BOSONIZATION OF
SUPERALGEBRA $U_q(\hat{sl}(N|1))$
FOR AN ARBITRARY LEVEL *

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Abstract

We give a bosonization of the quantum affine superalgebra $U_q(\hat{sl}(N|1))$ for an arbitrary level $k \in \mathbb{C}$. The bosonization of level $k \in \mathbb{C}$ is completely different from those of level $k = 1$. From this bosonization, we induce the Wakimoto realization whose character coincides with those of the Verma module. We give the screening that commute with $U_q(\hat{sl}(N|1))$. Using this screening, we propose the vertex operator that is the intertwiner among the Wakimoto realization and typical realization. We study non-vanishing property of the correlation function defined by a trace of the vertex operators.

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1. Introduction

Bosonizations provide a powerful method to construct correlation function of exactly solvable models. We construct a bosonization of the quantum affine superalgebra $U_q(\hat{sl}(N|1))$ ($N \geq 2$) for an arbitrary level $k \in \mathbb{C}$ [1, 2]. For the special level $k = 1$, bosonizations have been constructed for the quantum affine algebra $U_q(g)$ in many cases $g = (ADE)^{(r)}$, $(BC)^{(1)}$, $G_2^{(1)}$, $\hat{sl}(M|N)$, $osp(2|2)^{(2)}$ [3, 4, 5, 6, 7, 8, 9, 10]. Bosonizations of level $k \in \mathbb{C}$ are completely different from those of level $k = 1$. For an arbitrary level $k \in \mathbb{C}$ bosonizations have been studied only for $U_q(\hat{sl}_N)$ [11, 12] and $U_q(\hat{sl}(N|1))$ [1, 2]. Our construction is based on the ghost-boson system. We need more consideration to get the Wakimoto realization whose character coincides with those of the Verma module. Using $\xi$-$\eta$ system we construct the Wakimoto realization [13, 14] from our level $k$ bosonization. For an arbitrary level $k \neq -N + 1$ we construct the screening current that commutes with $U_q(\hat{sl}(N|1))$ modulo total difference. By using Jackson integral and the screening current, we construct the screening that commute with $U_q(\hat{sl}(N|1))$ [13, 15]. We propose the vertex operator that is the intertwiner among the Wakimoto realization and typical realization. By using the Gelfand-Zetlin basis, we have checked the intertwining property of the vertex operator for rank $N = 2, 3, 4$ [15]. We balance the background charge of the vertex operator by using the screening and propose the correlation function by a trace of them, which gives quantum and super generalization of Dotsenko-Fateev theory [16].

The paper is organized as follows. In section 2 we review bosonizations of $U_q(\hat{sl}_2)$. In section 3 we construct a bosonization of $U_q(\hat{sl}(N|1))$ for an arbitrary level $k \in \mathbb{C}$. We induce the Wakimoto realization by $\xi$-$\eta$ system. In section 4 we construct the screening that commute with $U_q(\hat{sl}(N|1))$ for an arbitrary level $k \neq -N + 1$. We propose the vertex operator and the correlation function.

2. Bosonization : Level $k = 1$ vs. Level $k \in \mathbb{C}$

In this section we review the bosonization of the quantum affine algebra $U_q(\hat{sl}_2)$. The purpose of this section is to make readers understand that the bosonization of level $k \in \mathbb{C}$ is complete different from those of level $k = 1$. In what follows let $q$ be a generic complex number $0 < |q| < 1$. We use the standard $q$-integer notation:

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

First we recall the definition of $U_q(\hat{sl}_2)$. We recall the Drinfeld realization of the quantum affine algebra $U_q(\hat{sl}_2)$.

Definition 2.1 [17] The generators of the quantum affine algebra $U_q(\hat{sl}_2)$
are \( x_{i,n}^\pm \), \( h_m \), \( h \), \( c \) \((n \in \mathbb{Z}, m \in \mathbb{Z} \neq 0)\). Defining relations are

\[
c : \text{central}, \quad [h, h_m] = 0,
\]

\[
[h_m, h_n] = \delta_{m+n,0} \frac{[2m]_q [cm]_q}{m},
\]

\[
[h, x^\pm(z)] = \pm 2x^\pm(z),
\]

\[
[h_m, x^\pm(z)] = \pm \frac{[2m]_q}{m} q^{\pm c m} z^m x^\pm(z),
\]

\[
(z_1 - q^{\pm 2} z_2) x^\pm(z_1) x^\pm(z_2) = (q^{\pm 2} z_1 - z_2) x^\pm(z_2) x^\pm(z_1),
\]

\[
[x^+(z_1), x^-(z_2)] = \frac{1}{(q - q^{-1}) z_1 z_2}
\]

\[
\times \left( \delta(q^{-c} z_1 / z_2) \psi^+(q^{\frac{c}{2}} z_2) - \delta(q^{c} z_1 / z_2) \psi^-(q^{-\frac{c}{2}} z_2) \right).
\]

where we have used \( \delta(z) = \sum_{n \in \mathbb{Z}} z^n \). We have set the generating function

\[
x^\pm(z) = \sum_{n \in \mathbb{Z}} x^\pm_n z^{-n-1},
\]

\[
\psi^\pm(q^{\frac{c}{2}} z) = q^{\pm h} e^{\pm (q-q^{-1}) \sum_{m>0} h_m z^m}.
\]

When the center \( c \) takes the complex number \( c = k \in \mathbb{C} \), we call it the level \( k \) representation. We call the realization by the differential operators the bosonization. Frenkel-Jing [3] constructed the level \( k = 1 \) bosonization of the quantum affine algebra \( U_q(\hat{g}) \) for simply-laced \( g = (ADE)^{(1)} \). Here we recall the level \( k = 1 \) bosonization of \( U_q(\hat{sl}_2) \). We introduce the boson \( a_n \) \((n \in \mathbb{Z} \neq 0)\) and the zero-mode operator \( \partial, \alpha \) by

\[
[a_m, a_n] = \frac{[2m]_q [m]_q}{m} \delta_{m+n,0}, \quad [\partial, \alpha] = 2.
\]

In what follows, in order to avoid divergences, we restrict ourselves to the Fock space of the bosons.

**Theorem 2.2** [3] A bosonization of the quantum affine algebra \( U_q(\hat{sl}_2) \) for the level \( k = 1 \) is given as follows.

\[
c = 1, \quad h = \partial, \quad h_n = a_n,
\]

\[
x^\pm(z) = e^{\mp \sum_{n \neq 0} \frac{a_n}{m q^{\pm \frac{c}{2}} z^{-n} (\alpha + \partial)}}.
\]

We have used the normal ordering symbol \( \langle \rangle \) :

\[
\langle a_k a_l \rangle := \begin{cases} 
  a_k a_l & (k < 0), \\
  a_l a_k & (k > 0), \\
  \alpha \partial & := \partial \alpha := \alpha \partial.
\end{cases}
\]
In this section we study the bosonization of the quantum superalgebra $U_q(\hat{sl}_2)$ [11]. We introduce the bosons and the zero-mode operator $a_n, b_n, c_n, Q_a, Q_b, Q_c$ ($n \in \mathbb{Z}$) as follows.

$$\begin{align*}
[a_m, a_n] &= \delta_{m+n,0} \frac{[2m]_q[(k+2)m]_q}{m}, \quad [\tilde{a}_0, Q_a] = 2(k+2), \\
[b_m, b_n] &= -\delta_{m+n,0} \frac{[2m]_q[2m]_q}{m}, \quad [\tilde{b}_0, Q_b] = -4, \\
[c_m, c_n] &= \delta_{m+n,0} \frac{[2m]_q[2m]_q}{m}, \quad [\tilde{c}_0, Q_c] = 4,
\end{align*}$$

where $\tilde{a}_0 = \frac{q^{-1}}{2 log q} a_0, \tilde{b}_0 = \frac{q^{-1}}{2 log q} b_0, \tilde{c}_0 = \frac{q^{-1}}{2 log q} c_0$. It is convenient to introduce the generating function $a(N|z;\alpha)$.

$$a(N|z;\alpha) = -\sum_{n \neq 0} \frac{a_n}{[Nn]_q} q^{\alpha n} z^{-n} + \frac{\tilde{a}_0}{N} \log z + \frac{Q_a}{N}.$$

In what follows, in order to avoid divergences, we restrict ourselves to the Fock space of the bosons.

**Theorem 2.3** [11] A bosonization of the quantum affine algebra $U_q(\hat{sl}_2)$ for the level $k \in \mathbb{C}$ is given as follows.

$$\begin{align*}
c &= k \in \mathbb{C}, \quad h = a_0 + b_0, \\
h_m &= q^{2m-|m|} a_m + q^{(k+2)m-\frac{k+2}{2}|m|} b_m, \\
x^+(z) &= \frac{-1}{(q-q^{-1})z} \left( e^{-b(2|q^{k-2}z|;1)} - c(2|q^{-1}z;0) ; - ; e^{-b(2|q^{-k-2}z|;1)} - c(2|q^{-1}z;0) ; \right), \\
x^-(z) &= \frac{1}{(q-q^{-1})z} \left( e^{a(k+2|q^{k-4}z|;\frac{k+2}{2}) - a(k+2|q^{-2}z|;\frac{k+2}{2}) + b(2|z|;1) + c(2|q^{-1}z;0) ;} - ; e^{a(k+2|q^{k-4}z|;\frac{k+2}{2}) - a(k+2|q^{-2}z|;\frac{k+2}{2}) + b(2|q^{-2k-4}z|;1) + c(2|q^{-1}z;0) ;} \right).
\end{align*}$$

The level $k = 1$ bosonization is given by "monomial". The level $k \in \mathbb{C}$ bosonization is given by "sum". They are completely different.

3. **Bosonization of Quantum Superalgebra $U_q(\hat{sl}(N|1))$**

In this section we study the bosonization of the quantum superalgebra $U_q(\hat{sl}(N|1))$ for an arbitrary level $k \in \mathbb{C}$.
3.1. Quantum Superalgebra \( U_q(\widehat{\mathfrak{sl}}(N|1)) \)

In this section we recall the definition of the quantum superalgebra \( U_q(\widehat{\mathfrak{sl}}(N|1)) \). We fix a generic complex number \( q \) such that \( 0 < |q| < 1 \). The Cartan matrix \( (A_{i,j})_{0 \leq i,j \leq N} \) of the affine Lie algebra \( \widehat{\mathfrak{sl}}(N|1) \) is given by

\[
A_{i,j} = (\nu_i + \nu_{i+1})\delta_{i,j} - \nu_i\delta_{i,j+1} - \nu_{i+1}\delta_{i+1,j}.
\]

Here we set \( \nu_1 = \cdots = \nu_N = +, \nu_{N+1} = \nu_0 = - \). We introduce the orthonormal basis \( \{\epsilon_i | i = 1, 2, \ldots, N + 1\} \) with the bilinear form, \( (\epsilon_i | \epsilon_j) = \nu_i \delta_{i,j} \). Define \( \bar{\epsilon}_i = \epsilon_i - \frac{\nu_i}{N+1} \sum_{j=1}^{N+1} \epsilon_j \). Note that \( \sum_{j=1}^{N+1} \bar{\epsilon}_j = 0 \). The classical simple roots \( \bar{\alpha}_i \) and the classical fundamental weights \( \bar{\Lambda}_i \) are defined by \( \bar{\alpha}_i = \nu_i\epsilon_i - \nu_{i+1}\epsilon_{i+1}, \bar{\Lambda}_i = \sum_{j=1}^{i} \bar{\epsilon}_j \) (\( 1 \leq i \leq N \)). Introduce the affine root \( \Lambda_0 \) and the null root \( \delta \) satisfying \( (\Lambda_0 | \Lambda_0) = (\delta | \delta) = 0 \), \( (\Lambda_0 | \epsilon_i) = 0 \), \( (\delta | \epsilon_i) = 0 \), \( (1 \leq i \leq N) \). The other affine weights and the affine roots are given by \( \alpha_0 = \delta - \sum_{j=1}^{N+1} \bar{\epsilon}_j \), \( \alpha_i = \bar{\alpha}_i \), \( \Lambda_i = \bar{\Lambda}_i + \Lambda_0 \) (\( 1 \leq i \leq N \)).

Let \( P = \bigoplus_{j=1}^{N} N\Lambda_j \oplus \mathbb{Z} \delta \) and \( P^* = \bigoplus_{j=1}^{N} N\mathbb{Z} \bar{\rho}_j \oplus \mathbb{Z} \mathbf{d} \) the affine \( \widehat{\mathfrak{sl}}(N|1) \) weight lattice and its dual lattice, respectively.

**Definition 3.1** \([18]\) The quantum affine superalgebra \( U_q(\widehat{\mathfrak{sl}}(N|1)) \) are generated by the generators \( h_i, \epsilon_i, f_i \) (\( 0 \leq i \leq N \)). The \( \mathbb{Z}_2 \)-grading of the generators are \( |e_0| = |f_0| = |e_N| = |f_N| = 1 \) and zero otherwise. The defining relations are given as follows.

The Cartan-Kac relations: For \( N \geq 2 \), \( 0 \leq i, j \leq N \), the generators subject to the following relations.

\[
[h_i, h_j] = 0, \quad [h_i, \epsilon_j] = A_{i,j} \epsilon_j, \quad [h_i, f_j] = -A_{i,j} f_j, \quad [\epsilon_i, f_j] = \delta_{i,j} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}.
\]

The Serre relations: For \( N \geq 2 \), the generators subject to the following relations for \( 1 \leq i \leq N - 1, 0 \leq j \leq N \) such that \( |A_{i,j}| = 1 \).

\[
[e_i, [e_i, \epsilon_j]_{q^{-1}}]_q = 0, \quad [f_i, [f_i, f_j]_{q^{-1}}]_q = 0.
\]

For \( N \geq 2 \), the generators subject to the following relations for \( 0 \leq i, j \leq N \) such that \( |A_{i,j}| = 0 \).

\[
[e_i, e_j] = 0, \quad [f_i, f_j] = 0.
\]

For \( N \geq 3 \), the Serre relations of fourth degree hold.

\[
[e_N, [e_0, [e_N, e_{N-1}]_{q^{-1}}]_{q^{-1}}]_{q^{-1}} = 0, \quad [e_0, [e_1, [e_0, e_N]_{q^{-1}}]_{q^{-1}} = 0,

[f_N, [f_0, [f_N, f_{N-1}]_{q^{-1}}]_{q^{-1}}]_{q^{-1}} = 0, \quad [f_0, [f_1, [f_0, f_N]_{q^{-1}}]_{q^{-1}}]_{q^{-1}} = 0.
\]

For \( N = 2 \), the extra Serre relations of fifth degree hold.

\[
[e_2, [e_0, [e_2, [e_0, e_1]_{q^{-1}}]]_{q^{-1}} = [e_0, [e_2, [e_0, [e_2, e_1]_{q^{-1}}]]_{q^{-1}},

[f_2, [f_0, [f_2, [f_0, f_1]_{q^{-1}}]]_{q^{-1}} = [f_0, [f_2, [f_0, [f_2, f_1]_{q^{-1}}]]_{q^{-1}}.
\]
Here and throughout this paper, we use the notations
\[ [X, Y]_\xi = XY - (-1)^{|X||Y|}\xi YX. \]
We write \([X, Y]_1\) as \([X, Y]\) for simplicity.

The quantum affine superalgebra \(U_q(\hat{sl}(N|1))\) has the \(\mathbb{Z}_2\)-graded Hopf-algebra structure. We take the following coproduct
\[ \Delta(e_i) = e_i \otimes 1 + q^{h_i} \otimes e_i, \quad \Delta(f_i) = f_i \otimes q^{-h_i} + 1 \otimes f_i, \quad \Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \]
and the antipode
\[ S(e_i) = -q^{-h_i}e_i, \quad S(f_i) = -f_iq^{h_i}, \quad S(h_i) = -h_i. \]

The coproduct \(\Delta\) satisfies an algebra automorphism \(\Delta(XY) = \Delta(X)\Delta(Y)\) and the antipode \(S\) satisfies a \(\mathbb{Z}_2\)-graded algebra anti-automorphism \(S(XY) = (-1)^{|X||Y|}S(Y)S(X)\). The multiplication rule for the tensor product is \(\mathbb{Z}_2\)-graded and is defined for homogeneous elements \(X, Y, X', Y' \in U_q(\hat{sl}(N|1))\) and \(v \in V, w \in W\) by \(X \otimes Y \cdot X' \otimes Y' = (-1)^{|Y||X'|}XX' \otimes YY'\) and \(X \otimes Y \cdot v \otimes w = (-1)^{|Y||v|}Xv \otimes Yw\), which extends to inhomogeneous elements through linearity.

**Definition 3.2** The quantum superalgebra \(U_q(\hat{sl}(N|1))\) is the subalgebra of \(U_q(\hat{sl}(N|1))\), that is generated by \(e_1, e_2, \ldots, e_N, f_1, f_2, \ldots, f_N, \) and \(h_1, h_2, \ldots, h_N\).

We recall the Drinfeld realization of \(U_q(\hat{sl}(N|1))\), that is convenient to construct bosonizations.

**Definition 3.3** [18] The generators of the quantum superalgebra \(U_q(\hat{sl}(N|1))\) are \(x_{i,n}^\pm, h_{i,m}, h, c (1 \leq i \leq N, n \in \mathbb{Z}, m \in \mathbb{Z}_{\neq 0})\). Defining relations are
\[
c : \text{central}, \quad [h_i, h_{j,m}] = 0, \\
[h_{i,m}, h_{j,n}] = [A_{i,j,m}]q^{-|cm|}q^{-c|m|}\delta_{m+n,0}, \\
[h_i, x_j^+(z)] = \pm A_{i,j}x_j^+(z), \\
[h_{i,m}, x_j^+(z)] = [A_{i,j,m}]q^{-|cm|}z^m x_j^+(z), \\
h_{i,m}, x_j^-(z) = -[A_{i,j,m}]q^{-|cm|}z^m x_j^-(z), \\
(z_1 - q^{A_{i,j}}z_2)x_i^+(z_1)x_j^+(z_2) = (q^{A_{i,j}}z_1 - z_2)x_j^+(z_2)x_i^+(z_1) \text{ for } |A_{i,j}| \neq 0, \\
x_i^+(z_1), x_j^+(z_2) = 0 \text{ for } |A_{i,j}| = 0, \\
x_i^-(z_1), x_j^-(z_2) = \frac{\delta_{i,j}}{(q - q^{-1})z_1z_2} \left(\delta(q^{-c}z_1/z_2)\psi_i^+(q^{q^{-c}}z_2) - \delta(q^c z_1/z_2)\psi_i^-(q^{-q^{-c}}z_2)\right),
\]
In this section we construct bosonizations of quantum superalgebra $U_q(\hat{\mathfrak{sl}}(N|1))$ for an arbitrary level $k \in \mathbb{C}$ [2]. We introduce the bosons and the zero-mode operators $a^i_m, Q^i_a (m \in \mathbb{Z}, 1 \leq j \leq N)$, $b^i_m, Q^i_b (m \in \mathbb{Z}, 1 \leq i < j \leq N+1)$, $c^i_m, Q^i_c (m \in \mathbb{Z}, 1 \leq i < j \leq N)$ which satisfy

\[
[a^i_m, a^j_n] = \frac{(k + N - 1)m}{m} [A_{i,j} m]_q \delta_{m+n,0}, \quad [a^i_0, Q^j_a] = (k + N - 1) A_{i,j},
\]

\[
[b^i_m, b^{i'}_{m'}] = -\nu_{i,i'} \frac{m^2}{m} \delta_{i,i'} \delta_{m+m',0}, \quad [b^i_0, Q^j_b] = -\nu_{i,i'} \delta_{i,i'} \delta_{j,j'},
\]

\[
[c^i_m, c^{i'}_{m'}] = \frac{m^2}{m} \delta_{i,i'} \delta_{m+m',0}, \quad [c^i_0, Q^j_c] = \delta_{i,i'} \delta_{j,j'},
\]

\[
[Q^i_b, Q^{i'}_{c}] = \delta_{i,N+1} \delta_{j',N+1} \pi \sqrt{-1} \quad (i, j) \neq (i', j').
\]

Other commutation relations are zero. In what follows we use the standard symbol of the normal orderings $\cdot$. It is convenient to introduce the generating function $b^{i,j}(z), c^{i,j}(z), b^{i,j}_c(z), a^{i,j}_d(z)$ and \((\frac{3}{m} \cdots \frac{3}{m'} a^{i,j}_d(z) |\alpha\rangle\) given by

\[
b^{i,j}(z) = - \sum_{m \neq 0} \frac{b^i_{m,j}}{|m| q} z^{-m} + Q^i_j + b^i_0 z \log z.
\]
\[ c_{i,j}(z) = - \sum_{m \neq 0} \frac{c_{m|q}}{|m|q} z^{-m} + Q_{c_{i,j}}^{i,j} + c_{0}^{i,j} \log z, \]
\[ b_{i,j}(z) = \pm (q - q^{-1}) \sum_{m > 0} b_{m|q}^{i,j} z^{-m} \pm b_{0}^{i,j} \log q, \]
\[ a_{i,j}^{\pm}(z) = \pm (q - q^{-1}) \sum_{m > 0} a_{m|q}^{i,j} z^{-m} \pm a_{0}^{i,j} \log q, \]
\[ \left( \frac{\gamma_1}{\beta_1} \cdots \frac{\gamma_r}{\beta_r} a^{i} \right) (z|\alpha) = - \sum_{m \neq 0} \frac{[\gamma_1 m|_q \cdots [\gamma_r m|_q a_{m|q}^{i} \alpha m|_q] z^{-m}}{|m|q} + \frac{\gamma_1 \cdots \gamma_r}{\beta_1 \cdots \beta_r} (Q_{a}^{i} + a_{0}^{i} \log z). \]

In order to avoid divergence we work on the Fock space defined below. We introduce the vacuum state \(|0\rangle \neq 0\) of the boson Fock space by
\[ a_{m}^{i|0} = b_{m}^{i,j|0} = c_{m}^{i,j|0} = 0 \quad (m \geq 0). \]

For \( p_{a}^{i} \in \mathbb{C} \quad (1 \leq i \leq N), \quad p_{b}^{i,j} \in \mathbb{C} \quad (1 \leq i < j \leq N + 1), \quad p_{c}^{i,j} \in \mathbb{C} \quad (1 \leq i < j \leq N), \) we set
\[ |p_{a}, p_{b}, p_{c}\rangle = e^{\sum_{i,j=1}^{N} \frac{\min(i,j)(N-1-\max(i,j))}{(N-1)(N-1)} p_{a}^{i} Q_{b}^{i,j} + \sum_{1 \leq i < j \leq N} p_{b}^{i,j} Q_{c}^{i,j} + \sum_{1 \leq i < j \leq N} p_{c}^{i,j} Q_{c}^{i,j} |0\rangle. \]

It satisfies
\[ a_{0}^{i}|p_{a}, p_{b}, p_{c}\rangle = p_{a}^{i}|p_{a}, p_{b}, p_{c}\rangle, \]
\[ b_{0}^{i,j}|p_{a}, p_{b}, p_{c}\rangle = p_{b}^{i,j}|p_{a}, p_{b}, p_{c}\rangle, \]
\[ c_{0}^{i,j}|p_{a}, p_{b}, p_{c}\rangle = p_{c}^{i,j}|p_{a}, p_{b}, p_{c}\rangle. \]

The boson Fock space \( F(p_{a}, p_{b}, p_{c}) \) is generated by the bosons \( a_{m}^{i}, b_{m}^{i,j}, c_{m}^{i,j} \) on the vector \( |p_{a}, p_{b}, p_{c}\rangle \). We set the space \( F(p_{a}) \) by
\[ F(p_{a}) = \bigoplus_{\begin{array}{c} p_{b}^{i,j} = -p_{c}^{i,j} \in \mathbb{Z} \quad (1 \leq i < j \leq N) \\ p_{b}^{i,N+1} \in \mathbb{Z} \quad (1 \leq i \leq N) \end{array}} F(p_{a}, p_{b}, p_{c}). \]

We impose the restriction \( p_{b}^{i,j} = -p_{c}^{i,j} \in \mathbb{Z} \quad (1 \leq i < j \leq N) \). We construct a bosonization on the space \( F(p_{a}) \).

**Theorem 3.4** [2] A bosonization of the quantum superalgebra \( U_{q}(\widehat{sl}(N|1)) \) for an arbitrary level \( k \in \mathbb{C} \) is given as follows.
\[ c = k \in \mathbb{C}, \]
\[
\begin{align*}
 h_i &= a_0^i + \sum_{l=1}^i (b_0^{l,i+1} - b_0^{l,i}) + \sum_{l=i+1}^N (b_0^{l,i} - b_0^{l+1,i}) + b_0^{N+1,i} - b_0^{i+1,N+1}, \\
 h_N &= a_0^N - \sum_{l=1}^{N-1} (b_0^l + b_0^{l+1,N+1}), \\
 h_{i,m} &= q^{-\frac{N-1}{2}|m|}a_i^m + \sum_{l=1}^i (q^{-\left(\frac{k}{2}+l-1\right)}b_{m}^{l,i+1} - q^{-\left(\frac{k}{2}+l\right)}b_{m}^{l,i}) \\
 &+ \sum_{l=i+1}^N (q^{-\left(\frac{k}{2}+l\right)}b_{m}^{l,i} - q^{-\left(\frac{k}{2}+l-1\right)}b_{m}^{l+1,i}) \\
 &+ q^{-\left(\frac{k}{2}+N\right)}b_{m}^{i,N+1} - q^{-\left(\frac{k}{2}+N-1\right)}b_{m}^{i+1,N+1}, \\
 h_{N,m} &= q^{-\frac{N-1}{2}|m|}a_N^m - \sum_{l=1}^{N-1} (q^{-\left(\frac{k}{2}+l\right)}b_{m}^l + q^{-\left(\frac{k}{2}+l\right)}b_{m}^{l,N+1}), \\
 x_i^+(z) &= \frac{1}{(q-q^{-1})z} \left( \sum_{j=1}^i \mathcal{E}(b+c)^{j,i}(q^{-1}z) + \sum_{j=1}^{i-1} \left( h_+^{j,i+1}(q^{-1}z) - h_-^{j,i}(q^{1/2}z) \right) \right) \\
 &+ \sum_{j=1}^{i} \left\{ h_+^{j,i+1}(q^{1/2}z) - (b+c)^{j,i+1}(q^{-1}z) - h_-^{j,i+1}(q^{1/2}z) - (b+c)^{j,i}(q^{-1}z) \right\}, \\
 x_i^-(z) &= q^{k+N-1} : e_{x_i}^+(q^{\frac{k+N-1}{2}z}) - b_{x_i}^{N+1} q^{k+N-1}(q^{N-1}z) - b_{x_i}^{N+1,N+1} q^{k+N-1}(q^{N-1}z) + b_{x_i}^{i+1,N+1}(q^{k+N}) : \\
 &+ \frac{1}{(q-q^{-1})z} \left( \sum_{j=1}^{i-1} \mathcal{E}(a_-^{j,i}(q^{\frac{k+N-1}{2}z} + (b+c)^{j,i+1}(q^{-1}z)) + \sum_{j=1}^{i} \left( b_-^{j,i}(q^{k+N-1}z) - b_-^{j,i+1}(q^{k+N-1}z) \right) \right) \\
 &+ \sum_{j=1}^{i} \left\{ b_-^{j,i}(q^{k+N-1}z) - (b+c)^{j,i}(q^{k+N-1}z) - b_-^{j,i+1}(q^{k+N-1}z) - (b+c)^{j,i+1}(q^{k+N-1}z) \right\} \\
 &+ \sum_{j=1}^{i} \left\{ b_-^{j,i}(q^{k+N-1}z) - (b+c)^{j,i+1}(q^{k+N-1}z) - b_-^{j,i+1}(q^{k+N-1}z) - (b+c)^{j,i+1}(q^{k+N-1}z) \right\} \\
 &+ \sum_{j=i+1}^{N} \left\{ b_-^{j,i}(q^{k+N-1}z) - (b+c)^{j,i+1}(q^{k+N-1}z) - b_-^{j,i+1}(q^{k+N-1}z) - (b+c)^{j,i+1}(q^{k+N-1}z) \right\} \\
 &+ \sum_{j=i+1}^{N} \left\{ b_-^{j,i}(q^{k+N-1}z) - (b+c)^{j,i+1}(q^{k+N-1}z) - b_-^{j,i+1}(q^{k+N-1}z) - (b+c)^{j,i+1}(q^{k+N-1}z) \right\} \\
 &+ \sum_{j=i+1}^{N} \left\{ b_-^{j,i}(q^{k+N-1}z) - (b+c)^{j,i+1}(q^{k+N-1}z) - b_-^{j,i+1}(q^{k+N-1}z) - (b+c)^{j,i+1}(q^{k+N-1}z) \right\} \right.
\end{align*}
\]

\textit{LEVEL $k$ BOSONIZATION OF SUPERALGEBRA $U_q(\widehat{\mathfrak{su}}(N|1))$}
We fix parameters \( \theta \) and let us recall the Heisenberg realization of quantum superalgebra \( \mathfrak{sl}(N|1) \) introduced in \cite{1}. We introduce the coordinates \( x_{i,j} \) by

\[
x_{i,j} = \begin{cases} 
  z_{i,j} & (1 \leq i < j \leq N), \\
  \theta_{i,j} & (1 \leq i \leq N, j = N + 1).
\end{cases}
\]

Here \( z_{i,j} \) are complex variables and \( \theta_{i,N+1} \) are the Grassmann odd variables that satisfy \( \theta_{i,N+1}\theta_{i,N+1} = 0 \) and \( \theta_{i,N+1}\theta_{i,j,N+1} = -\theta_{j,N+1}\theta_{i,N+1} \) (i.e. \( i \neq j \)).

We introduce the differential operators \( \partial_{i,j} = x_{i,j}\frac{\partial}{\partial x_{i,j}} \), (1 \leq i < j \leq N + 1).

**Theorem 3.5** \cite{1} We fix parameters \( \lambda_i \in \mathbb{C} \) (1 \leq i \leq N). The Heisenberg realization of \( U_q(\mathfrak{sl}(N|1)) \) is given as follows.

\[
h_i = \sum_{j=1}^{i-1} (\nu_i \partial_{j,i} - \nu_{i+1} \partial_{j,i+1}) + \lambda_i - (\nu_i + \nu_{i+1}) \partial_{i,i+1} + \sum_{j=i+1}^{N} (\nu_{i+1} \partial_{i+1,j+1} - \nu_i \partial_{i,j+1}),
\]

\[
e_i = \sum_{j=1}^{i} \frac{x_{j,i}}{x_{j,i+1}} [\partial_{j,i+1}]_q q^{\sum_{l=j}^{i-1} (\nu_{i} \partial_{l,i} - \nu_{i+1} \partial_{l,i+1})},
\]

\[
f_i = \sum_{j=1}^{i} \nu_j \frac{x_{j,i+1}}{x_{j,i}} [\partial_{j,i}]_q q^{\sum_{l=j}^{i-1} (\nu_{i} \partial_{l,i} - \nu_{i+1} \partial_{l,i+1}) - \lambda_i + (\nu_i + \nu_{i+1}) \partial_{i,i+1} + \sum_{l=i+1}^{N+1} (\nu_i \partial_{i,l} - \nu_{i+1} \partial_{i+1,l})}
\]

\[
+ x_{i,i+1} \left[ \lambda_i - \nu_i \partial_{i,i+1} - \sum_{l=i+2}^{N+1} (\nu_i \partial_{i,l} - \nu_{i+1} \partial_{i+1,l}) \right]_q
\]

\[
- \sum_{j=i+1}^{N} \nu_{i+1} \frac{x_{i,j+1}}{x_{i+1,j+1}} [\partial_{i+1,j+1}]_q q^{\lambda_i + \sum_{l=i+1}^{N+1} (\nu_{i+1} \partial_{i+1,l} - \nu_i \partial_{i,l})}.
\]

Here we read \( x_{i,i} = 1 \) and, for Grassmann odd variables \( x_{i,j} \), the expression \( \frac{1}{x_{i,j}} \) stands for the derivative \( \frac{\partial}{\partial x_{i,j}} \).
We study how to recover the bosonization of the affine superalgebra $U_q(\hat{sl}(N|1))$ from the Heisenberg realization of $U_q(sl(N|1))$. We make the following replacement with suitable argument.

\[
\begin{align*}
\vartheta_{i,j} & \rightarrow -b_{i,j}^i(z)/\log q \quad (1 \leq i < j \leq N + 1), \\
[\vartheta_{i,j}]_q & \rightarrow \begin{cases} 
\frac{e^{b_{i,j}^i(z)} - e^{-b_{i,j}^j(z)}}{(q - q^{-1})z} & (j \neq N + 1), \\
1 & (j = N + 1).
\end{cases} \\
x_{i,j} & \rightarrow \begin{cases} 
: e^{(b+c)^i,j(z)} : & (j \neq N + 1), \\
: e^{-b_{i,j}^j(z)} : \text{ or } : e^{-b_{i,j}^i(z)} & (j = N + 1).
\end{cases} \\
\lambda_i & \rightarrow a_i^i(z)/\log q \quad (1 \leq i \leq N), \\
[\lambda_i]_q & \rightarrow \frac{e^{a_i^i(z)} - e^{-a_i^i(z)}}{(q - q^{-1})z} \quad (1 \leq i \leq N).
\end{align*}
\]

From the above replacement, the element $h_i$ of the Heisenberg realization is replaced as following.

\[
q^{h_i} \rightarrow \begin{cases}
e^{a_i^i(z)} + \sum_{l=1}^{N-1} (b_{i,l}^{i,l+1}(z) - b_{i,l}^{i+1,l}(z)) + \sum_{l=1}^{N-1} (b_{i,l}^{i,l}(z) - b_{i,l}^{i+1,l+1}(z)) & (1 \leq i \leq N - 1), \\
e^{a_i^i(z)} - \sum_{l=1}^{N-1} (b_{i,l}^{i,l}(z) + b_{i,l}^{i+1,l+1}(z)) & (i = N).
\end{cases}
\]

We impose $q$-shift to variable $z$ of the operators $a_i^i(z)$, $b_{i,j}^i(z)$. For instance, we have to replace $a_i^i(z) \rightarrow a_i^i(q^{c+1}z)$. Bridging the gap by the $q$-shift, we have the bosonizations $\psi_1^\pm(q^{\pm\frac{1}{2}}z) \in U_q(\hat{sl}(N|1))$ from $q^{h_i} \in U_q(sl(N|1))$.

\[
\begin{align*}
\psi_1^\pm(q^{\pm\frac{1}{2}}z) & = e^{a_i^i(z)} + \sum_{l=1}^{N-1} (b_{i,l}^{i,l+1}(q^{c+1}z) - b_{i,l}^{i+1,l}(q^{c+1}z)) \\
& \times \sum_{l=1}^{N-1} (b_{i,l}^{i,l}(q^{c+1}z) + b_{i,l}^{i+1,l+1}(q^{c+1}z) - b_{i,l}^{i,l+1}(q^{c+1}z)), \\
\psi_N^\pm(q^{\pm\frac{1}{2}}z) & = e^{a_i^i(z)} + \sum_{l=1}^{N-1} (b_{i,l}^{i,l+1}(q^{c+1}z) + b_{i,l}^{i+1,l+1}(q^{c+1}z)).
\end{align*}
\]

In this replacement, one element $q^{h_i}$ goes to two elements $\psi_1^\pm(q^{\pm\frac{1}{2}}z)$. Hence this replacement is not a map. Replacements from $e_i$, $f_i$ to $x_i^\pm(z)$ are given by similar way, however they are more complicated. See details in [2].

### 3.4. Wakimoto Realization

In this section we give the Wakimoto realization $F(p_a)$ whose character coincides with those of the Verma module [14]. We introduce the operators $\xi_m^i$ and $\eta_m^i$ $(1 \leq i < j \leq N, m \in \mathbb{Z})$ by

\[
\eta_{m}^{i,j}(z) = \sum_{m \in \mathbb{Z}} \eta_{m}^{i,j} z^{m-1} =: e^{c^{i,j}(z)}; \quad \xi_{m}^{i,j}(z) = \sum_{m \in \mathbb{Z}} \xi_{m}^{i,j} z^{-m} =: e^{-c^{i,j}(z)};
\]


The Fourier components $\eta_{m}^{i,j} = \oint \frac{dz}{2\pi\sqrt{-1}} z^{m} \eta^{i,j}(z)$, $\xi_{m}^{i,j} = \oint \frac{dz}{2\pi\sqrt{-1}} z^{m-1} \xi^{i,j}(z)$ ($m \in \mathbb{Z}$) are well defined on the space $F(p_{a})$. We focus our attention on the operators $\eta_{0}^{i,j}, \xi_{0}^{i,j}$ satisfying $(\eta_{0}^{i,j})^{2} = 0, (\xi_{0}^{i,j})^{2} = 0$. They satisfy

$$\text{Im}(\eta_{0}^{i,j}) = \text{Ker}(\eta_{0}^{i,j}), \quad \text{Im}(\xi_{0}^{i,j}) = \text{Ker}(\xi_{0}^{i,j}), \quad \eta_{0}^{i,j} \xi_{0}^{i,j} + \xi_{0}^{i,j} \eta_{0}^{i,j} = 1.$$ 

We have a direct sum decomposition.

$$F(p_{a}) = \eta_{0}^{i,j} \xi_{0}^{i,j} F(p_{a}) \oplus \xi_{0}^{i,j} \eta_{0}^{i,j} F(p_{a}),$$

$$\text{Ker}(\eta_{0}^{i,j}) = \eta_{0}^{i,j} \xi_{0}^{i,j} F(p_{a}), \quad \text{Coker}(\eta_{0}^{i,j}) = \xi_{0}^{i,j} \eta_{0}^{i,j} F(p_{a}) = F(p_{a})/(\eta_{0}^{i,j} \xi_{0}^{i,j}) F(p_{a}).$$

We set the operator $\eta_{0}, \xi_{0}$ by

$$\eta_{0} = \prod_{1 \leq i < j \leq N} \eta_{0}^{i,j}, \quad \xi_{0} = \prod_{1 \leq i < j \leq N} \xi_{0}^{i,j}.$$ 

**Definition 3.6** [14] We introduce the subspace $F(p_{a})$ by

$$F(p_{a}) = \eta_{0} \xi_{0} F(p_{a}).$$

We call $F(p_{a})$ the Wakimoto realization.

4. Screening and Vertex Operator

In this section we give the screening that commutes with the quantum superalgebra $U_{q}(\widehat{sl}(N|1))$. We propose the vertex operators and the correlation functions.

4.1. Screening

In this section we give the screening $Q_{i}$ ($1 \leq i \leq N$) that commutes with the quantum superalgebra $U_{q}(\widehat{sl}(N|1))$ for an arbitrary level $k \neq -N + 1$ [15]. The Jackson integral with parameter $p \in \mathbb{C}$ ($|p| < 1$) and $s \in \mathbb{C}^{*}$ is defined by

$$\int_{0}^{s \infty} f(z) dz = s(1 - p) \sum_{m \in \mathbb{Z}} f(s p^{m}) p^{m}.$$ 

In order to avoid divergence we work in the Fock space.

**Theorem 4.1** [15] The screening $Q_{i}$ commutes with the quantum superalgebra.

$$[Q_{i}, U_{q}(\widehat{sl}(N|1))] = 0 \quad (1 \leq i \leq N).$$
We have introduced the screening operators $Q_i$ ($1 \leq i \leq N$) as follows.

$$Q_i = \int_0^{s_N^{\infty}} e^{-\left(\frac{b+b^+}{2}\right)(z^{\frac{h+N-1}{2}})} \tilde{S}_i(z) : d_p \tilde{z}, \quad (p = q^{2(k+N-1)}).$$

Here we have set the bosonic operators $\tilde{S}_i(z)$ ($1 \leq i \leq N$) by

$$\tilde{S}_i(z) = \frac{1}{(q - q^{-1})z} \sum_{j=i+1}^N \left( e^{-b^+_j (q^{N-1-j} z) - (b+c)^{i,j} (q^{N-1} z)} - e^{-b^+_j (q^{N-1-j} z) - (b+c)^{i,j} (q^{N-1} z)} \right)$$

$$\times \left( e^{(b+c)^{i,j} (q^{N-1} z)} + \sum_{j=i+1}^N (b_j^{i,j} (q^{N-j} z) - b_j^{i,j} (q^{N-j} z)) + b^{i,j} (q^{N-1} z) - b^{i,j} (q^{N-1} z) \right)$$

$$+ q : e^{b^{i,j} (q^{N-1} z) + b^{i,j} (q^{N-1} z)} :$$

$$\tilde{S}_N(z) = -q^{-1} : e^{b^{N,N+1} (z)} :$$

4.2. Vertex Operator

In this section we introduce the vertex operators $\Phi(z)$, $\Phi^*(z)$ [15]. Let $F$ and $F'$ be $U_q(\hat{s}l(N|1))$ representation for an arbitrary level $k \neq -N+1$. Let $V_\alpha$ and $V_\alpha^* = 2^N$-dimensional typical representation with a parameters $\alpha$ [21]. Let $\{v_j\}_{j=1}^N$ be the basis of $V_\alpha$. Let $\{v_j^*\}_{j=1}^N$ be the dual basis of $V_\alpha^*$, satisfying $(v_i | v_j^*) = \delta_{i,j}$. Let $V_{\alpha,z}$ and $V_{\alpha,z}^*$ be the evaluation module and its dual of the typical representation. For instance, the 8-dimensional representation $V_{\alpha,z}$ of $U_q(\hat{s}l(3|1))$ is given by

$$h_1 = E_{3,3} - E_{4,4} + E_{5,5} - E_{6,6},$$
$$h_2 = E_{2,2} - E_{3,3} + E_{6,6} - E_{7,7},$$
$$h_3 = \alpha(E_{1,1} + E_{2,2}) + (\alpha + 1)(E_{3,3} + E_{4,4} + E_{5,5} + E_{6,6}) + (\alpha + 2)(E_{7,7} + E_{8,8}),$$
$$e_1 = E_{3,4} + E_{5,6},$$
$$e_2 = E_{2,3} + E_{6,7},$$
$$e_3 = \sqrt{[\alpha]} q E_{1,2} - \sqrt{[\alpha + 1]} q(E_{3,5} + E_{4,6}) + \sqrt{[\alpha + 2]} q E_{7,8},$$
$$f_1 = E_{4,3} + E_{5,6},$$
$$f_2 = E_{3,2} + E_{7,6},$$
$$f_3 = \sqrt{[\alpha]} q E_{2,1} - \sqrt{[\alpha + 1]} q(E_{5,3} + E_{6,4}) + \sqrt{[\alpha + 2]} q E_{8,7},$$
$$h_0 = -\alpha(E_{1,1} + E_{4,4}) - (\alpha + 1)(E_{2,2} + E_{3,3} + E_{6,6} + E_{7,7}) - (\alpha + 2)(E_{5,5} + E_{8,8}),$$
$$e_0 = -z(\sqrt{[\alpha]} q E_{4,1} - \sqrt{[\alpha + 1]} q(E_{6,2} + E_{7,3}) + \sqrt{[\alpha + 2]} q E_{8,5}),$$
$$f_0 = z^{-1}(\sqrt{[\alpha]} q E_{1,4} - \sqrt{[\alpha + 1]} q(E_{2,6} + E_{3,7}) + \sqrt{[\alpha + 2]} q E_{5,8}).$$
Consider the following intertwiners of $U_q(\widehat{sl}(N|1))$-representation [20]:
\[
\Phi(z) : \mathcal{F} \rightarrow \mathcal{F}' \otimes V_{\alpha,z}, \quad \Phi^*(z) : \mathcal{F} \rightarrow \mathcal{F}' \otimes V_{\alpha,z}^*.
\]
They are intertwiners in the sense that for any $x \in U_q(\widehat{sl}(N|1))$,
\[
\Phi(z) \cdot x = \Delta(x) \cdot \Phi(z), \quad \Phi^*(z) \cdot x = \Delta(x) \cdot \Phi^*(z).
\]
We expand the intertwining operators.
\[
\Phi(z) = \sum_{j=1}^{2N} \Phi_j(z) \otimes v_j, \quad \Phi^*(z) = \sum_{j=1}^{2N} \Phi_j^*(z) \otimes v_j^*.
\]
We set the $\mathbb{Z}_2$-grading of the intertwiner be $|\Phi(z)| = |\Phi^*(z)| = 0$. For $l_a = (l_{a1}, l_{a2}, \ldots, l_{aN}) \in \mathbb{C}^N$ and $\beta \in \mathbb{C}$, we set the bosonic operator $\phi^{l_a}(z|\beta)$ by
\[
\phi^{l_a}(z|\beta) = e^{\sum_{i,j=1}^{N} \left( \frac{l_{ai}}{k+1} \cdot \frac{\min(i,j)}{N-1} \cdot \frac{\max(i,j)}{1} \right) a^i (z|\beta)}.
\]
In order to balance the background charge of the vertex operators, we introduce the product of the screenings $Q^{(t)}$ for $t = (t_1, t_2, \ldots, t_N) \in \mathbb{N}^N$.
\[
Q^{(t)} = Q_1^{t_1} Q_2^{t_2} \cdots Q_N^{t_N}.
\]
The screening operator $Q^{(t)}$ give rise to the map,
\[
Q^{(t)} : \mathcal{F}(p_a) \rightarrow \mathcal{F}(p_a + \hat{t}).
\]
Here $\hat{t} = (\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_N)$ where $\hat{t}_i = \sum_{j=1}^{N} A_{i,j} t_j$.

**Theorem 4.2** [15] For $k = \alpha \neq 0, -1, -2, \ldots, -N + 1$, bosonizations of the special components of the vertex operators $\Phi^{(t)}(z)$ and $\Phi^{*(t)}(z)$ are given by
\[
\Phi^{(t)}_{2\alpha}(z) = Q^{(t)} \phi^j \left( q^{k+N-1}z \mid -\frac{k+N-1}{2} \right),
\]
\[
\Phi^{*(t)}_{1\alpha}(z) = Q^{(t)} \phi^{*j} \left( q^k z \mid -\frac{k+N-1}{2} \right),
\]
where we have used $\hat{t} = -(0, \ldots, 0, \alpha + N - 1)$, $\hat{t}^* = (0, \ldots, 0, \alpha)$ and $t = (t_1, t_2, \ldots, t_N) \in \mathbb{N}^N$. The other components $\Phi^{(t)}_j(z)$ and $\Phi^{*(t)}_j(z) (1 \leq j \leq 2^N)$ are determined by the intertwining property and are represented by multiple contour integrals of Drinfeld currents and the special components $\Phi^{(t)}_{2\alpha}(z)$ and $\Phi^{*(t)}_{1\alpha}(z)$. We have checked this theorem for $N = 2, 3, 4$. 
Here we give additional explanation on the above theorem. The explicit formulae of the intertwining properties $\Phi^{(t)}(z) \cdot x = \Delta(x) \cdot \Phi^{(t)}(z)$ for $U_q(\widehat{\mathfrak{sl}}(3|1))$ are summarized as follows. We have set the $\mathbb{Z}_2$-grading of $V_\alpha$ as follows: $|v_1| = |v_5| = |v_6| = |v_7| = 0$, and $|v_2| = |v_3| = |v_4| = |v_8| = 1$.

\[
\begin{align*}
\Phi_3^{(t)}(z) &= [\Phi_4^{(t)}(z), f_1]_q, \quad \Phi_5^{(t)}(z) = [\Phi_6^{(t)}(z), f_1]_q, \\
\Phi_2^{(t)}(z) &= [\Phi_3^{(t)}(z), f_2]_q, \quad \Phi_6^{(t)}(z) = [\Phi_7^{(t)}(z), f_2]_q, \\
\Phi_1^{(t)}(z) &= \frac{1}{\sqrt{|\alpha|}_q} [\Phi_2^{(t)}(z), f_3]_{q^{-\alpha}}, \quad \Phi_3^{(t)}(z) = \frac{-1}{\sqrt{|\alpha+1|}_q} [\Phi_5^{(t)}(z), f_3]_{q^{-\alpha-1}}, \\
\Phi_4^{(t)}(z) &= \frac{-1}{\sqrt{|\alpha+1|}_q} [\Phi_6^{(t)}(z), f_3]_{q^{-\alpha-1}}, \quad \Phi_7^{(t)}(z) = \frac{1}{\sqrt{|\alpha+2|}_q} [\Phi_8^{(t)}(z), f_3]_{q^{-\alpha-2}}.
\end{align*}
\]

The elements $f_j$ are written by contour integral of the Drinfeld current $f_j = \oint \frac{dw}{2\pi i} x_j^-(w)$. Hence the components $\Phi_j^{(t)} (1 \leq j \leq 8)$ are represented by multiple contour integrals of Drinfeld currents $x_j^-(w) (1 \leq j \leq 3)$ and the special component $\Phi_8^{(t)}(z)$.

### 4.3. Correlation Function

In this section we study the correlation function as an application of the vertex operators. We study non-vanishing property of the correlation function which is defined to be the trace of the vertex operators over the Wakimoto module of $U_q(\widehat{\mathfrak{sl}}(N|1))$. We propose the $q$-Virasoro operator $L_0$ for $k = \alpha \neq -N + 1$ as follows.

\[
L_0 = \frac{1}{2} \sum_{i,j=1}^{N} \sum_{m \in \mathbb{Z}} : a_{-m}^i m^2 [\min(i,j)]_q [((N-1)-\max(i,j))]_q a_{jm}^i : \\
+ \sum_{i,j=1}^{N} \min(i,j)(N-1-\max(i,j)) \frac{a_{jm}^i}{(k+N-1)(N-1)} \\
- \frac{1}{2} \sum_{1 \leq i < j \leq N} \sum_{m \in \mathbb{Z}} : b_{-m}^{i,j} m^2 [m]_q^2 b_{jm}^{i,j} : + \frac{1}{2} \sum_{1 \leq i < j \leq N} \sum_{m \in \mathbb{Z}} : c_{-m}^{i,j} m^2 [m]_q^2 c_{jm}^{i,j} : \\
+ \frac{1}{2} \sum_{1 \leq i < j \leq N} \sum_{m \in \mathbb{Z}} : b_{-m}^{i,N+1} m^2 [m]_q^2 b_{jm}^{i,N+1} : + \frac{1}{2} \sum_{1 \leq i < j \leq N} b_{jm}^{i,N+1}.
\]

The $L_0$ eigenvalue of $|l_0, 0, 0\rangle$ is $\frac{1}{2(k+N+1)} (\hat{\lambda} | 2 \hat{\rho} |$, where $\hat{\rho} = \sum_{i=1}^{N} \hat{\Lambda}_i$ and $\hat{\lambda} = \sum_{i=1}^{N} \bar{\lambda}_i \bar{\Lambda}_i$.

**Theorem 4.3 [15]** For $k = \alpha \neq 0, -1, -2, \cdots, -N + 1$, the correlation...
function of the vertex operators,
\[ \text{Tr}_{F(l_0)} \left( q^{L_0} \Phi_{i_1}^{(y(1))}(w_1) \cdots \Phi_{i_m}^{(y(m))}(w_m) \Phi_{j_1}^{(x(1))}(z_1) \cdots \Phi_{j_n}^{(x(n))}(z_n) \right) \neq 0, \]
if and only if \( x_s = (x_{s,1}, x_{s,2}, \ldots, x_{s,N}) \in \mathbb{N}^N (1 \leq s \leq n) \) and \( y_s = (y_{s,1}, y_{s,2}, \ldots, y_{s,N}) \in \mathbb{N}^N (1 \leq s \leq m) \) satisfy the following condition.

\[ \sum_{s=1}^{n} x_{s,i} + \sum_{s=1}^{m} y_{s,i} = \frac{(n-m)i}{N-1} + n \cdot i \quad (1 \leq i \leq N). \]

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