delta_{m+n}\frac{[2m]_q[2m]_q}{m}, \ [\tilde{c}_0, Q_c] = 4,$$

where $\tilde{a}_0 = \frac{q-q^{-1}}{2\log q}a_0$, $\tilde{b}_0 = \frac{q-q^{-1}}{2\log q}b_0$, $\tilde{c}_0 = \frac{q-q^{-1}}{2\log q}c_0$. It is convenient to introduce the generating function $a(N|z;\alpha)$.

$$a(N|z;\alpha) = -\sum_{n\neq 0}\frac{a_n}{[Nn]_q}q^{|n|\alpha}z^{-n} + \frac{\tilde{a}_0}{N}\log z + \frac{Q_a}{N}.$$

In what follows, in order to avoid divergences, we restrict ourselves to the Fock space of the bosons.

Theorem 2.3 *[11] A bosonization of the quantum affine algebra $U_q(\widehat{sl}_2)$ for the level $k \in \mathbf{C}$ is given as follows.*

$$\begin{aligned}
c &= k \in \mathbf{C}, \quad h = a_0 + b_0,\\
h_m &= q^{2m-|m|}a_m + q^{(k+2)m-\frac{k+2}{2}|m|}b_m,\\
x^+(z) &= \frac{-1}{(q-q^{-1})z}\Big(: e^{-b(2|q^{-k-2}z;1)-c(2|q^{-k-1}z;0)} :\\
&\qquad\qquad - : e^{-b(2|q^{-k-2}z;1)-c(2|q^{-k-3}z;0)} :\Big),\\
x^-(z) &= \frac{1}{(q-q^{-1})z}\Big(: e^{a(k+2|q^k z,-\frac{k+2}{2})-a(k+2|q^{-2}z;\frac{k+2}{2})+b(2|z;-1)+c(2|q^{-1}z;0)} :\\
&\quad - : e^{a(k+2|q^{-k-4}z;-\frac{k+2}{2})-a(k+2|q^{-2}z;\frac{k+2}{2})+b(2|q^{-2k-4}z;-1)+c(2|q^{-2k-3}z;0)} :\Big).
\end{aligned}$$

The level $k = 1$ bosonization is given by "monomial". The level $k \in \mathbf{C}$ bosonization is given by "sum". They are completely different.

3. Bosonization of Quantum Superalgebra $U_q(\widehat{sl}(N|1))$

In this section we study the bosonization of the quantum superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \in \mathbf{C}$.

3.1. Quantum Superalgebra $U_q(\widehat{sl}(N|1))$

In this section we recall the definition of the quantum superalgebra $U_q(\widehat{sl}(N|1))$. We fix a generic complex number q such that $0<|q|<1$. The Cartan matrix $(A_{i,j})_{0\leq i,j\leq N}$ of the affine Lie algebra $\widehat{sl}(N|1)$ is given by

$$A_{i,j}=(\nu_i+\nu_{i+1})\delta_{i,j}-\nu_i\delta_{i,j+1}-\nu_{i+1}\delta_{i+1,j}.$$

Here we set $\nu_1=\cdots=\nu_N=+,\nu_{N+1}=\nu_0=-$. We introduce the orthonormal basis $\{\epsilon_i|i=1,2,\cdots,N+1\}$ with the bilinear form, $(\epsilon_i|\epsilon_j)=\nu_i\delta_{i,j}$. Define $\bar{\epsilon}_i=\epsilon_i-\frac{\nu_i}{N-1}\sum_{j=1}^{N+1}\epsilon_j$. Note that $\sum_{j=1}^{N}\bar{\epsilon}_j=0$. The classical simple roots $\bar{\alpha}_i$ and the classical fundamental weights $\bar{\Lambda}_i$ are defined by $\bar{\alpha}_i=\nu_i\epsilon_i-\nu_{i+1}\epsilon_{i+1}$, $\bar{\Lambda}_i=\sum_{j=1}^{i}\bar{\epsilon}_j$ $(1\leq i\leq N)$. Introduce the affine weight Λ_0 and the null root δ satisfying $(\Lambda_0|\Lambda_0)=(\delta|\delta)=0$, $(\Lambda_0|\delta)=1$, $(\Lambda_0|\epsilon_i)=0$, $(\delta|\epsilon_i)=0$, $(1\leq i\leq N)$. The other affine weights and the affine roots are given by $\alpha_0=\delta-\sum_{j=1}^{N}\bar{\alpha}_j$, $\alpha_i=\bar{\alpha}_i$, $\Lambda_i=\bar{\Lambda}_i+\Lambda_0$, $(1\leq i\leq N)$. Let $P=\oplus_{j=1}^{N}\mathbf{Z}\Lambda_j\oplus\mathbf{Z}\delta$ and $P^*=\oplus_{j=1}^{N}\mathbf{Z}h_j\oplus\mathbf{Z}d$ the affine $\widehat{sl}(N|1)$ weight lattice and its dual lattice, respectively.

Definition 3.1 *[18] The quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ are generated by the generators h_i,e_i,f_i $(0\leq i\leq N)$. The $\mathbf{Z}_2$-grading of the generators are $|e_0|=|f_0|=|e_N|=|f_N|=1$ and zero otherwise. The defining relations are given as follows.*
The Cartan-Kac relations : *For $N\geq 2$, $0\leq i,j\leq N$, the generators subject to the following relations.*

$$[h_i,h_j]=0,\ [h_i,e_j]=A_{i,j}e_j,\ [h_i,f_j]=-A_{i,j}f_j,\ [e_i,f_j]=\delta_{i,j}\frac{q^{h_i}-q^{-h_i}}{q-q^{-1}}.$$

The Serre relations : *For $N\geq 2$, the generators subject to the following relations for $1\leq i\leq N-1$, $0\leq j\leq N$ such that $|A_{i,j}|=1$.*

$$[e_i,[e_i,e_j]_{q^{-1}}]_q=0,\quad [f_i,[f_i,f_j]_{q^{-1}}]_q=0.$$

For $N\geq 2$, the generators subject to the following relations for $0\leq i,j\leq N$ such that $|A_{i,j}|=0$.

$$[e_i,e_j]=0,\quad [f_i,f_j]=0.$$

For $N\geq 3$, the Serre relations of fourth degree hold.

$$\begin{array}{ll}[e_N,[e_0,[e_N,e_{N-1}]_{q^{-1}}]_q]=0, & [e_0,[e_1,[e_0,e_N]_q]_{q^{-1}}]=0,\\ [f_N,[f_0,[f_N,f_{N-1}]_{q^{-1}}]_q]=0, & [f_0,[f_1,[f_0,f_N]_q]_{q^{-1}}]=0.\end{array}$$

For $N=2$, the extra Serre relations of fifth degree hold.

$$\begin{array}{l}[e_2,[e_0,[e_2,[e_0,e_1]_q]]]_{q^{-1}}=[e_0,[e_2,[e_0,[e_2,e_1]_q]]]_{q^{-1}},\\ [f_2,[f_0,[f_2,[f_0,f_1]_q]]]_{q^{-1}}=[f_0,[f_2,[f_0,[f_2,f_1]_q]]]_{q^{-1}}.\end{array}$$

Here and throughout this paper, we use the notations

$$[X,Y]_\xi = XY - (-1)^{|X||Y|}\xi YX.$$

We write $[X,Y]_1$ *as* $[X,Y]$ *for simplicity.*
The quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ has the $\mathbf{Z}_2$-graded Hopf-algebra structure. We take the following coproduct

$$\Delta(e_i) = e_i \otimes 1 + q^{h_i} \otimes e_i,\ \Delta(f_i) = f_i \otimes q^{-h_i} + 1 \otimes f_i,\ \Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i,$$

and the antipode

$$S(e_i) = -q^{-h_i} e_i,\ S(f_i) = -f_i q^{h_i},\ S(h_i) = -h_i.$$

The coproduct Δ satisfies an algebra automorphism $\Delta(XY) = \Delta(X)\Delta(Y)$ and the antipode S satisfies a $\mathbf{Z}_2$-graded algebra anti-automorphism $S(XY) = (-1)^{|X||Y|}S(Y)S(X)$. The multiplication rule for the tensor product is $\mathbf{Z}_2$-graded and is defined for homogeneous elements $X, Y, X', Y' \in U_q(\widehat{sl}(N|1))$ and $v \in V, w \in W$ by $X \otimes Y \cdot X' \otimes Y' = (-1)^{|Y||X'|} XX' \otimes YY'$ and $X \otimes Y \cdot v \otimes w = (-1)^{|Y||v|} Xv \otimes Yw$, which extends to inhomogeneous elements through linearity.

Definition 3.2 *The quantum superalgebra* $U_q(\widehat{sl}(N|1))$ *is the subalgebra of* $U_q(\widehat{sl}(N|1))$*, that is generated by* $e_1, e_2, \cdots, e_N$*,* $f_1, f_2, \cdots, f_N$*, and* $h_1, h_2, \cdots, h_N$*.*

We recall the Drinfeld realization of $U_q(\widehat{sl}(N|1))$, that is convenient to construct bosonizations.

Definition 3.3 *[18] The generators of the quantum superalgebra* $U_q(\widehat{sl}(N|1))$ *are* $x_{i,n}^{\pm}$*,* $h_{i,m}$*,* h*,* c $(1 \le i \le N, n \in \mathbf{Z}, m \in \mathbf{Z}_{\neq 0})$*. Defining relations are*

$$\begin{aligned}
&c : \text{central},\ [h_i, h_{j,m}] = 0,\\
&[h_{i,m}, h_{j,n}] = \frac{[A_{i,j}m]_q[cm]_q}{m} q^{-c|m|}\delta_{m+n,0},\\
&[h_i, x_j^{\pm}(z)] = \pm A_{i,j} x_j^{\pm}(z),\\
&[h_{i,m}, x_j^{+}(z)] = \frac{[A_{i,j}m]_q}{m} q^{-c|m|} z^m x_j^{+}(z),\\
&[h_{i,m}, x_j^{-}(z)] = -\frac{[A_{i,j}m]_q}{m} z^m x_j^{-}(z),\\
&(z_1 - q^{\pm A_{i,j}} z_2) x_i^{\pm}(z_1) x_j^{\pm}(z_2) = (q^{\pm A_{j,i}} z_1 - z_2) x_j^{\pm}(z_2) x_i^{\pm}(z_1) \text{ for } |A_{i,j}| \neq 0,\\
&[x_i^{\pm}(z_1), x_j^{\pm}(z_2)] = 0 \text{ for } |A_{i,j}| = 0,\\
&[x_i^{+}(z_1), x_j^{-}(z_2)] = \frac{\delta_{i,j}}{(q-q^{-1})z_1 z_2}\left(\delta(q^{-c}z_1/z_2)\psi_i^{+}(q^{\frac{c}{2}}z_2) - \delta(q^{c}z_1/z_2)\psi_i^{-}(q^{-\frac{c}{2}}z_2)\right),
\end{aligned}$$

$$\Big(x_i^\pm(z_1)x_i^\pm(z_2)x_j^\pm(z)-(q+q^{-1})x_i^\pm(z_1)x_j^\pm(z)x_i^\pm(z_2)+x_j^\pm(z)x_i^\pm(z_1)x_i^\pm(z_2)\Big)$$
$$+(z_1\leftrightarrow z_2)=0 \quad \text{for } |A_{i,j}|=1,\ i\neq N,$$

where we have used $\delta(z)=\sum_{m\in\mathbf{Z}}z^m$. *Here we have used the generating function*

$$x_j^\pm(z)=\sum_{m\in\mathbf{Z}}x_{j,m}^\pm z^{-m-1},$$
$$\psi_i^\pm(q^{\pm\frac{c}{2}}z)=q^{\pm h_i}e^{\pm(q-q^{-1})\sum_{m>0}h_{i,\pm m}z^{\mp m}}.$$

The relation between two definitions of $U_q(\widehat{sl}(N|1))$ are given by

$$h_0=c-(h_1+\cdots+h_N),\quad e_i=x_{i,0}^+,\quad f_i=x_{i,0}^-\quad \text{for}\quad 1\le i\le N,$$
$$e_0=(-1)[x_{N,0}^-\cdots,[x_{3,0}^-,[x_{2,0}^-,x_{1,1}^-]_{q^{-1}}]_{q^{-1}}\cdots]_{q^{-1}}q^{-h_1-h_2-\cdots-h_N},$$
$$f_0=q^{h_1+h_2+\cdots+h_N}[\cdots[[x_{1,-1}^+,x_{2,0}^+]_q,x_{3,0}^+]_q,\cdots x_{N,0}^+]_q.$$

For instance we have the coproduct as follows.

$$\begin{aligned}\Delta(h_{i,m}) &= h_{i,m}\otimes q^{\frac{cm}{2}}+q^{\frac{3cm}{2}}\otimes h_{i,m}\quad (m>0),\\ \Delta(h_{i,-m}) &= h_{i,-m}\otimes q^{-\frac{3cm}{2}}+q^{-\frac{cm}{2}}\otimes h_{i,-m}\quad (m>0).\end{aligned}$$

3.2. Bosonization

In this section we construct bosonizations of quantum superalgebra $U_q(\hat{sl}(N|1))$ for an arbitrary level $k\in\mathbf{C}$ [2]. We introduce the bosons and the zero-mode operators a_m^j, Q_a^j $(m\in\mathbf{Z}, 1\le j\le N)$, $b_m^{i,j}, Q_b^{i,j}$ $(m\in\mathbf{Z}, 1\le i<j\le N+1)$, $c_m^{i,j}, Q_c^{i,j}$ $(m\in\mathbf{Z}, 1\le i<j\le N)$ which satisfy

$$[a_m^i,a_n^j]=\frac{[(k+N-1)m]_q[A_{i,j}m]_q}{m}\delta_{m+n,0},\ [a_0^i,Q_a^j]=(k+N-1)A_{i,j},$$
$$[b_m^{i,j},b_n^{i',j'}]=-\nu_i\nu_j\frac{[m]_q^2}{m}\delta_{i,i'}\delta_{j,j'}\delta_{m+n,0},\ [b_0^{i,j},Q_b^{i',j'}]=-\nu_i\nu_j\delta_{i,i'}\delta_{j,j'},$$
$$[c_m^{i,j},c_n^{i',j'}]=\frac{[m]_q^2}{m}\delta_{i,i'}\delta_{j,j'}\delta_{m+n,0},\ [c_0^{i,j},Q_c^{i',j'}]=\delta_{i,i'}\delta_{j,j'},$$
$$[Q_b^{i,j},Q_b^{i',j'}]=\delta_{j,N+1}\delta_{j',N+1}\pi\sqrt{-1}\qquad (i,j)\neq(i',j').$$

Other commutation relations are zero. In what follows we use the standard symbol of the normal orderings ::. It is convenient to introduce the generating function $b^{i,j}(z), c^{i,j}(z), b_\pm^{i,j}(z), a_\pm^j(z)$ and $\left(\frac{\gamma_1}{\beta_1}\cdots\frac{\gamma_r}{\beta_r}\ a^i\right)(z|\alpha)$ given by

$$b^{i,j}(z)\ =\ -\sum_{m\neq0}\frac{b_m^{i,j}}{[m]_q}z^{-m}+Q_b^{i,j}+b_0^{i,j}\log z,$$

$$\begin{aligned}
c^{i,j}(z) &= -\sum_{m\neq 0}\frac{c_m^{i,j}}{[m]_q}z^{-m}+Q_c^{i,j}+c_0^{i,j}\log z,\\
b_\pm^{i,j}(z) &= \pm(q-q^{-1})\sum_{\pm m>0}b_m^{i,j}z^{-m}\pm b_0^{i,j}\log q,\\
a_\pm^{j}(z) &= \pm(q-q^{-1})\sum_{\pm m>0}a_m^{j}z^{-m}\pm a_0^{j}\log q,\\
\left(\frac{\gamma_1}{\beta_1}\cdots\frac{\gamma_r}{\beta_r}\ a^i\right)(z|\alpha) &= -\sum_{m\neq 0}\frac{[\gamma_1 m]_q\cdots[\gamma_r m]_q}{[\beta_1 m]_q\cdots[\beta_r m]_q}\frac{a_m^i}{[m]_q}q^{-\alpha|m|}z^{-m}\\
&+ \frac{\gamma_1\cdots\gamma_r}{\beta_1\cdots\beta_r}(Q_a^i+a_0^i\log z).
\end{aligned}$$

In order to avoid divergence we work on the Fock space defined below. We introduce the vacuum state $|0\rangle\neq 0$ of the boson Fock space by

$$a_m^i|0\rangle=b_m^{i,j}|0\rangle=c_m^{i,j}|0\rangle=0\quad(m\geq 0).$$

For $p_a^i\in\mathbf{C}$ $(1\leq i\leq N)$, $p_b^{i,j}\in\mathbf{C}$ $(1\leq i<j\leq N+1)$, $p_c^{i,j}\in\mathbf{C}$ $(1\leq i<j\leq N)$, we set

$$\begin{aligned}
|p_a,p_b,p_c\rangle &= e^{\sum_{i,j=1}^N\frac{\mathrm{Min}(i,j)(N-1-\mathrm{Max}(i,j))}{(N-1)(k+N-1)}p_a^iQ_a^j}\\
&\times\ e^{-\sum_{1\leq i<j\leq N+1}p_b^{i,j}Q_b^{i,j}+\sum_{1\leq i<j\leq N}p_c^{i,j}Q_c^{i,j}}|0\rangle.
\end{aligned}$$

It satisfies

$$\begin{aligned}
&a_0^i|p_a,p_b,p_c\rangle=p_a^i|p_a,p_b,p_c\rangle,\\
&b_0^{i,j}|p_a,p_b,p_c\rangle=p_b^{i,j}|p_a,p_b,p_c\rangle,\ \ c_0^{i,j}|p_a,p_b,p_c\rangle=p_c^{i,j}|p_a,p_b,p_c\rangle.
\end{aligned}$$

The boson Fock space $F(p_a,p_b,p_c)$ is generated by the bosons $a_m^i,b_m^{i,j},c_m^{i,j}$ on the vector $|p_a,p_b,p_c\rangle$. We set the space $F(p_a)$ by

$$F(p_a)=\bigoplus_{\substack{p_b^{i,j}=-p_c^{i,j}\in\mathbf{Z}\ (1\leq i<j\leq N)\\ p_b^{i,N+1}\in\mathbf{Z}\ (1\leq i\leq N)}}F(p_a,p_b,p_c).$$

We impose the restriction $p_b^{i,j}=-p_c^{i,j}\in\mathbf{Z}$ $(1\leq i<j\leq N)$. We construct a bosonization on the space $F(p_a)$.

Theorem 3.4 *[2] A bosonization of the quantum superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k\in\mathbf{C}$ is given as follows.*

$$c\ =\ k\in\mathbf{C},$$

$$
\begin{aligned}
h_i &= a_0^i + \sum_{l=1}^{i}(b_0^{l,i+1} - b_0^{l,i}) + \sum_{l=i+1}^{N}(b_0^{i,l} - b_0^{i+1,l}) + b_0^{i,N+1} - b_0^{i+1,N+1},\\
h_N &= a_0^N - \sum_{l=1}^{N-1}(b_0^{l,N} + b_0^{l,N+1}),\\
h_{i,m} &= q^{-\frac{N-1}{2}|m|}a_m^i + \sum_{l=1}^{i}(q^{-(\frac{k}{2}+l-1)|m|}b_m^{l,i+1} - q^{-(\frac{k}{2}+l)|m|}b_m^{l,i})\\
&+ \sum_{l=i+1}^{N}(q^{-(\frac{k}{2}+l)|m|}b_m^{i,l} - q^{-(\frac{k}{2}+l-1)|m|}b_m^{i+1,l})\\
&+ q^{-(\frac{k}{2}+N)|m|}b_m^{i,N+1} - q^{-(\frac{k}{2}+N-1)|m|}b_m^{i+1,N+1},\\
h_{N,m} &= q^{-\frac{N-1}{2}|m|}a_m^N - \sum_{l=1}^{N-1}(q^{-(\frac{k}{2}+l)|m|}b_m^{l,N} + q^{-(\frac{k}{2}+l)|m|}b_m^{l,N+1}),\\
x_i^+(z) &= \frac{1}{(q-q^{-1})z} : \sum_{j=1}^{i} e^{(b+c)^{j,i}(q^{j-1}z)+\sum_{l=1}^{j-1}(b_+^{l,i+1}(q^{l-1}z)-b_+^{l,i}(q^lz))} \times\\
&\times \left\{e^{b_+^{j,i+1}(q^{j-1}z)-(b+c)^{j,i+1}(q^jz)} - e^{b_-^{j,i+1}(q^{j-1}z)-(b+c)^{j,i+1}(q^{j-2}z)}\right\} :,\\
x_N^+(z) &= : \sum_{j=1}^{N} e^{(b+c)^{j,N}(q^{j-1}z)+b^{j,N+1}(q^{j-1}z)-\sum_{l=1}^{j-1}(b_+^{l,N+1}(q^lz)+b_+^{l,N}(q^lz))} :,\\
x_i^-(z) &= q^{k+N-1} : e^{a_+^i(q^{\frac{k+N-1}{2}}z)-b^{i,N+1}(q^{k+N-1}z)-b_+^{i+1,N+1}(q^{k+N-1}z)+b^{i+1,N+1}(q^{k+N}z)} :\\
&+ \frac{1}{(q-q^{-1})z} : \left\{\sum_{j=1}^{i-1} e^{a_-^i(q^{-\frac{k+N-1}{2}}z)+(b+c)^{j,i+1}(q^{-k-j}z)+b_-^{i,n+1}(q^{-k-n}z)-b_-^{i+1,n+1}(q^{-k-n+1}z)}\right.\\
&\times e^{\sum_{l=j+1}^{i}(b_-^{l,i+1}(q^{-k-l+1}z)-b_-^{l,i}(q^{-k-l}z))+\sum_{l=i+1}^{N}(b_-^{i,l}(q^{-k-l}z)-b_-^{i+1,l}(q^{-k-l+1}z))}\\
&\times \left(e^{-b_-^{j,i}(q^{-k-j}z)-(b+c)^{j,i}(q^{-k-j+1}z)} - e^{-b_+^{j,i}(q^{-k-j}z)-(b+c)^{j,i}(q^{-k-j-1}z)}\right)\\
&+ e^{a_-^i(q^{-\frac{k+N-1}{2}}z)+(b+c)^{i,i+1}(q^{-k-i}z)}\\
&\times e^{\sum_{l=i+1}^{N}(b_-^{i,l}(q^{-k-l}z)-b_-^{i+1,l}(q^{-k-l+1}z))+b_-^{i,N+1}(q^{-k-N}z)-b_-^{i+1,N+1}(q^{-k-N+1}z)}\\
&- e^{a_+^i(q^{\frac{k+N-1}{2}}z)+(b+c)^{i,i+1}(q^{k+i}z)}\\
&\times e^{\sum_{l=i+1}^{N}(b_+^{i,l}(q^{k+l}z)-b_+^{i+1,l}(q^{k+l-1}z))+b_+^{i,N+1}(q^{k+N}z)-b_+^{i+1,N+1}(q^{k+N-1}z)}\\
&- \sum_{j=i+1}^{N-1} e^{a_+^i(q^{\frac{k+N-1}{2}}z)+(b+c)^{i,j+1}(q^{k+j}z)}\\
&\times e^{b_+^{i,N+1}(q^{k+N}z)-b_+^{i+1,N+1}(q^{k+N-1}z)+\sum_{l=j+1}^{N}(b_+^{i,l}(q^{k+l}z)-b_+^{i+1,l}(q^{k+l-1}z))}
\end{aligned}
$$

$$\times \quad \left(e^{b_+^{i+1,j+1}(q^{k+j}z)-(b+c)^{i+1,j+1}(q^{k+j+1}z)}-e^{b_-^{i+1,j+1}(q^{k+j}z)-(b+c)^{i+1,j+1}(q^{k+j-1}z)}\right)\Big\}:.$$

$$\begin{aligned}
x_N^-(z) &= \frac{1}{(q-q^{-1})z}:\left\{\sum_{j=1}^{N-1} e^{a_-^N(q^{-\frac{k+N-1}{2}}z)-b_+^{j,N+1}(q^{-k-j}z)-b^{j,N+1}(q^{-k-j-1}z)}\right.\\
&\times \; e^{-\sum_{l=j+1}^{N-1}(b_-^{l,N}(q^{-k-l}z)+b_-^{l,N+1}(q^{-k-l}z))}\\
&\times \; q^{j-1}\left(e^{-b_+^{j,N}(q^{-k-j}z)-(b+c)^{j,N}(q^{-k-j-1}z)}-e^{-b_-^{j,N}(q^{-k-j}z)-(b+c)^{j,N}(q^{-k-j+1}z)}\right)\Big\}:\\
&+ \; q^{N-1}:\left(e^{a_+^N(q^{\frac{k+N-1}{2}}z)-b^{N,N+1}(q^{k+N-1}z)}-e^{a_-^N(q^{-\frac{k+N-1}{2}}z)-b^{N,N+1}(q^{-k-N+1}z)}\right):.
\end{aligned}$$

3.3. Replacement from $U_q(sl(N|1))$ to $U_q(\widehat{sl}(N|1))$

In this section we study the relation between $U_q(sl(N|1))$ and $U_q(\widehat{sl}(N|1))$. Let us recall the Heisenberg realization of quantum superalgebra $U_q(sl(N|1))$ [1]. We introduce the coordinates $x_{i,j}$, $(1\leq i<j\leq N+1)$ by

$$x_{i,j}=\begin{cases} z_{i,j} & (1\leq i<j\leq N),\\ \theta_{i,j} & (1\leq i\leq N, j=N+1).\end{cases} \tag{3..1}$$

Here $z_{i,j}$ are complex variables and $\theta_{i,N+1}$ are the Grassmann odd variables that satisfy $\theta_{i,N+1}\theta_{i,N+1}=0$ and $\theta_{i,N+1}\theta_{j,N+1}=-\theta_{j,N+1}\theta_{i,N+1}$, $(i\neq j)$. We introduce the differential operators $\vartheta_{i,j}=x_{i,j}\frac{\partial}{\partial x_{i,j}}$, $(1\leq i<j\leq N+1)$.

Theorem 3.5 *[1] We fix parameters $\lambda_i\in\mathbf{C}$ $(1\leq i\leq N)$. The Heisenberg realization of $U_q(sl(N|1))$ is given as follows.*

$$\begin{aligned}
h_i &= \sum_{j=1}^{i-1}(\nu_i\vartheta_{j,i}-\nu_{i+1}\vartheta_{j,i+1})+\lambda_i-(\nu_i+\nu_{i+1})\vartheta_{i,i+1}+\sum_{j=i+1}^{N}(\nu_{i+1}\vartheta_{i+1,j+1}-\nu_i\vartheta_{i,j+1}),\\
e_i &= \sum_{j=1}^{i}\frac{x_{j,i}}{x_{j,i+1}}[\vartheta_{j,i+1}]_q\; q^{\sum_{l=1}^{j-1}(\nu_i\vartheta_{l,i}-\nu_{i+1}\vartheta_{l,i+1})},\\
f_i &= \sum_{j=1}^{i-1}\nu_i\frac{x_{j,i+1}}{x_{j,i}}[\vartheta_{j,i}]_q q^{\sum_{l=j+1}^{i-1}(\nu_{i+1}\vartheta_{l,i+1}-\nu_i\vartheta_{l,i})-\lambda_i+(\nu_i+\nu_{i+1})\vartheta_{i,i+1}+\sum_{l=i+2}^{N+1}(\nu_i\vartheta_{i,l}-\nu_{i+1}\vartheta_{i+1,l})}\\
&+ \; x_{i,i+1}\left[\lambda_i-\nu_i\vartheta_{i,i+1}-\sum_{l=i+2}^{N+1}(\nu_i\vartheta_{i,l}-\nu_{i+1}\vartheta_{i+1,l})\right]_q\\
&- \; \sum_{j=i+1}^{N}\nu_{i+1}\frac{x_{i,j+1}}{x_{i+1,j+1}}[\vartheta_{i+1,j+1}]_q q^{\lambda_i+\sum_{l=j+1}^{N+1}(\nu_{i+1}\vartheta_{i+1,l}-\nu_i\vartheta_{i,l})}.
\end{aligned}$$

Here we read $x_{i,i}=1$ and, for Grassmann odd variables $x_{i,j}$, the expression $\frac{1}{x_{i,j}}$ stands for the derivative $\frac{1}{x_{i,j}}=\frac{\partial}{\partial x_{i,j}}$.

We study how to recover the bosonization of the affine superalgebra $U_q(\widehat{sl}(N|1))$ from the Heisenberg realization of $U_q(sl(N|1))$. We make the following replacement with suitable argument.

$$\begin{aligned}
\vartheta_{i,j} &\to -b_{\pm}^{i,j}(z)/\log q \qquad (1 \le i < j \le N+1),\\
[\vartheta_{i,j}]_q &\to \begin{cases} \dfrac{e^{\pm b_+^{i,j}(z)} - e^{\pm b_-^{i,j}(z)}}{(q-q^{-1})z} & (j \ne N+1),\\ 1 & (j = N+1). \end{cases}\\
x_{i,j} &\to \begin{cases} :e^{(b+c)^{i,j}(z)}: & (j \ne N+1),\\ :e^{-b^{i,j}(z)}: \text{ or } :e^{-b_{\pm}^{i,j}(q^{\pm 1}z) - b^{i,j}(z)}: & (j = N+1). \end{cases}\\
\lambda_i &\to a_{\pm}^i(z)/\log q \quad (1 \le i \le N),\\
[\lambda_i]_q &\to \frac{e^{\pm a_+^i(z)} - e^{\pm a_-^i(z)}}{(q-q^{-1})z} \quad (1 \le i \le N).
\end{aligned}$$

From the above replacement, the element h_i of the Heisenberg realization is replaced as following.

$$q^{h_i} \to \begin{cases} e^{a_{\pm}^i(z) + \sum_{l=1}^{i}(b_{\pm}^{l,i+1}(z) - b_{\pm}^{l,i}(z)) + \sum_{l=i+1}^{N}(b_{\pm}^{i,l}(z) - b_{\pm}^{i+1,l}(z))} & (1 \le i \le N-1),\\ e^{a_{\pm}^N(z) - \sum_{l=1}^{N-1}(b_{\pm}^{l,N}(z) + b_{\pm}^{l,N+1}(z))} & (i = N). \end{cases}$$

We impose q-shift to variable z of the operators $a_{\pm}^i(z)$, $b_{\pm}^{i,j}(z)$. For instance, we have to replace $a_{\pm}^i(z) \to a_{\pm}^i(q^{\pm\frac{c+N-1}{2}}z)$. Bridging the gap by the q-shift, we have the bosonizations $\psi_i^{\pm}(q^{\pm\frac{c}{2}}z) \in U_q(\widehat{sl}(N|1))$ from $q^{h_i} \in U_q(sl(N|1))$.

$$\begin{aligned}
\psi_i^{\pm}(q^{\pm\frac{c}{2}}z) &= e^{a_{\pm}^i(q^{\pm\frac{c+N-1}{2}}z) + \sum_{l=1}^{i}(b_{\pm}^{l,i+1}(q^{\pm(l+c-1)}z) - b_{\pm}^{l,i}(q^{\pm(l+c)}z))}\\
&\times\ e^{\sum_{l=i+1}^{N}(b_{\pm}^{i,l}(q^{\pm(c+l)}z) - b_{\pm}^{i-1,l}(q^{\pm(c+l-1)}z)) + b_{\pm}^{i,N+1}(q^{\pm(c+N)}z) - b_{\pm}^{i+1,N+1}(q^{\pm(c+N-1)}z)},\\
\psi_N^{\pm}(q^{\pm\frac{c}{2}}z) &= e^{a_{\pm}^N(q^{\pm\frac{c+N-1}{2}}z) - \sum_{l=1}^{N-1}(b_{\pm}^{l,N}(q^{\pm(c+l)}z) + b_{\pm}^{l,N+1}(q^{\pm(c+l)}z))}.
\end{aligned}$$

In this replacement, one element q^{h_i} goes to two elements $\psi_i^{\pm}(q^{\pm\frac{c}{2}}z)$. Hence this replacement is not a map. Replacements from e_i, f_i to $x_i^{\pm}(z)$ are given by similar way, however they are more complicated. See details in [2].

3.4. Wakimoto Realization

In this section we give the Wakimoto realization $\mathcal{F}(p_a)$ whose character coincides with those of the Verma module [14]. We introduce the operators $\xi_m^{i,j}$ and $\eta_m^{i,j}$ $(1 \le i < j \le N, m \in \mathbf{Z})$ by

$$\eta^{i,j}(z) = \sum_{m \in \mathbf{Z}} \eta_m^{i,j} z^{-m-1} =: e^{c^{i,j}(z)}:, \quad \xi^{i,j}(z) = \sum_{m \in \mathbf{Z}} \xi_m^{i,j} z^{-m} =: e^{-c^{i,j}(z)}: .$$

The Fourier components $\eta_m^{i,j} = \oint \frac{dz}{2\pi\sqrt{-1}} z^m \eta^{i,j}(z)$, $\xi_m^{i,j} = \oint \frac{dz}{2\pi\sqrt{-1}} z^{m-1} \xi^{i,j}(z)$ ($m \in \mathbf{Z}$) are well defined on the space $F(p_a)$. We focus our attention on the operators $\eta_0^{i,j}, \xi_0^{i,j}$ satisfying $(\eta_0^{i,j})^2 = 0$, $(\xi_0^{i,j})^2 = 0$. They satisfy

$$\mathrm{Im}(\eta_0^{i,j}) = \mathrm{Ker}(\eta_0^{i,j}),\ \mathrm{Im}(\xi_0^{i,j}) = \mathrm{Ker}(\xi_0^{i,j}),\ \eta_0^{i,j}\xi_0^{i,j} + \xi_0^{i,j}\eta_0^{i,j} = 1.$$

We have a direct sum decomposition.

$$\begin{aligned}
&F(p_a) = \eta_0^{i,j}\xi_0^{i,j}F(p_a) \oplus \xi_0^{i,j}\eta_0^{i,j}F(p_a),\\
&\mathrm{Ker}(\eta_0^{i,j}) = \eta_0^{i,j}\xi_0^{i,j}F(p_a),\ \mathrm{Coker}(\eta_0^{i,j}) = \xi_0^{i,j}\eta_0^{i,j}F(p_a) = F(p_a)/(\eta_0^{i,j}\xi_0^{i,j})F(p_a).
\end{aligned}$$

We set the operator η_0, ξ_0 by

$$\eta_0 = \prod_{1 \le i < j \le N} \eta_0^{i,j}, \quad \xi_0 = \prod_{1 \le i < j \le N} \xi_0^{i,j}.$$

Definition 3.6 *[14] We introduce the subspace $\mathcal{F}(p_a)$ by*

$$\mathcal{F}(p_a) = \eta_0 \xi_0 F(p_a).$$

We call $\mathcal{F}(p_a)$ the Wakimoto realization.

4. Screening and Vertex Operator

In this section we give the screening that commutes with the quantum superalgebra $U_q(\widehat{sl}(N|1))$. We propose the vertex operators and the correlation functions.

4.1. Screening

In this section we give the screening $\mathcal{Q}_i$ $(1 \le i \le N)$ that commutes with the quantum superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \neq -N+1$ [15]. The Jackson integral with parameter $p \in \mathbf{C}$ $(|p| < 1)$ and $s \in \mathbf{C}^*$ is defined by

$$\int_0^{s\infty} f(z) d_p z = s(1-p) \sum_{m \in \mathbf{Z}} f(sp^m) p^m.$$

In order to avoid divergence we work in the Fock space.

Theorem 4.1 *[15] The screening $\mathcal{Q}_i$ commutes with the quantum superalgebra.*

$$[\mathcal{Q}_i, U_q(\widehat{sl}(N|1))] = 0 \quad (1 \le i \le N).$$

We have introduced the screening operators $\mathcal{Q}_i$ $(1\leq i\leq N)$ as follows.

$$\mathcal{Q}_i=\int_0^{s\infty}:e^{-\left(\frac{1}{k+N-1}a^i\right)\left(z\left|\frac{k+N-1}{2}\right.\right)}\widetilde{S}_i(z):d_pz,\quad (p=q^{2(k+N-1)}).$$

Here we have set the bosonic operators $\widetilde{S}_i(z)$ $(1\leq i\leq N)$ by

$$\begin{aligned}\widetilde{S}_i(z) &= \frac{1}{(q-q^{-1})z}\sum_{j=i+1}^{N}:\left(e^{-b_-^{i,j}(q^{N-1-j}z)-(b+c)^{i,j}(q^{N-j}z)}-e^{-b_+^{i,j}(q^{N-1-j}z)-(b+c)^{i,j}(q^{N-j-2}z)}\right)\\ &\times\ e^{(b+c)^{i+1,j}(q^{N-1-j}z)+\sum_{l=j+1}^{N}(b_-^{i+1,l}(q^{N-l}z)-b_-^{i,l}(q^{N-l-1}z))+b_-^{i+1,N+1}(z)-b_-^{i,N+1}(q^{-1}z)}:\\ &+\ q:e^{b^{i,N+1}(z)+b_+^{i+1,N+1}(z)-b^{i+1,N+1}(qz)}:\qquad (1\leq i\leq N-1),\\ \widetilde{S}_N(z) &= -q^{-1}:e^{b^{N,N+1}(z)}:.\end{aligned}$$

4.2. Vertex Operator

In this section we introduce the vertex operators $\Phi(z)$, $\Phi^*(z)$ [15]. Let $\mathcal{F}$ and $\mathcal{F}'$ be $U_q(\widehat{sl}(N|1))$ representation for an arbitrary level $k\neq -N+1$. Let V_α and V_α^{*S} be 2^N-dimensional typical representation with a parameters α [21]. Let $\{v_j\}_{j=1}^{2^N}$ be the basis of V_α. Let $\{v_j^*\}_{j=1}^{2^N}$ be the dual basis of V_α^{*S}, satisfying $(v_i|v_j^*)=\delta_{i,j}$. Let $V_{\alpha,z}$ and $V_{\alpha,z}^{*S}$ be the evaluation module and its dual of the typical representation. For instance, the 8-dimensional representation $V_{\alpha,z}$ of $U_q(\widehat{sl}(3|1))$ is given by

$$\begin{aligned}h_1 &= E_{3,3}-E_{4,4}+E_{5,5}-E_{6,6},\\ h_2 &= E_{2,2}-E_{3,3}+E_{6,6}-E_{7,7},\\ h_3 &= \alpha(E_{1,1}+E_{2,2})+(\alpha+1)(E_{3,3}+E_{4,4}+E_{5,5}+E_{6,6})+(\alpha+2)(E_{7,7}+E_{8,8}),\\ e_1 &= E_{3,4}+E_{5,6},\\ e_2 &= E_{2,3}+E_{6,7},\\ e_3 &= \sqrt{[\alpha]_q}E_{1,2}-\sqrt{[\alpha+1]_q}(E_{3,5}+E_{4,6})+\sqrt{[\alpha+2]_q}E_{7,8},\\ f_1 &= E_{4,3}+E_{6,5},\\ f_2 &= E_{3,2}+E_{7,6},\\ f_3 &= \sqrt{[\alpha]_q}E_{2,1}-\sqrt{[\alpha+1]_q}(E_{5,3}+E_{6,4})+\sqrt{[\alpha+2]_q}E_{8,7},\\ h_0 &= -\alpha(E_{1,1}+E_{4,4})-(\alpha+1)(E_{2,2}+E_{3,3}+E_{6,6}+E_{7,7})-(\alpha+2)(E_{5,5}+E_{8,8}),\\ e_0 &= -z(\sqrt{[\alpha]_q}E_{4,1}-\sqrt{[\alpha+1]_q}(E_{6,2}+E_{7,3})+\sqrt{[\alpha+2]_q}E_{8,5}),\\ f_0 &= z^{-1}(\sqrt{[\alpha]_q}E_{1,4}-\sqrt{[\alpha+1]_q}(E_{2,6}+E_{3,7})+\sqrt{[\alpha+2]_q}E_{5,8}).\end{aligned}$$

Consider the following intertwiners of $U_q(\widehat{sl}(N|1))$-representation [20].

$$\Phi(z) : \mathcal{F} \longrightarrow \mathcal{F}' \otimes V_{\alpha,z}, \quad \Phi^*(z) : \mathcal{F} \longrightarrow \mathcal{F}' \otimes V_{\alpha,z}^{*S}.$$

They are intertwiners in the sense that for any $x \in U_q(\widehat{sl}(N|1))$,

$$\Phi(z) \cdot x = \Delta(x) \cdot \Phi(z), \quad \Phi^*(z) \cdot x = \Delta(x) \cdot \Phi^*(z).$$

We expand the intertwining operators.

$$\Phi(z) = \sum_{j=1}^{2^N} \Phi_j(z) \otimes v_j, \quad \Phi^*(z) = \sum_{j=1}^{2^N} \Phi_j^*(z) \otimes v_j^*.$$

We set the $\mathbf{Z}_2$-grading of the intertwiner be $|\Phi(z)| = |\Phi^*(z)| = 0$. For $l_a = (l_a^1, l_a^2, \cdots, l_a^N) \in \mathbf{C}^N$ and $\beta \in \mathbf{C}$, we set the bosonic operator $\phi^{l_a}(z|\beta)$ by

$$\phi^{l_a}(z|\beta) =: e^{\sum_{i,j=1}^N \left(\frac{l_a^i}{k+N-1} \frac{\mathrm{Min}(i,j)}{N-1} \frac{N-1-\mathrm{Max}(i,j)}{1} a^j\right)(z|\beta)} : .$$

In order to balance thegbackground chargeh of the vertex operators, we introduce the product of the screenings $\mathcal{Q}^{(t)}$ for $t = (t_1, t_2, \cdots, t_N) \in \mathbf{N}^N$.

$$\mathcal{Q}^{(t)} = \mathcal{Q}_1^{t_1} \mathcal{Q}_2^{t_2} \cdots \mathcal{Q}_N^{t_N}.$$

The screening operator $\mathcal{Q}^{(t)}$ give rise to the map,

$$\mathcal{Q}^{(t)} : \mathcal{F}(p_a) \to \mathcal{F}(p_a + \hat{t}).$$

Here $\hat{t} = (\hat{t}_1, \hat{t}_2, \cdots, \hat{t}_N)$ where $\hat{t}_i = \sum_{j=1}^N A_{i,j} t_j$.

Theorem 4.2 *[15] For $k = \alpha \neq 0, -1, -2, \cdots, -N+1$, bosonizations of the special components of the vertex operators $\Phi^{(t)}(z)$ and $\Phi^{*(t)}(z)$ are given by*

$$\begin{aligned} \Phi_{2^N}^{(t)}(z) &= \mathcal{Q}^{(t)} \phi^{\hat{l}} \left(q^{k+N-1} z \left| -\frac{k+N-1}{2} \right. \right), \\ \Phi_1^{*(t)}(z) &= \mathcal{Q}^{(t)} \phi^{\hat{l}^*} \left(q^k z \left| -\frac{k+N-1}{2} \right. \right), \end{aligned}$$

where we have used $\hat{l} = -(0, \cdots, 0, \alpha + N - 1)$, $\hat{l}^* = (0, \cdots, 0, \alpha)$ and $t = (t_1, t_2, \cdots, t_N) \in \mathbf{N}^N$. The other components $\Phi_j^{(t)}(z)$ and $\Phi_j^{*(t)}(z)$ $(1 \leq j \leq 2^N)$ are determined by the intertwining property and are represented by multiple contour integrals of Drinfeld currents and the special components $\Phi_{2^N}^{(t)}(z)$ and $\Phi_1^{*(t)}(z)$. We have checked this theorem for $N = 2, 3, 4$.

Here we give additional explanation on the above theorem. The explicit formulae of the intertwining properties $\Phi^{(t)}(z)\cdot x=\Delta(x)\cdot\Phi^{(t)}(z)$ for $U_q(\widehat{sl}(3|1))$ are summarized as follows. We have set the $\mathbf{Z}_2$-grading of V_α as follows : $|v_1|=|v_5|=|v_6|=|v_7|=0$, and $|v_2|=|v_3|=|v_4|=|v_8|=1$.

$$\begin{aligned}
\Phi_3^{(t)}(z) &= [\Phi_4^{(t)}(z),f_1]_q,\ \Phi_5^{(t)}(z)=[\Phi_6^{(t)}(z),f_1]_q,\\
\Phi_2^{(t)}(z) &= [\Phi_3^{(t)}(z),f_2]_q,\ \Phi_6^{(t)}(z)=[\Phi_7^{(t)}(z),f_2]_q,\\
\Phi_1^{(t)}(z) &= \frac{1}{\sqrt{[\alpha]_q}}[\Phi_2^{(t)}(z),f_3]_{q^{-\alpha}},\ \Phi_3^{(t)}(z)=\frac{-1}{\sqrt{[\alpha+1]_q}}[\Phi_5^{(t)}(z),f_3]_{q^{-\alpha-1}},\\
\Phi_4^{(t)}(z) &= \frac{-1}{\sqrt{[\alpha+1]_q}}[\Phi_6^{(t)}(z),f_3]_{q^{-\alpha-1}},\ \Phi_7^{(t)}(z)=\frac{1}{\sqrt{[\alpha+2]_q}}[\Phi_8^{(t)}(z),f_3]_{q^{-\alpha-2}}.
\end{aligned}$$

The elements f_j are written by contour integral of the Drinfeld current $f_j=\oint\frac{dw}{2\pi\sqrt{-1}}x_j^-(w)$. Hence the components $\Phi_j^{(t)}$ $(1\leq j\leq 8)$ are represented by multiple contour integrals of Drinfeld currents $x_j^-(w)$ $(1\leq j\leq 3)$ and the special component $\Phi_8^{(t)}(z)$.

4.3. Correlation Function

In this section we study the correlation function as an application of the vertex operators . We study non-vanishing property of the correlation function which is defined to be the trace of the vertex operators over the Wakimoto module of $U_q(\widehat{sl}(N|1))$. We propose the q-Virasoro operator L_0 for $k=\alpha\neq -N+1$ as follows.

$$\begin{aligned}
L_0 &= \frac{1}{2}\sum_{i,j=1}^{N}\sum_{m\in\mathbf{Z}} : a_{-m}^i\frac{m^2[\mathrm{Min}(i,j)m]_q[(N-1-\mathrm{Max}(i,j))m]_q}{[m]_q[(k+N-1)m]_q[(N-1)m]_q[m]_q}a_m^j :\\
&+\sum_{i,j=1}^{N}\frac{\mathrm{Min}(i,j)(N-1-\mathrm{Max}(i,j))}{(k+N-1)(N-1)}a_0^j\\
&-\frac{1}{2}\sum_{1\leq i<j\leq N}\sum_{m\in\mathbf{Z}} : b_{-m}^{i,j}\frac{m^2}{[m]_q^2}b_m^{i,j} : +\frac{1}{2}\sum_{1\leq i<j\leq N}\sum_{m\in\mathbf{Z}} : c_{-m}^{i,j}\frac{m^2}{[m]_q^2}c_m^{i,j} :\\
&+\frac{1}{2}\sum_{1\leq i\leq N}\sum_{m\in\mathbf{Z}} : b_{-m}^{i,N+1}\frac{m^2}{[m]_q^2}b_m^{i,N+1} : +\frac{1}{2}\sum_{1\leq i\leq N}b_0^{i,N+1}.
\end{aligned}$$

The L_0 eigenvalue of $|l_a,0,0\rangle$ is $\frac{1}{2(k+N-1)}(\bar{\lambda}|\bar{\lambda}+2\bar{\rho})$, where $\bar{\rho}=\sum_{i=1}^N\bar{\Lambda}_i$ and $\bar{\lambda}=\sum_{i=1}^N l_a^i\bar{\Lambda}_i$.

Theorem 4.3 *[15] For $k=\alpha\neq 0,-1,-2,\cdots,-N+1$, the correlation* *function of the vertex operators,*

$$\mathrm{Tr}_{\mathcal{F}(l_a)}\left(q^{L_0}\Phi_{i_1}^{*(y_{(1)})}(w_1)\cdots\Phi_{i_m}^{*(y_{(m)})}(w_m)\Phi_{j_1}^{(x_{(1)})}(z_1)\cdots\Phi_{j_n}^{(x_{(n)})}(z_n)\right)\neq 0,$$

if and only if $x_{(s)}=(x_{(s),1},x_{(s),2},\cdots,x_{(s),N})\in\mathbf{N}^N$ $(1\leq s\leq n)$ *and* $y_{(s)}=(y_{(s),1},y_{(s),2},\cdots,y_{(s),N})\in\mathbf{N}^N$ $(1\leq s\leq m)$ *satisfy the following condition.*

$$\sum_{s=1}^{n}x_{(s),i}+\sum_{s=1}^{m}y_{(s),i}=\frac{(n-m)i}{N-1}\alpha+n\cdot i\qquad(1\leq i\leq N).$$

Acknowledgements.
This work is supported by the Grant-in-Aid for Scientific Research **C** (21540228) from Japan Society for Promotion of Science. The author would like to thank Professor Branko Dragovich and the organizing committee of the 7-th Mathematical Physics Meeting for an invitation.

References

[1] H.Awata, S.Odake, J.Shiraishi, *Lett.Math.Phys.*__42__ (1997) 271.
[2] T.Kojima, *J.Math.Phys.*__53__ (2012) 013515.
[3] I.B.Frenkel and N.Jing, *Proc. Nat. Acad. Sci. U.S.A.*__85__(1988) 9373.
[4] D.Bernard, *Lett.Math.Phys.*__17__ (1989) 239.
[5] N.Jing, Y.Koyama and K.Misra, *Selecta Math.*__5__ (1999) 243.
[6] N.Jing, *Invent.Math.*__102__, 663-690 (1990).
[7] N.Jing, *Proc.Amer.Math.Soc.*__127__(1999) 21.
[8] K.Kimura, J.Shiraishi and J.Uchiyama, *Comm. Math. Phys.* **188** (1997) 367.
[9] Y.-Z.Zhang, *J.Math.Phys.*__40__(1999) 6110.
[10] W.-L.Yang and Y.-Z.Zhang, *Phys.Lett.*__A261__ (1999) 252.
[11] J.Shiraishi, *Phys.Lett.*__A171__(1992) 243.
[12] H.Awata, S.Odake and J.Shiraishi, *Commun.Math.Phys.* **162** (1994) 61.
[13] Y.-Z.Zhang and M.D.Gould, *J.Math.Phys.*__41__ (2000) 5577.
[14] T.Kojima, to appear in *Springer Proceedings : Mathematics and Statistics* (2013).
[15] T.Kojima, *J.Math.Phys.* **53** (2012) 083503.
[16] Vl.S.Dotsenko and V.A.Fateev, *Nuclear Phys.*__B240__ (1984) 312.
[17] V.G. Drinfeld, *Sov.Math.Dokl.* **36** (1988) 212.
[18] H.Yamane, *Publ.Res.Inst.Math.Sci.* **35**(1999) 321.
[19] M.Wakimoto, *Commun.Math.Phys.*__104__(1986) 605.
[20] I.B.Frenkel and N.Yu.Reshetikhin, *Commun.Math.Phys.*__146__ (1992) 8.
[21] T.D.Palev and V.N.Tolstoy, *Commun.Math.Phys.*__141__ (1991) 549.