AIP | Journal of Mathematical Physics

Diagonalization of transfer matrix of supersymmetry Uq((M+1|N+1)) chain with a boundary

Takeo Kojima

Citation: J. Math. Phys. **54**, 043507 (2013); doi: 10.1063/1.4799933
View online: http://dx.doi.org/10.1063/1.4799933
View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v54/i4
Published by the American Institute of Physics.

JOURNAL OF MATHEMATICAL PHYSICS **54**, 043507 (2013)

Diagonalization of transfer matrix of supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ chain with a boundary

Takeo Kojima[a)]
Faculty of Engineering, Yamagata University, Jonan 4-3-16, Yonezawa 992-8510, Japan

(Received 30 November 2012; accepted 21 March 2013; published online 15 April 2013)

We study the supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ analogue of the supersymmetric *t-J* model with a boundary. Our approach is based on the algebraic analysis method of solvable lattice models. We diagonalize the commuting transfer matrix by using the bosonizations of the vertex operators associated with the quantum affine supersymmetry $U_q(\widehat{sl}(M+1|N+1))$. © *2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4799933]

I. INTRODUCTION

There have been many developments in the exactly solvable models. Various methods were invented to solve models. The algebraic analysis method[1] provides a powerful method to study exactly solvable models. This paper is devoted to the algebraic analysis method to open boundary problem of exactly solvable model. In this paper we study the supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ analogue of the supersymmetric *t-J* model with a boundary, where $M, N = 0, 1, 2, \ldots$ such that $M \neq N$. The supersymmetric *t-J* model was proposed in an attempt to understand high-temperature superconductivity. In the framework of the quantum inverse scattering method,[3,4] the investigations of the supersymmetric *t-J* model and the $U_q(\widehat{sl}(M+1|N+1))$ analogue of it have been carried out in several papers.[8,9,11–15] In the framework of the algebraic analysis method,[1,2,19,23] the $U_q(\widehat{sl}(M+1|N+1))$ chain "without boundaries" has been studied in few papers.[16–18] In this paper we focus our attention on the boundary condition of the exactly solvable model. We study the supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ chain "with a boundary" in the framework of the algebraic analysis method.[1,2,24] We diagonalize the commuting transfer matrix of the supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ chain with a boundary. Several solvable models with a boundary have been studied by means of algebraic analysis method.[24–30] Here we would like to draw reader's attention to new technical aspect in our problem. Generally speaking, in the algebraic analysis method, the transfer matrix $T_B(z)$ of the solvable model with a boundary is written by the vertex operators $\Phi_j^*(z)$ and $\Phi_j(z)$ associated with the quantum affine symmetry

$$T_B^{(i)}(z) = g \sum_{j=1}^{M+N+2} \Phi_j^*(z^{-1}) K^{(i)}(z)_j^j \Phi_j(z)(-1)^{[v_j]}.$$

Here $K^{(i)}(z)_j^j$ is the matrix element of the boundary K-matrix. The key of diagonalization of the transfer matrix $T_B(z)$ is the bosonization of the boundary state ${}_{(i)}\langle B|$ that satisfies the following condition:

$${}_{(i)}\langle B|T_B^{(i)}(z) = {}_{(i)}\langle B|.$$

By using the bosonizations of the vertex operators, we construct the boundary state ${}_{(i)}\langle B|$. Hence our calculations depend heavily on the bosonization formulae of the vertex operators. For solvable models that are governed by the quantum symmetry $U_q(\widehat{sl}(N))$, $U_q(A_2^{(2)})$ or the elliptic symmetry

a)kojima@yz.yamagata-u.ac.jp

$U_{q,p}(\widehat{sl}(N))$,[24–29] the bosonizations of the vertex operators are realized by "monomial." However, the bosonizations of the vertex operators for the quantum supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ are realized by "sum." For instance, the bosonizations of the vertex operators $\Phi^*_{M+1+j}(z)$ $(j=1,2,\ldots,N+1)$ are written by the sum $\sum_{\epsilon_1,\epsilon_2,\cdots,\epsilon_j}$ as follows (see (4.74)):

$$\Phi^*_{M+1+j}(z) = \sum_{\epsilon_1,\epsilon_2,\cdots,\epsilon_j=\pm} e^{\frac{M\pi\sqrt{-1}}{M-N}} q^{j-1}(q-q^{-1})^M (qz)^{-1} \prod_{k=1}^{j}\epsilon_k \prod_{k=1}^{M+j}\int_{C_{M+1+j}} \frac{dw_k}{2\pi\sqrt{-1}w_k}$$
$$\times \frac{1}{\prod_{k=0}^{M}(1-qw_k/w_{k+1})(1-qw_{k+1}/w_k)\prod_{k=1}^{j-1}(1-q^{\epsilon_k}w_{M+k}/w_{M+k+1})}$$
$$\times\ \eta_0\xi_0 : \phi_1^*(z)X_1^-(qw_1)\cdots X_M^-(qw_M)X^-_{M+1,\epsilon_1}(qw_{M+1})\cdots X^-_{M+j,\epsilon_j}(qw_{M+j}) : \eta_0\xi_0.$$

Technically this is cool part of our problem. Surprisingly we shall conclude that the bosonization of the boundary state ${}_{(i)}\langle B|$ is realized by "monomial," though those of the vertex operator is realized by "sum." The bosonization of the boundary state ${}_{(i)}\langle B|$ is constructed by acting a monomial of exponential $e^{G^{(i)}}$ on the highest weight vector $v^*_{\Lambda_{M+1}} \in L^*(\Lambda_{M+1})$ of the quantum supersymmetry $U_q(\widehat{sl}(M+1|N+1))$,

$${}_{(i)}\langle B| = v^*_{\Lambda_{M+1}} \cdot e^{G^{(i)}}.$$

Here $G^{(i)}$ is quadratic in the bosonic operators (see (5.5)). We would like to give a comment on the earlier study in the framework of the algebraic analysis method.[30] The supersymmetric *t-J* model with a boundary (the supersymmetry $U_q(\widehat{sl}(2|1))$ chain with a boundary) was studied and the bosonization conjecture of the boundary state was given in Ref. 30. However, their conjecture of the boundary state is different from our bosonization upon the special case of $M=1, N=0$. In this paper we give not only the bosonization formulae of the boundary state, but also give detailed proof that the vector ${}_{(i)}\langle B|$ becomes the eigenvector of the transfer matrix $T_B^{(i)}(z)$. Moreover, we classify the boundary K-matrix and find a new solution that has three different diagonal elements (see (A11)). Of-course we construct the boundary state associated with this new K-matrix.

The text is organized as follows. In Sec. II we introduce the supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ analogue of finite *t-J* model with double boundaries. We introduce the $U_q(\widehat{sl}(M+1|N+1))$ analogue of semi-infinite *t-J* model as the limit of the finite chain. In Sec. III we give mathematical formulation of the supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ chain with a boundary. This formulation is based on the representation theory of the quantum supersymmetry $U_q(\widehat{sl}(M+1|N+1))$. This formulation is free from the difficulty of divergence. In Sec. IV we review the bosonizations of the vertex operators and give the integral representations of the vertex operators. In Sec. V we give the bosonizations of the boundary state, that is the main result of this paper. We give detailed proof of the bosonization of the boundary state. In Appendix A we classify the diagonal solutions of the boundary Yang-Baxter equation associated with $U_q(\widehat{sl}(M+1|N+1))$. New solution is presented even for the small rank case $U_q(\widehat{sl}(2|1))$. In Appendix B we summarize the normal orderings, that we use in Secs. IV and VI.

II. $U_q(\widehat{sl}(M+1|N+1))$ CHAIN WITH A BOUNDARY

In this section we introduce the $U_q(\widehat{sl}(M+1|N+1))$ analogue of the supersymmetric *t-J* model with a boundary. We fix a complex number $0<|q|<1$ and two natural numbers $M, N = 0, 1, 2, \ldots$ such that $M \neq N$.

A. Finite $U_q(\widehat{sl}(M+1|N+1))$ chain

In this section we introduce the finite $U_q(\widehat{sl}(M+1|N+1))$ chain with double boundaries. We follow the general scheme given by Refs. 15, 20–22, and 30. In what follows we use the standard

notation of the q-integer

$$[a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}. \tag{2.1}$$

Let us introduce the signatures ν_i $(i = 1, 2, \ldots, M + N + 2)$ by

$$\nu_1 = \nu_2 = \cdots = \nu_{M+1} = +, \nu_{M+2} = \nu_{M+3} = \cdots = \nu_{M+N+2} = -. \tag{2.2}$$

Let us set the vector spaces $V_1 = \oplus_{j=1}^{M+1}\mathbf{C}v_j$ and $V_0 = \oplus_{j=1}^{N+1}\mathbf{C}v_{M+1+j}$. We set $V = V_1 \oplus V_0$. The $\mathbf{Z}_2$-grading of the basis $\{v_j\}_{1\le j\le M+N+2}$ of V is chosen to be $\left[v_j\right] = \frac{\nu_j+1}{2}$ $(j = 1, 2, \ldots, M + N + 2)$. A linear operator $S \in \mathrm{End}(V)$ is represented in the form of a $(M + N + 2) \times (M + N + 2)$ matrix : $Sv_j = \sum_{i=1}^{M+N+2} v_i S_{i,j}$. The $\mathbf{Z}_2$-grading of $(M + N + 2) \times (M + N + 2)$ matrix $(S_{i,j})_{1\le i,j\le M+N+2}$ is defined by $[S] = [v_i] + [v_j](mod.2)$ if RHS of the equation does not depend on i and j such that $S_{i,j} \neq 0$. All $(M + N + 2) \times (M + N + 2)$ matrix $S = (S_{i,j})_{1\le i,j\le M+N+2}$ are divided into blocks : $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, D are $(M + 1) \times (M + 1), (M + 1) \times (N + 1)$, $(N + 1) \times (M + 1), (N + 1) \times (N + 1)$ matrices, respectively. We introduce the supertrace of $S \in \mathrm{End}(V)$ by

$$\mathrm{str}_V(S) = \mathrm{tr}_{V_1}(A) - \mathrm{tr}_{V_0}(D) = \sum_{j=1}^{M+N+2} (-1)^{[v_j]} S_{j,j}, \tag{2.3}$$

where $\mathrm{tr}_{V_1}(A)$, $\mathrm{tr}_{V_0}(D)$ represent ordinary trace. We introduce supertranspose "*st*" by

$$S^{st} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{st} = \begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix}, \tag{2.4}$$

where A^t, B^t, C^t, D^t represent ordinary transpose of matrices. We consider the tensor product $V \otimes V \otimes \cdots \otimes V$ of n space and define the action of the operator $S_1 \otimes S_2 \otimes \ldots \otimes S_n$ where $S_j \in \mathrm{End}(V)$ have $\mathbf{Z}_2$-grading. We define

$$\begin{aligned} &S_1 \otimes S_2 \otimes \cdots \otimes S_n \cdot v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_n} \\ &= \exp\left(\pi\sqrt{-1}\sum_{k=1}^{n}[S_k]\sum_{l=1}^{k-1}[v_{j_l}]\right) S_1 v_{j_1} \otimes S_2 v_{j_2} \otimes \cdots \otimes S_n v_{j_n}. \end{aligned} \tag{2.5}$$

Definition 2.1: We set the R-matrix $R(z) \in \mathrm{End}(V \otimes V)$ associated with the quantum supersymmetry $U_q(\widehat{sl}(M + 1|N + 1))$ as follows:[8,9]

$$R(z) = r(z)\bar{R}(z), \bar{R}(z)v_{j_1} \otimes v_{j_2} = \sum_{k_1,k_2=1}^{M+N+2} v_{k_1} \otimes v_{k_2}\bar{R}(z)_{k_1,k_2}^{j_1,j_2}. \tag{2.6}$$

Here we have set

$$\bar{R}(z)_{j,j}^{j,j} = \begin{cases} -1 & (1 \le j \le M + 1), \\ -\dfrac{(q^2 - z)}{(1 - q^2 z)} & (M + 2 \le j \le M + N + 2), \end{cases} \tag{2.7}$$

$$\bar{R}(z)_{i,j}^{i,j} = \frac{(1 - z)q}{(1 - q^2 z)}(1 \le i \ne j \le M + N + 2), \tag{2.8}$$

$$\bar{R}(z)_{i,j}^{j,i} = \begin{cases} (-1)^{[v_i][v_j]}\dfrac{(1 - q^2)}{(1 - q^2 z)} & (1 \le i < j \le M + N + 2), \\ (-1)^{[v_i][v_j]}\dfrac{(1 - q^2)z}{(1 - q^2 z)} & (1 \le j < i \le M + N + 2), \end{cases} \tag{2.9}$$

$$\bar{R}(z)_{i,j}^{i,j} = 0 \text{ otherwise.} \tag{2.10}$$

Here we have set

$$r(z) = z^{\frac{1-M+N}{M-N}} \exp\left(-\sum_{m=1}^{\infty} \frac{[(M-N-1)m]_q}{m[(M-N)m]_q} q^m (z^m - z^{-m})\right). \tag{2.11}$$

The R-matrix $R(z)$ satisfies the weight conservation condition: $R(z)_{k_1,k_2}^{j_1,j_2} \neq 0$ only when $[j_1] + [j_2] + [k_1] + [k_2] = 0(mod.2)$. For instance, the function $r(z)$ is written as following in the case $M > N$:

$$r(z) = z^{\frac{1-M+N}{M-N}} \frac{(q^2 z; q^{2(M-N)})_\infty (q^{2(M-N)} z^{-1}; q^{2(M-N)})_\infty}{(q^{2(M-N)} z; q^{2(M-N)})_\infty (q^2 z^{-1}; q^{2(M-N)})_\infty}. \tag{2.12}$$

Here we have used the infinite product

$$(z; p)_\infty = \prod_{n=0}^{\infty} (1 - p^n z)(|p| < 1).$$

The R-matrix $R(z)$ satisfies the graded Yang-Baxter equation.

$$R_{12}(z_1/z_2) R_{13}(z_1/z_3) R_{23}(z_2/z_3) = R_{23}(z_2/z_3) R_{13}(z_1/z_3) R_{12}(z_1/z_2), \tag{2.13}$$

where $R_{12}(z)$, $R_{13}(z)$, and $R_{23}(z)$ act in $V \otimes V \otimes V$ with $R_{12}(z) = R(z) \otimes 1$, $R_{23}(z) = 1 \otimes R(z)$, etc. Multiplying the signature $(-1)^{[v_{k_1}][v_{k_2}]}$ to our R-matrix $R(z)_{k_1,k_2}^{j_1,j_2}$, we have the R-matrix $R^{PS}(z)$ of the Perk-Schultz model:[9] $R^{PS}(z)_{k_1,k_2}^{j_1,j_2} = (-1)^{[v_{k_1}][v_{k_2}]} R(z)_{k_1,k_2}^{j_1,j_2}$. The R-matrix $R^{PS}(z)$ of the Perk-Schultz model[9] satisfies the ungraded Yang-Baxter equation. The R-matrix $R(z)$ satisfies the initial condition $R(1) = P$ where P is the graded permutation operator: $P_{k_1,k_2}^{j_1,j_2} = \delta_{j_1,k_2} \delta_{j_2,k_1} (-1)^{[v_{k_1}][v_{k_2}]}$. The R-matrix $R(z)$ satisfies the unitary condition

$$R_{12}(z) R_{21}(1/z) = 1, \tag{2.14}$$

where $R_{21}(z) = P R_{12}(z) P$. The R-matrix $R(z)$ satisfies and the crossing symmetry:

$$R_{12}^{-1,st_1}(z) M_1 R_{12}^{st_1}(q^{2(N-M)} z) M_1^{-1} = 1, \tag{2.15}$$

where supertranspose "*st*" is introduced in (2.4). Here we have set the matrix $M \in \mathrm{End}(V)$ defined by

$$M_{i,j} = \delta_{i,j} M_j,\ M_j = \begin{cases} q^{-2(j-1)} & (1 \le j \le M+1), \\ q^{-2(2M+2-j)} & (M+2 \le j \le M+N+2). \end{cases} \tag{2.16}$$

We have the commutativity $[M \otimes M, R(z)] = 0$. In Appendix A we classify the boundary K-matrix that satisfies the graded boundary Yang-Baxter equation (2.28), and find new diagonal solution (see (A11)).

Definition 2.2: We set the diagonal K-matrix $K^{(i)}(z) \in \mathrm{End}(V)$ $(i = 1, 2, 3)$ *associated with the quantum supersymmetry* $U_q(\widehat{sl}(M+1|N+1))$ *as follows:*

$$K^{(i)}(z) = z^{-\frac{2M}{M-N}} \frac{\varphi^{(i)}(z)}{\varphi^{(i)}(z^{-1})} \bar{K}^{(i)}(z) (i = 1, 2, 3), \tag{2.17}$$

$$\bar{K}^{(i)}(z) v_j = \sum_{k=1}^{M+N+2} v_k \delta_{j,k} \bar{K}^{(i)}(z)_j^j. \tag{2.18}$$

The $\bar{K}^{(i)}(z)_j^j$ *and* $\varphi^{(i)}(z)$ *are given in the following conditions 1,2,3.*

Condition 1: $\bar{K}^{(1)}(z)_j^j$ *are* $\varphi^{(1)}(z)$ *are defined as follows:*

$$\bar{K}^{(1)}(z)_j^j = 1 (j = 1, 2, \cdots, M+N+2), \tag{2.19}$$

and

$$\varphi^{(1)}(z) = \exp\left(\sum_{m=1}^{\infty} \frac{[2(N+1)m]_q}{m[2(M-N)m]_q} z^{2m} + \sum_{j=1}^{M}\sum_{m=1}^{\infty} \frac{[2(M-N-j)m]_q}{2m[2(M-N)m]_q}(1-q^{2m})z^{2m} \right.$$
$$\left. + \sum_{j=M+2}^{M+N+1}\sum_{m=1}^{\infty} \frac{[2(-M-N-2-j)m]_q}{2m[2(M-N)m]_q}(1+q^{2m})z^{2m} - \sum_{m=1}^{\infty} \frac{[(M-N-1)m]_q}{2m[(M-N)m]_q} q^m z^{2m}\right). \tag{2.20}$$

Condition 2: $\bar{K}^{(2)}(z)^j_j$ and $\varphi^{(2)}(z)$ are defined by followings. We fix a natural number $L = 1, 2, \ldots, M + N + 1$ and a complex number $r \in \mathbf{C}$.

$$\bar{K}^{(2)}(z)^j_j = \begin{cases} 1 & (1 \le j \le L), \\ \dfrac{1-r/z}{1-rz} & (L+1 \le j \le M+N+2). \end{cases} \tag{2.21}$$

Condition 2.1: For $L \le M + 1$ we set

$$\varphi^{(2)}(z) = \varphi^{(1)}(z) \times \exp\left(-\sum_{m=1}^{\infty} \frac{[(M-N-L)m]_q}{m[(M-N)m]_q}(rq^{-L}z)^m\right). \tag{2.22}$$

Condition 2.2: For $M + 2 \le L \le M + N + 2$ we set

$$\varphi^{(2)}(z) = \varphi^{(1)}(z) \times \exp\left(-\sum_{m=1}^{\infty} \frac{[(-M-N-2+L)m]_q}{m[(M-N)m]_q}(rq^{L-2M-2}z)^m\right). \tag{2.23}$$

Condition 3: $\bar{K}^{(3)}(z)^j_j$ and $\varphi^{(3)}(z)$ are defined by followings. We fix two natural numbers L, $K = 1, 2, \ldots, M + N$ such that $L + K \le M + N + 1$. We fix a complex number $r \in \mathbf{C}$.

$$\bar{K}^{(3)}(z)^j_j = \begin{cases} 1 & (1 \le j \le L), \\ \dfrac{1-r/z}{1-rz} & (L+1 \le j \le L+K), \\ z^{-2} & (L+K+1 \le j \le M+N+2), \end{cases} \tag{2.24}$$

Condition 3.1: For $L + K \le M + 1$ we set

$$\varphi^{(3)}(z) = \varphi^{(1)}(z)$$
$$\times \exp\left(\sum_{m=1}^{\infty}\left\{\frac{[(-M+N+L)m]_q}{m[(N-M)m]_q}(rq^{-L}z)^m + \frac{[(-M+N+L+K)m]_q}{m[(M-N)m]_q}(q^{L-K}z/r)^m\right\}\right). \tag{2.25}$$

Condition 3.2: For $L \le M + 1 \le L + K + 1$ we set

$$\varphi^{(3)}(z) = \varphi^{(1)}(z) \tag{2.26}$$
$$\times \exp\left(\sum_{m=1}^{\infty}\left\{\frac{[(-M+N+L)m]_q}{m[(M-N)m]_q}(rq^{-L}z)^m + \frac{[(M+N+L+2-L-K)m]_q}{m[(M-N)m]_q}(q^{3L+K-2M-2}z/r)^m\right\}\right).$$

Condition 3.3: For $M+1 \le L-1$ we set

$$\varphi^{(3)}(z) = \varphi^{(1)}(z) \tag{2.27}$$
$$\times \exp\left(\sum_{m=1}^{\infty}\left\{\frac{[(M+N+2-L)m]_q}{m[(M-N)m]_q}(rq^{L-2M-2}z)^m + \frac{[(M+N+L+2-L-K)m]_q}{m[(M-N)m]_q}(q^{L-2M-2}z/r)^m\right\}\right).$$

In what follows we sometimes just write $K(z)$ by dropping the suffix "(i)" from $K^{(i)}(z)$. For classification of K-matrix, see Appendix A and Refs. 10–12. The K-matrix $K(z) \in \mathrm{End}(V)$ satisfies the graded boundary Yang-Baxter equation,

$$K_2(z_2)R_{21}(z_1z_2)K_1(z_1)R_{12}(z_1/z_2) = R_{21}(z_1/z_2)K_1(z_1)R_{12}(z_1z_2)K_2(z_2). \tag{2.28}$$

The K-matrix $K(z)$ satisfies $K(1) = 1$. The K-matrix $K(z)$ satisfies the boundary unitary condition

$$K(z)K(1/z) = 1. \tag{2.29}$$

We set the dual K-matrix $K^+(z) \in \mathrm{End}(V)$ by

$$K^+(z) = K(1/q^{N-M}z)^{st}M, \tag{2.30}$$

where supertranspose "st" is introduced in (2.4). The dual K-matrix $K^+(z)$ satisfies the dual graded boundary Yang-Baxter equation,

$$\begin{aligned} &K_2^+(z_2)^{st_2}M_2^{-1}R_{2,1}(1/q^{2(M-N)}z_1z_2)M_2K_1^+(z_1)^{st_1}R_{12}(z_2/z_1) \\ &= R_{21}(z_2/z_1)K_1^+(z_1)^{st_1}M_1^{-1}R_{12}(1/q^{2(M-N)}z_1z_2)M_1K_2^+(z_2)^{st_2}. \end{aligned} \tag{2.31}$$

We set the monodromy matrix $\mathcal{T}(z)$ by

$$\mathcal{T}(z) = R_{01}(z)R_{02}(z)\cdots R_{0,P}(z) \in \mathrm{End}(V_P \otimes \cdots \otimes V_1 \otimes V_0), \tag{2.32}$$

where V_j are copies of V.

Definition 2.3: We introduce the transfer matrix $T_B^{fin}(z)$ by

$$T_B^{fin}(z) = \mathrm{str}_{V_0}(K^+(z)\mathcal{T}(z^{-1})^{-1}K(z)\mathcal{T}(z)), \tag{2.33}$$

where the supertrace str is defined in (2.3).

Proposition 2.4: The transfer matrix $T_B^{fin}(z)$ form a commutative family,

$$[T_B^{fin}(z_1), T_B^{fin}(z_2)] = 0 \text{ for any } z_1, z_2. \tag{2.34}$$

The commutativity is proved by using unitarity and cross-symmetry, boundary Yang-Baxter equation, and dual boundary Yang-Baxter equation.[15,21,22,30] We set the Hamiltonian H_B^{fin} by

$$H_B^{fin} = \frac{d}{dz}T_B^{fin}(z)|_{z=1} = \sum_{j=1}^{P-1} h_{j,j+1} + \frac{1}{2}\frac{d}{dz}K_1(z)|_{z=1} + \frac{\mathrm{str}_{V_0}(K_0^+(1)h_{0,P})}{\mathrm{str}_{V_0}(K_0^+(1))}, \tag{2.35}$$

where $h_{j,j+1} = P_{j,j+1}\frac{d}{dz}R_{j,j+1}(z)|_{z=1}$.

B. Semi-infinite $U_q(\widehat{sl}(M+1|N+1))$ chain

In this section we introduce the semi-infinite $U_q(\widehat{sl}(M+1|N+1))$ chain with a boundary. We consider the Hamiltonian (2.35) in the semi-infinite limit,

$$H_B^{(i)} = \lim_{P\to\infty} H_B^{fin} = \lim_{P\to\infty}\frac{d}{dz}T_B^{fin}(z)|_{z=1} = \sum_{j=1}^{\infty} h_{j,j+1} + \frac{1}{2}\frac{d}{dz}K_1^{(i)}(z)|_{z=1}, \tag{2.36}$$

which acts formally on the left-infinite tensor product space.

$$\cdots \otimes V \otimes V \otimes V. \tag{2.37}$$

We would like to diagonalize the Hamiltonian $H_B^{(i)}$ in the semi-infinite limit. It is convenient to study the transfer matrix $\widetilde{T}_B^{(i)}(z) = \lim_{P\to\infty} T_B^{fin}(z)$, including the spectral parameter z, instead of the Hamiltonian $H_B^{(i)}$. The transfer matrix $T_B^{fin}(z)$ in Sec. II A is defined by finite product of the R-matrix. Hence it is free from the difficulty of divergence. The semi-infinite transfer matrix $\widetilde{T}_B^{(i)}(z)$ is given by infinite product of the R-matrix. Generally speaking it is not free from the difficulty of divergence. We know two useful concepts to study the infinite product of the R-matrix. One is the corner transfer matrix (CTM) introduced by Baxter.[19] The other is the vertex operator introduced by Baxter[32] and Jimbo, Miwa, and Nakayashiki.[33] It has been checked for some cases that we can avoid the divergence of the CTM by multiplying normalization factor. In Sec. III we will give mathematical formulation of the supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ chain with a boundary, that is free from the difficulty of divergence. The CTM for the supersymmetry $U_q(\widehat{sl}(2|1))$[31] will give a supporting argument for the mathematical formulation in Sec. III. In this section we do not study up the details of the divergence problem of infinite product of the R-matrix. Following the strategy summarized in Refs. 1, 2, and 33, we only study heuristic argument of the vertex operators and the semi-infinite transfer matrix. Here we introduce the vertex operators $\widetilde{\Phi}_j(z)$ and $\widetilde{\Phi}_j^*(z)$ that act on the left-infinite tensor product space: $\cdots \otimes V \otimes V \otimes V$, as the limit of the monodromy matrix.[1,2,33] The matrix elements of the vertex operator $\widetilde{\Phi}_j(z)$ are given as follows:

$$\left(\widetilde{\Phi}_j(z)\right)_{\cdots,k_n,\cdots,k_2,k_1}^{\cdots,k'_n,\cdots,k'_2,k_1} = \lim_{P\to\infty} \sum_{j_1,j_2,\cdots,j_P=1}^{M+N+2} \prod_{s=1}^{P} R(z)_{j_{s-1},k_s}^{j_s,k'_s} (j = j_0). \tag{2.38}$$

The matrix elements of the dual vertex operator $\widetilde{\Phi}_j^*(z)$ are given as follows:

$$\left(\widetilde{\Phi}_j^*(z)\right)_{\cdots,k_n,\cdots,k_2,k_1}^{\cdots,k'_n,\cdots,k'_2,k'_1} = \lim_{P\to\infty} \sum_{j_1,j_2,\cdots,j_P=1}^{M+N+2} \prod_{s=1}^{P} R(z)_{k_s,j_s}^{k'_s,j_{s-1}} (j = j_0). \tag{2.39}$$

We expect that the vertex operator $\widetilde{\Phi}_j(z)$ and the dual vertex operator $\widetilde{\Phi}_j^*(z)$ give rise to well-defined operators. From heuristic argument by using the R-matrix,[1,33] the vertex operators $\widetilde{\Phi}_j(z)$ and $\widetilde{\Phi}_j^*(z)$ are expected to satisfy the following commutation relations:

$$\widetilde{\Phi}_{j_2}(z_2)\widetilde{\Phi}_{j_1}(z_1) = \sum_{k_1,k_2=1}^{M+N+2} R(z_1/z_2)_{j_1,j_2}^{k_1,k_2} \widetilde{\Phi}_{k_1}(z_1)\widetilde{\Phi}_{k_2}(z_2)(-1)^{[v_{j_1}][v_{j_2}]}, \tag{2.40}$$

$$\widetilde{\Phi}_{j_2}^*(z_2)\widetilde{\Phi}_{j_1}^*(z_1) = \sum_{k_1,k_2=1}^{M+N+2} R(z_1/z_2)_{k_1,k_2}^{j_1,j_2} \widetilde{\Phi}_{k_1}^*(z_1)\widetilde{\Phi}_{k_2}^*(z_2)(-1)^{[v_{j_1}][v_{j_2}]}, \tag{2.41}$$

$$\widetilde{\Phi}_{j_2}(z_2)\widetilde{\Phi}_{j_1}^*(z_1) = \sum_{k_1,k_2=1}^{M+N+2} R^{-1,st_1}(z_1/z_2)_{j_1,j_2}^{k_1,k_2} \widetilde{\Phi}_{k_1}^*(z_1)\widetilde{\Phi}_{k_2}(z_2)(-1)^{[v_{k_1}][v_{k_2}]}. \tag{2.42}$$

The transfer matrix $\widetilde{T}_B^{(i)}(z)$ is rewritten by the vertex operators $\widetilde{\Phi}_j(z)$ and $\widetilde{\Phi}_j^*(z)$ as follows.[24,34]

$$\widetilde{T}_B^{(i)}(z) = \sum_{j=1}^{M+N+2} \widetilde{\Phi}_j^*(z^{-1}) K^{(i)}(z)_j^j \widetilde{\Phi}_j(z)(-1)^{[v_j]}. \tag{2.43}$$

From the graded boundary Yang-Baxter relation (2.28) and the commutation relations of the vertex operators, we have the commutativity of the transfer matrix $\widetilde{T}_B^{(i)}(z)$.

$$[\widetilde{T}_B^{(i)}(z_1), \widetilde{T}_B^{(i)}(z_2)] = 0 \text{ for any } z_1, z_2. \tag{2.44}$$

In order to diagonalize the transfer matrix $\widetilde{T}_B^{(i)}(z)$, we follow the strategy that we call the algebraic analysis method.[1]

III. MATHEMATICAL FORMULATION

In this section we give mathematical formulation of the supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ chain with a boundary, that is free from the difficulty of divergence.[1,2,19,24]

A. Quantum supersymmetry $U_q(\widehat{sl}(M+1|N+1))$

In this section we review the definition of the quantum supersymmetry $U_q(\widehat{sl}(M+1|N+1))$.[5,7] The Cartan matrix of the affine superalgebra $\widehat{sl}(M+1|N+1)$ is given by

$$(A_{i,j})_{0\le i,j\le M+N+1} = \begin{pmatrix} 0 & -1 & 0 & \cdots & & & & \cdots & 0 & 1 \\ -1 & 2 & -1 & \cdots & & & & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & & & & & & \cdots \\ & \cdots & \cdots & \cdots & -1 & & & & & \\ & & & -1 & 2 & -1 & & & & \\ & & & & -1 & 0 & 1 & & & \\ & & & & & 1 & -2 & 1 & & \\ & & & & & & 1 & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & & & & & \cdots & -2 & 1 \\ 1 & 0 & \cdots & & & & & \cdots & 1 & -2 \end{pmatrix}. \tag{3.1}$$

Here the diagonal part is $(A_{i,i})_{0\le i\le M+N+1} = (0, \overbrace{2,\cdots,2}^{M}, 0, \overbrace{-2,\cdots,-2}^{N})$. Let us introduce orthonormal basis $\{\epsilon_j | j = 1, 2, \ldots, M+N+2\}$ with the bilinear form $(\epsilon_i|\epsilon_j) = \nu_i\delta_{i,j}$, where the signature $\nu_i = \pm$ is given in (2.2). Define $\bar{\epsilon}_i = \epsilon_i - \nu_i \frac{1}{M-N}\sum_{j=1}^{M+N+2}\epsilon_j$. The classical simple roots $\bar{\alpha}_i$ ($i = 1, 2, \ldots, M+N+1$) and the classical fundamental weights $\bar{\Lambda}_i$ ($i = 1, 2, \ldots, M+N+1$) are defined by

$$\bar{\alpha}_i = \nu_i\epsilon_i - \nu_{i+1}\epsilon_{i+1},\ \bar{\Lambda}_i = \sum_{j=1}^{i}\bar{\epsilon}_j (i = 1, 2, \cdots, M+N+1). \tag{3.2}$$

Introduce the affine weight Λ_0 and the null root δ having $(\Lambda_0|\epsilon_i) = (\delta|\epsilon_i) = 0$ for $i = 1, 2, \ldots, M+N+2$ and $(\Lambda_0|\lambda_0) = (\delta|\delta) = 0$, $(\Lambda_0|\delta) = 1$. The affine roots α_i ($i = 0, 1, 2, \ldots, M+N+1$) and the affine fundamental weights Λ_i ($i = 0, 1, 2, \ldots, M+N+1$) are given by

$$\alpha_0 = \delta - \sum_{j=1}^{M+N+1}\alpha_j,\ \alpha_i = \bar{\alpha}_i (i = 1, 2, \cdots, M+N+1), \tag{3.3}$$

$$\Lambda_0 = \Lambda_0,\ \Lambda_i = \Lambda_0 + \bar{\Lambda}_i (i = 1, 2, \cdots, M+N+1). \tag{3.4}$$

Definition 3.1: The quantum supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ ($M, N = 0, 1, 2, \ldots,$ and $M \neq N$) is a q-analogue of the universal enveloping algebra $\widehat{sl}(M+1|N+1)$ generated by the Chevalley generators $\{e_i, f_i, h_i | i = 0, 1, 2, \ldots, M+N+1\}$. The $\mathbf{Z}_2$-grading of the generators are $[e_0] = [f_0] = [e_{M+1}] = [f_{M+1}] = 1$ and zero otherwise.

The Cartan-Kac relations: *For $i, j = 0, 1, \ldots, M+N+1$, the generators subject to the following relations:*

$$[h_i, h_j] = 0,\ [h_i, e_j] = A_{i,j}e_j,\ [h_i, f_j] = -A_{i,j}f_j,\ [e_i, f_j] = \delta_{i,j}\frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}. \tag{3.5}$$

For $i, j = 0, 1, \ldots, M+N+1$ such that $|A_{i,j}| = 0$, the generators subject to the following relations:

$$[e_i, e_j] = 0,\ [f_i, f_j] = 0. \tag{3.6}$$

The Serre relations : *For $i, j = 0, 1, \ldots, M + N + 1$ such that $|A_{i,j}| = 1$ and $i \neq 0, M + 1$, the generators subject to the following relations:*

$$[e_i, [e_i, e_j]_{q^{-1}}]_q = 0, [f_i, [f_i, f_j]_{q^{-1}}]_q = 0. \tag{3.7}$$

For $M + N \geq 2$, the Serre relations of the fourth degree hold,

$$\begin{aligned}&[[[e_i, e_j]_q, e_k]_{q^{-1}}, e_j] = 0, [[[f_i, f_j]_q, f_k]_{q^{-1}}, f_j] = 0,\\ &(i, j, k) = (M + N - 1, 0, 1), (M - 1, M, M + 1).\end{aligned} \tag{3.8}$$

For $(M, N) = (1, 0)$ the extra Serre relations of the fifth degree hold,

$$[e_0, [e_2, [e_0, [e_2, e_1]_q]]]_{q^{-1}} = [e_2, [e_0, [e_2, [e_0, e_1]_q]]]_{q^{-1}}, \tag{3.9}$$

$$[f_0, [f_2, [f_0, [f_2, f_1]_q]]]_{q^{-1}} = [f_2, [f_0, [f_2, [f_0, f_1]_q]]]_{q^{-1}}. \tag{3.10}$$

For $(M, N) = (0, 1)$ the extra Serre relations of the fifth degree hold,

$$[e_0, [e_1, [e_0, [e_1, e_2]_q]]]_{q^{-1}} = [e_1, [e_0, [e_1, [e_0, e_2]_q]]]_{q^{-1}}, \tag{3.11}$$

$$[f_0, [f_1, [f_0, [f_1, f_2]_q]]]_{q^{-1}} = [f_1, [f_0, [f_1, [f_0, f_2]_q]]]_{q^{-1}}. \tag{3.12}$$

Here and throughout this paper, we use the notations,

$$[X, Y]_\xi = XY - (-1)^{[X][Y]}\xi YX. \tag{3.13}$$

We write $[X, Y]_1$ as $[X, Y]$ for simplicity. The quantum supersymmetry $U_q(\widehat{sl}(M + 1|N + 1))$ has the $\mathbf{Z}_2$-graded Hopf-algebra structure. We take the following coproduct:

$$\Delta(e_i) = e_i \otimes 1 + q^{h_i} \otimes e_i, \Delta(f_i) = f_i \otimes q^{-h_i} + 1 \otimes f_i, \Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \tag{3.14}$$

and the antipode

$$S(e_i) = -q^{-h_i}e_i, S(f_i) = -f_i q^{h_i}, S(h_i) = -h_i. \tag{3.15}$$

The coproduct Δ satisfies an algebra automorphism $\Delta(XY) = \Delta(X)\ \Delta\ (Y)$ and the antipode S satisfies a $\mathbf{Z}_2$-graded algebra anti-automorphism $S(XY) = (-1)^{[X][Y]}S(Y)S(X)$. The multiplication rule for the tensor product is $\mathbf{Z}_2$-graded and is defined for homogeneous elements $X, Y, X', Y' \in U_q(\widehat{sl}(N|1))$ and $v \in V$, $w \in W$ by $X \otimes Y \cdot X' \otimes Y' = (-1)^{[Y][X']}XX' \otimes YY'$ and $X \otimes Y \cdot v \otimes w = (-1)^{[Y][v]}Xv \otimes Yw$, which extends to inhomogeneous elements through linearity.

B. Mathematical formulation

In this section we give mathematical formulation of our problem. We introduce the evaluation representation V_z of the $(M + N + 2)$ dimensional basic representation $V = \oplus_{j=1}^{M+N+2}\mathbf{C}v_j$. Let $E_{i,j}$ be the $(M + N + 2) \times (M + N + 2)$ matrix whose (i, j)-elements is unity and zero elsewhere: $E_{i,j}v_k = \delta_{j,k}v_i$. For $i = 1, 2, \ldots, M + N + 2$, we set the evaluation representation V_z with the vectors $\{v_i \otimes z^n | i = 1, 2, \cdots, M + N + 2; n \in \mathbf{Z}\}$.

$$\begin{aligned}&e_i = E_{i,i+1}, f_i = \nu_i E_{i+1,i}, h_i = \nu_i E_{i,i} - \nu_{i+1}E_{i+1,i+1},\\ &e_0 = -zE_{M+N+2,1}, f_0 = z^{-1}E_{1,M+N+2}, h_0 = -E_{1,1} - E_{M+N+2,M+N+2}.\end{aligned} \tag{3.16}$$

Let V_z^* the dual space of V_z with vectors $\{v_i^* \otimes z^n | i = 1, 2, \cdots, M + N + 2; n \in \mathbf{Z}\}$ such that $(v_i^* \otimes z^m | v_j \otimes z^n) = \delta_{i,j}\delta_{m+n,0}$. The $U_q(\widehat{sl}(M + 1|N + 1))$-module structure is given by $(xv|w) = (v|(-1)^{|x||v|}S(x)w)$ for $v \in V_z^*$, $w \in V_z$ and we call the module as V_z^{*S}. For $i = 1, 2, \ldots, M + N + 2$, we have the explicit action on V_z^{*S} as follows:

$$\begin{aligned}&e_i = -\nu_i\nu_{i+1}q^{-\nu_i}E_{i+1,i}, \quad f_i = -\nu_i q^{\nu_i}E_{i,i+1}, \quad h_i = -\nu_i E_{i,i} + \nu_{i+1}E_{i+1,i+1},\\ &e_0 = qzE_{1,M+N+2}, \quad f_0 = q^{-1}z^{-1}E_{M+N+2,1}, \quad h_0 = E_{1,1} + E_{M+N+2,M+N+2}.\end{aligned} \tag{3.17}$$

Let $L(\lambda)$ the irreducible highest weight $U_q(\widehat{sl}(M+1|N+1))$-module with the level-1 highest weight λ.

Definition 3.2: We define the type-I vertex operators $\Phi(z)$ and $\Phi^*(z)$ as the intertwiners of $U_q(\widehat{sl}(M+1|N+1))$-module if they exist,

$$\Phi(z) : L(\lambda) \to L(\mu) \otimes V_z,\ \Phi^*(z) : L(\mu) \to L(\lambda) \otimes V_z^{*S}, \tag{3.18}$$

$$\Phi(z) \cdot x = \Delta(x) \cdot \Phi(z),\ \Phi^*(z) \cdot x = \Delta(x) \cdot \Phi^*(z), \tag{3.19}$$

for $x \in U_q(\widehat{sl}(M+1|N+1))$. The operators $\Phi(z)$ and $\Phi^*(z)$ depend on the weights λ and μ.

We expand the vertex operators $\Phi(z) = \sum_{j=1}^{M+N+2} \Phi_j(z) \otimes v_j$ and $\Phi^*(z) = \sum_{j=1}^{M+N+2} \Phi_j^*(z) \otimes v_j^*$. The vertex operators $\Phi_j(z)$ and $\Phi_j^*(z)$ are expected to satisfy the following commutation relations. See conjectures for the special case $U_q(\widehat{sl}(N|1))$ in Ref. 18:

$$\Phi_{j_2}(z_2)\Phi_{j_1}(z_1) = \sum_{k_1,k_2=1}^{M+N+2} R(z_1/z_2)_{j_1,j_2}^{k_1,k_2} \Phi_{k_1}(z_1)\Phi_{k_2}(z_2)(-1)^{[v_{j_1}][v_{j_2}]}, \tag{3.20}$$

$$\Phi_{j_2}^*(z_2)\Phi_{j_1}^*(z_1) = \sum_{k_1,k_2=1}^{M+N+2} R(z_1/z_2)_{k_1,k_2}^{j_1,j_2} \Phi_{k_1}^*(z_1)\Phi_{k_2}^*(z_2)(-1)^{[v_{j_1}][v_{j_2}]}, \tag{3.21}$$

$$\Phi_{j_2}(z_2)\Phi_{j_1}^*(z_1) = \sum_{k_1,k_2=1}^{M+N+2} R^{-1,st_1}(z_1/z_2)_{j_1,j_2}^{k_1,k_2} \Phi_{k_1}^*(z_1)\Phi_{k_2}(z_2)(-1)^{[v_{k_1}][v_{k_2}]}. \tag{3.22}$$

The vertex operators satisfy the inversion relations

$$\Phi_i(z)\Phi_j^*(z) = g^{-1}(-1)^{[v_i]}\delta_{i,j},\ \sum_{k=1}^{M+N+2} (-1)^{[v_k]}\Phi_k^*(z)\Phi_k(z) = g^{-1}. \tag{3.23}$$

Here we have used

$$g = e^{\frac{\pi\sqrt{-1}M}{2(M-N)}} \exp\left(-\sum_{m=1}^{\infty} \frac{[(M-N-1)m]_q}{m[(M-N)m]_q} q^m\right). \tag{3.24}$$

Definition 3.3: We set the transfer matrix $T_B^{(i)}(z)$ by

$$T_B^{(i)}(z) = g \sum_{j=1}^{M+N+2} \Phi_j^*(z^{-1}) K^{(i)}(z)_j^j \Phi_j(z)(-1)^{[v_j]}. \tag{3.25}$$

From the graded boundary Yang-Baxter relation (2.28) and the commutation relations of the vertex operators, we have the commutativity of the transfer matrix $T_B^{(i)}(z)$.

Proposition 3.4: The transfer matrix $T_B^{(i)}(z)$ forms a commutative family,

$$[T_B^{(i)}(z_1), T_B^{(i)}(z_2)] = 0 \text{ for any } z_1, z_2. \tag{3.26}$$

In what follows we focus our attention on the vertex operators acting on the irreducible highest weight module $L(\lambda)$ with the level 1 dominant integrable highest weight λ. Following the strategy proposed in Refs. 1 and 24 and the CTM argument for $U_q(\widehat{sl}(2|1))$, we consider our problem upon the following identification:

$$T_B^{(i)}(z) = \widetilde{T}_B^{(i)}(z),\ \Phi_j(z) = \widetilde{\Phi}_j(z),\ \Phi_j^*(z) = \widetilde{\Phi}_j^*(z). \tag{3.27}$$

The point of using the vertex operators $\Phi_j(z)$, $\Phi_j^*(z)$ is that they are well-defined objects, free from the difficulty of divergence. Let us set $L^*(\lambda)$ the restricted dual module of the irreducible highest weight module $L(\lambda)$ with level-1 dominant integrable highest weight λ.

Proposition 3.5: The vector ${}_{(i)}\langle B| \in L^*(\lambda)(i = 1, 2, 3)$ *becomes the eigenvector of the transfer matrix* $T_B^{(i)}(z)$ *with eigenvalue 1:*

$$ {}_{(i)}\langle B|T_B^{(i)}(z) = {}_{(i)}\langle B|, \tag{3.28}$$

if and only if the following condition holds,

$$ {}_{(i)}\langle B|\Phi_j^*(z^{-1})K^{(i)}(z)_j^j = {}_{(i)}\langle B|\Phi_j^*(z)(j = 1, 2, \cdots, M + N + 2). \tag{3.29}$$

We call this vector ${}_{(i)}\langle B| \in L^*(\lambda)$ *the boundary state.*

Proof. Multiplying the vertex operator $\Phi_j^*(z)$ from the right of (3.28) and using the inversion relation (3.23), we have (3.29). □

In order to construct the boundary state ${}_{(i)}\langle B|$, it is convenient to introduce the bosonizations of the vertex operators $\Phi_j^*(z)$.

IV. VERTEX OPERATOR

In this section we review the bosonization of the vertex operators. We give the integral representation of the vertex operators, which are convenient for the construction of the boundary state ${}_{(i)}\langle B|$.

A. Drinfeld realization

In order to give the bosonizations, it is convenient to introduce the Drinfeld realization of the quantum supersymmetry $U_q(\widehat{sl}(M + 1|N + 1))$.[6,7]

Definition 4.1 (Ref. 7): The generators of the quantum supersymmetry $U_q(\widehat{sl}(M + 1|N + 1))$, *which we call the Drinfeld generators, are given by*

$$ X_{i,m}^{\pm}, h_{i,n}, h_i, c, (i = 1, 2, \cdots, M + N + 1, m \in \mathbf{Z}, n \in \mathbf{Z}_{\neq 0}). \tag{4.1}$$

The $\mathbf{Z}_2$*-grading of the Drinfeld generators are* $|X_m^{\pm,M+1}| = 1$ for $m \in \mathbf{Z}$ *and zero otherwise. For* $i, j = 1, 2, \ldots, M + N + 1$, *the Drinfeld generators are subject to the following relations:*

$$ c : \text{central}, [h_i, h_{j,m}] = 0, \tag{4.2}$$

$$ [h_{i,m}, h_{j,n}] = \frac{[A_{i,j}m]_q[cm]_q}{m}\delta_{m+n,0}, \tag{4.3}$$

$$ [h_i, X_j^{\pm}(z)] = \pm A_{i,j}X_j^{\pm}(z), \tag{4.4}$$

$$ [h_{i,m}, X_j^{+}(z)] = \frac{[A_{i,j}m]_q}{m}q^{-c|m|/2}z^m X_j^{+}(z), \tag{4.5}$$

$$ [h_{i,m}, X_j^{-}(z)] = -\frac{[A_{i,j}m]_q}{m}q^{c|m|/2}z^m X_j^{-}(z), \tag{4.6}$$

$$ (z_1 - q^{\pm A_{i,j}}z_2)X_i^{\pm}(z_1)X_j^{\pm}(z_2) = (q^{\pm A_{j,i}}z_1 - z_2)X_j^{\pm}(z_2)X_i^{\pm}(z_1), \text{ for } |A_{i,j}| \neq 0, \tag{4.7}$$

$$[X_i^{\pm}(z_1), X_j^{\pm}(z_2)] = 0, \text{ for } |A_{i,j}| = 0, \tag{4.8}$$

$$[X_i^{+}(z_1), X_j^{-}(z_2)] = \frac{\delta_{i,j}}{(q-q^{-1})z_1 z_2}\left(\delta(q^{-c}z_1/z_2)\psi_i^{+}(q^{\frac{c}{2}}z_2) - \delta(q^{c}z_1/z_2)\psi_i^{-}(q^{-\frac{c}{2}}z_2)\right), \tag{4.9}$$

$$\begin{aligned}&\Big(X_i^{\pm}(z_1)X_i^{\pm}(z_2)X_j^{\pm}(z) - (q+q^{-1})X_i^{\pm}(z_1)X_j^{\pm}(z)X_i^{\pm}(z_2) + X_j^{\pm}(z)X_i^{\pm}(z_1)X_i^{\pm}(z_2)\Big)\\ &+(z_1 \leftrightarrow z_2) = 0, \text{ for } |A_{i,j}| = 1, i \neq M+1,\end{aligned} \tag{4.10}$$

$$\begin{aligned}&\big(X_{M+1}^{\pm}(z_1)X_{M+2}^{\pm}(w_1)X_{M+1}^{\pm}(z_2)X_M^{\pm}(w_2) - q^{-1}X_{M+1}^{\pm}(z_1)X_{M+2}^{\pm}(w_1)X_M^{\pm}(w_2)X_{M+1}^{\pm}(z_2)\\ &-qX_{M+1}^{\pm}(z_1)X_{M+1}^{\pm}(z_2)X_M^{\pm}(w_2)X_{M+2}^{\pm}(w_1) + X_{M+1}^{\pm}(z_1)X_M^{\pm}(w_2)X_{M+1}^{\pm}(z_2)X_{M+2}^{\pm}(w_1)\\ &+X_{M+2}^{\pm}(w_1)X_{M+1}^{\pm}(z_2)X_M^{\pm}(w_2)X_{M+1}^{\pm}(z_1) - q^{-1}X_{M+2}^{\pm}(w_1)X_M^{\pm}(w_2)X_{M+1}^{\pm}(z_2)X_{M+1}^{\pm}(z_1)\\ &-qX_{M+1}^{\pm}(z_2)X_M^{\pm}(w_2)X_{M+2}^{\pm}(w_1)X_{M+1}^{\pm}(z_1) + X_M^{\pm}(w_2)X_{M+1}^{\pm}(z_2)X_{M+2}^{\pm}(w_1)X_{M+1}^{\pm}(z_1)\big)\\ &+(z_1 \leftrightarrow z_2) = 0,\end{aligned} \tag{4.11}$$

where we have used $\delta(z) = \sum_{m\in\mathbf{Z}} z^m$. *Here we have set the generating functions*

$$X_j^{\pm}(z) = \sum_{m\in\mathbf{Z}} X_{j,m}^{\pm} z^{-m-1}, \tag{4.12}$$

$$\psi_i^{+}(z) = q^{h_i}\exp\left((q-q^{-1})\sum_{m=1}^{\infty} h_{i,m}z^{-m}\right), \tag{4.13}$$

$$\psi_i^{-}(z) = q^{-h_i}\exp\left(-(q-q^{-1})\sum_{m=1}^{\infty} h_{i,-m}z^{m}\right). \tag{4.14}$$

The relation between the Chevalley generators and the Drinfeld generators are obtained as follows:

$$e_i = X_{i,0}^{+},\ f_i = X_{i,0}^{-}(i = 1, 2, \cdots, M+N+1), \tag{4.15}$$

$$h_0 = c - h_1 - h_2 - \cdots - h_{M+N+1}, \tag{4.16}$$

$$\begin{aligned}e_0 = (-1)^{N+1}&[X_{M+N+1,0}^{-}\cdots, [X_{M+2,0}^{-}, [X_{M+1,0}^{-}\cdots, [X_{2,0}^{-}, X_{1,0}^{-}]_{q^{-1}}\cdots]_{q^{-1}}]_q\cdots]_q\\ &\times q^{-h_1-h_2-\cdots-h_{M+N+1}},\end{aligned} \tag{4.17}$$

$$\begin{aligned}f_0 = q^{h_1+h_2+\cdots+h_{M+N+1}}&\\ \times [\cdots[[\cdots[X_{1,-1}^{+}, X_{2,0}^{+}]_q, \cdots, X_{M+1,0}^{+}]_q, X_{M+2,0}^{+}]_{q^{-1}}, \cdots X_{M+N+1,0}^{+}]_{q^{-1}}.&\end{aligned} \tag{4.18}$$

B. Bosonization

In this section we review the bosonizations of the Drinfeld realizations of the quantum supersymmetry $U_q(\widehat{sl}(M+1|N+1))$.[16] In what follows we assume the level $c = 1$, where we have the simplest realization. Let us introduce the bosons

$$a_n^k, b_n^l, c_n^l, Q_{a^k}, Q_{b^l}, Q_{c^l}, (n \in \mathbf{Z}, k = 1, 2, \cdots, M+1, l = 1, 2, \cdots, N+1), \tag{4.19}$$

satisfying the following commutation relations.

$$[a_m^i, a_n^j] = \delta_{i,j}\delta_{m+n,0}\frac{[m]_q^2}{m}, [a_0^i, Q_{a^j}] = \delta_{i,j}, [a_0^i, a_0^j] = 0(1 \le i, j \le M+1), \tag{4.20}$$

$$[b_m^i, b_n^j] = -\delta_{i,j}\delta_{m+n,0}\frac{[m]_q^2}{m}, [b_0^i, Q_{b^j}] = -\delta_{i,j}, [b_0^i, b_0^j] = 0(1 \le i, j \le N+1), \tag{4.21}$$

$$[c_m^i, c_n^j] = \delta_{i,j}\delta_{m+n,0}\frac{[m]_q^2}{m}, [c_0^i, Q_{c^j}] = \delta_{i,j}, [c_0^i, c_0^j] = 0(1 \le i, j \le N+1). \tag{4.22}$$

Other commutation relations vanish. We set $h_i = h_{i,0}$. For $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, N$, we set

$$Q_{h_i} = Q_{a^i} - Q_{a^{i+1}},\ Q_{h_{M+1}} = Q_{a^{M+1}} + Q_{b^1},\ Q_{h_{M+1+j}} = -Q_{b^j} + Q_{b^{j+1}}. \tag{4.23}$$

It is convenient to introduce the generating function $h^i(z;\beta)$ by

$$h^i(z;\beta) = -\sum_{n\neq 0}\frac{h_{i,n}}{[n]_q}q^{-\beta|n|}z^{-n} + Q_{h_i} + h_{i,0}\log z(\beta \in \mathbf{R}). \tag{4.24}$$

We introduce the q-difference operator defined by

$$\partial_z f(z) = \frac{f(qz) - f(q^{-1}z)}{(q-q^{-1})z}. \tag{4.25}$$

In what follows we use the standard normal ordering : :. We set

$$: a_m^i a_n^i := \begin{cases} a_m^i a_n^i & (m \le 0) \\ a_n^i a_m^i & (m > 0), \end{cases} \quad : a_0^i Q_{a^i} :=: Q_{a^i} a_0^i := Q_{a^i} a_0^i, \tag{4.26}$$

$$: b_m^i b_n^i := \begin{cases} b_m^i b_n^i & (m \le 0) \\ b_n^i b_m^i & (m > 0), \end{cases} \quad : b_0^i Q_{b^i} :=: Q_{b^i} b_0^i := Q_{b^i} b_0^i, \tag{4.27}$$

$$: c_m^i c_n^i := \begin{cases} c_m^i c_n^i & (m \le 0) \\ c_n^i c_m^i & (m > 0), \end{cases} \quad : c_0^i Q_{c^i} :=: Q_{c^i} c_0^i := Q_{c^i} c_0^i. \tag{4.28}$$

Theorem 4.2 (Ref. 16): *The Drinfeld generators for the level $c = 1$ are realized as follows:*

$$c = 1, \tag{4.29}$$

$$h_{i,m} = a_m^i q^{-|m|/2} - a_m^{i+1} q^{|m|/2}, \tag{4.30}$$

$$h_{M+1,m} = a_m^{M+1} q^{-|m|/2} + b_m^1 q^{-|m|/2}, \tag{4.31}$$

$$h_{M+1+j,m} = -b_m^j q^{|m|/2} + b_m^{j+1} q^{-|m|/2}, \tag{4.32}$$

$$X_i^+(z) = : e^{h^i(z;1/2)} : e^{\pi\sqrt{-1}a_0^i}, \tag{4.33}$$

$$X_{M+1}^+(z) = : e^{h^{M+1}(z;1/2)} e^{c^1(z;0)} : \prod_{i=1}^{M} e^{-\pi\sqrt{-1}a_0^i}, \tag{4.34}$$

$$X_{M+1+j}^+(z) = : e^{h^{M+1+j}(z;1/2)}[\partial_z e^{-c^j(z;0)}]e^{c^{j+1}(z;0)} :, \tag{4.35}$$

$$X_i^-(z) = - : e^{-h^i(z;-1/2)} : e^{-\pi\sqrt{-1}a_0^i}, \tag{4.36}$$

$$X_{M+1}^-(z) = : e^{-h^{M+1}(z;-1/2)}[\partial_z e^{-c^1(z;0)}] : \prod_{i=1}^{M} e^{\pi\sqrt{-1}a_0^i}, \tag{4.37}$$

$$X_{M+1+j}^-(z) = - : e^{-h^{M+1+j}(z;-1/2)} e^{c^j(z;0)}[\partial_z e^{-c^{j+1}(z;0)}] :, \tag{4.38}$$

for $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, N$. Here we read $h_i = h_{i,0}$.

C. Highest weight module

In this section we study the irreducible highest weight module $L(\lambda)$ with the level 1 highest weight λ. We introduce the vacuum vector $|0\rangle$ by

$$a_n^i|0\rangle = b_n^j|0\rangle = c_n^j|0\rangle = 0(n \geq 0, i = 1, 2, \cdots, M+1, \ j = 1, 2, \cdots, N+1). \tag{4.39}$$

For $\lambda_a^i, \lambda_b^j, \lambda_c^j \in \mathbf{C}(i = 1, \ldots, M+1, j = 1, \ldots, N+1)$, we set the vector

$$|\lambda_a^1, \cdots, \lambda_a^{M+1}, \lambda_b^1, \cdots, \lambda_b^{N+1}, \lambda_c^1, \cdots, \lambda_c^{N+1}\rangle = e^{\sum_{i=1}^{M+1} \lambda_a^i Q_{a^i} + \sum_{j=1}^{N+1} \lambda_b^j Q_{b^j} + \sum_{j=1}^{N+1} \lambda_c^j Q_{c^j}}|0\rangle. \tag{4.40}$$

The Fock module $\mathcal{F}_{\lambda_a^1,\cdots,\lambda_a^{M+1},\lambda_b^1,\cdots,\lambda_b^{N+1},\lambda_c^1,\cdots,\lambda_c^{N+1}}$ is generated by acting creation operators $a_{-n}^i, b_{-n}^j, c_{-n}^j$ $(n > 0)$ over the vector $|\lambda_a^1, \cdots, \lambda_a^{M+1}, \lambda_b^1, \cdots, \lambda_b^{N+1}, \lambda_c^1, \cdots, \lambda_c^{N+1}\rangle$. We would like to obtain the highest weight vectors v_λ for the level 1 highest weight $\lambda = \sum_{j=0}^{M+N+1} \lambda_j \Lambda_j$, where Λ_j are the affine fundamental weights and λ_j satisfy $\sum_{j=0}^{M+N+1} \lambda_j = 1$. When the vector $|\lambda_a^1, \cdots, \lambda_a^{M+1}, \lambda_b^1, \cdots, \lambda_b^{N+1}, \lambda_c^1, \cdots, \lambda_c^{N+1}\rangle$ satisfies the following conditions:

$$h_i|\lambda_a^1, \cdots, \lambda_a^{M+1}, \lambda_b^1, \cdots, \lambda_b^{N+1}, \lambda_c^1, \cdots, \lambda_c^{N+1}\rangle = \lambda_i|\lambda_a^1, \cdots, \lambda_a^{M+1}, \lambda_b^1, \cdots, \lambda_b^{N+1}, \lambda_c^1, \cdots, \lambda_c^{N+1}\rangle,$$
$$e_i|\lambda_a^1, \cdots, \lambda_a^{M+1}, \lambda_b^1, \cdots, \lambda_b^{N+1}, \lambda_c^1, \cdots, \lambda_c^{N+1}\rangle = 0(i = 1, 2, \cdots, M+N+1),$$

we identify the highest weight vector $v_\lambda = |\lambda_a^1, \cdots, \lambda_a^{M+1}, \lambda_b^1, \cdots, \lambda_b^{N+1}, \lambda_c^1, \cdots, \lambda_c^{N+1}\rangle$. Solving these equations, we have two classes of solutions.

(1) $v_{\Lambda_i}(i = 1, 2, \cdots, M+N+1)$.

For $i = 1, 2, \ldots, M+1$, we identify

$$v_{\Lambda_i} = |\overbrace{\beta+1, \cdots, \beta+1}^{i}, \overbrace{\beta, \cdots, \beta}^{M+N+2-i}, \overbrace{0, \cdots, 0}^{N+1}\rangle(\beta \in \mathbf{C}). \tag{4.41}$$

For $j = 1, \ldots, N+1$, we identify

$$v_{\Lambda_{M+1+j}} = |\overbrace{\beta+1, \cdots, \beta+1}^{M+1+j}, \overbrace{\beta, \cdots, \beta}^{N+1-j}, \overbrace{0, \cdots, 0}^{j}, \overbrace{-1, \cdots, -1}^{N+1-j}\rangle(\beta \in \mathbf{C}). \tag{4.42}$$

(2) $v_{(1-\alpha)\Lambda_0+\alpha\Lambda_{M+1}}$ for $\alpha \in \mathbf{C}$. We identify

$$v_{(1-\alpha)\Lambda_0+\alpha\Lambda_{M+1}} = |\overbrace{\beta, \cdots, \beta}^{M+1}, \overbrace{\beta-\alpha, \cdots, \beta-\alpha}^{N+1}, \overbrace{-\alpha, \cdots, -\alpha}^{N+1}\rangle(\beta \in \mathbf{C}). \tag{4.43}$$

For $i = 1, 2, \ldots, M+1, j = 1, 2, \ldots, N+1$ and $\alpha, \beta \in \mathbf{C}$, we set the spaces.

$$\mathcal{F}_{(\Lambda_i,\beta)} = \bigoplus_{i_1,\cdots,i_{M+N+1}\in\mathbf{Z}} \mathcal{F}_{(\overbrace{\beta+1, \cdots, \beta+1}^{i}, \overbrace{\beta, \cdots, \beta}^{M+N+2-i}, \overbrace{0, \cdots, 0}^{N+1})\circ(i_1,i_2,\cdots,i_{M+N+1})}, \tag{4.44}$$

$$\mathcal{F}_{(\Lambda_{M+1+j},\beta)} = \bigoplus_{i_1,\cdots,i_{M+N+1}\in\mathbf{Z}} \mathcal{F}_{(\overbrace{\beta+1, \cdots, \beta+1}^{M+1+j}, \overbrace{\beta, \cdots, \beta}^{N+1-j}, \overbrace{0, \cdots, 0}^{j}, \overbrace{-1, \cdots, -1}^{N+1-j})\circ(i_1,i_2,\cdots,i_{M+N+1})} \tag{4.45}$$

$$\mathcal{F}_{((1-\alpha)\Lambda_0+\alpha\Lambda_{M+1},\beta)} = \bigoplus_{i_1,\cdots,i_{M+N+1}\in\mathbf{Z}} \mathcal{F}_{(\overbrace{\beta, \cdots, \beta}^{M+1}, \overbrace{\beta-\alpha, \cdots, \beta-\alpha}^{N+1}, \overbrace{-\alpha, \cdots, -\alpha}^{N+1})\circ(i_1,i_2,\cdots,i_{M+N+1})} \tag{4.46}$$

Here we have used the following abbreviation:

$$\begin{aligned}
&(\lambda_a^1, \cdots, \lambda_a^{M+1}, \lambda_b^1, \cdots, \lambda_b^{N+1}, \lambda_c^1, \cdots, \lambda_c^{N+1}) \circ (i_1, i_2, \cdots, i_{M+N+1}) \\
&= (\lambda_a^1, \cdots, \lambda_a^{M+1}, \lambda_b^1, \cdots, \lambda_b^{N+1}, \lambda_c^1, \cdots, \lambda_c^{N+1}) \\
&+ (i_1, i_2 - i_1, \cdots, i_{M+1} - i_M, i_{M+1} - i_{M+2}, \cdots, i_{M+N} - i_{M+N+1}, i_{M+N+1} \\
&\quad , i_{M+1} - i_{M+2}, \cdots, i_{M+N} - i_{M+N+1}, i_{M+N+1}).
\end{aligned} \tag{4.47}$$

The spaces $\mathcal{F}_{(\Lambda_i,\beta)}$ and $\mathcal{F}_{(1-\alpha)\Lambda_0+\alpha\Lambda_{M+1},\beta)}$ are $U_q(\widehat{sl}(M+1|N+1))$-module and $v_{\Lambda_i} \in \mathcal{F}_{(\Lambda_i,\beta)}$ and $v_{(1-\alpha)\Lambda_0+\alpha\Lambda_{M+1}} \in \mathcal{F}_{(1-\alpha)\Lambda_0+\alpha\Lambda_{M+1},\beta}$. However, these modules are not irreducible in general. In order

to obtain irreducible module, we introduce ξ-η system. We introduce the operators ξ^j_m and η^j_m ($j = 1, 2, \ldots, N + 1; m \in \mathbf{Z}$) by

$$\xi^j(z) = \sum_{m\in\mathbf{Z}} \xi^j_m z^{-m} =: e^{-c^j(z)} :, \eta^j(z) = \sum_{m\in\mathbf{Z}} \eta^j_m z^{-m-1} =: e^{c^j(z)} : . \tag{4.48}$$

The Fourier components $\xi^j_m = \oint \frac{dz}{2\pi\sqrt{-1}} z^{m-1}\xi^j(z)$ and $\eta^j_m = \oint \frac{dz}{2\pi\sqrt{-1}} z^m \eta^j(z)$ are well-defined on the spaces $\mathcal{F}_{(\Lambda_i,\beta)}$, $\mathcal{F}_{(\alpha,\beta)}$ for $\alpha \in \mathbf{Z}$. They satisfy the anti-commutation relations.

$$\{\xi^j_m, \eta^j_n\} = \delta_{m+n,0}, \{\xi^j_m, \xi^j_n\} = \{\eta^j_m, \eta^j_n\} = 0(j = 1, 2, \cdots, N + 1). \tag{4.49}$$

Here we have used $\{a, b\} = ab + ba$. They commute with each other.

$$[\xi^j_m, \eta^{j'}_n] = [\xi^j_m, \xi^{j'}_n] = [\eta^j_m, \eta^{j'}_n] = 0(1 \le j \ne j' \le N + 1). \tag{4.50}$$

We focus our attention on the operators η^j_0, ξ^j_0 satisfying $(\eta^j_0)^2 = 0$, $(\xi^j_0)^2 = 0$. They satisfy

$$\mathrm{Im}(\eta^j_0) = \mathrm{Ker}(\eta^j_0), \mathrm{Im}(\xi^j_0) = \mathrm{Ker}(\xi^j_0). \tag{4.51}$$

The products $\eta^j_0\xi^j_0$ and $\xi^j_0\eta^j_0$ are orthogonal projection operators, which satisfy

$$\eta^j_0\xi^j_0 + \xi^j_0\eta^j_0 = 1. \tag{4.52}$$

Hence we have a direct sum decomposition for $i = 1, 2, \ldots, M + N + 1, j = 1, 2, \ldots, N + 1$.

$$\mathcal{F}_{(\Lambda_i,\beta)} = \eta^j_0\xi^j_0\mathcal{F}_{(\Lambda_i,\beta)} \oplus \xi^j_0\eta^j_0\mathcal{F}_{(\Lambda_i,\beta)}, \mathcal{F}_{(\alpha,\beta)} = \eta^j_0\xi^j_0\mathcal{F}_{(\alpha,\beta)} \oplus \xi^j_0\eta^j_0\mathcal{F}_{(\alpha,\beta)}, \tag{4.53}$$

and

$$\mathrm{Ker}(\eta^j_0) = \eta^j_0\xi^j_0\mathcal{F}_{(\Lambda_i,\beta)} \text{ or } \eta^j_0\xi^j_0\mathcal{F}_{(\alpha,\beta)}, \mathrm{Coker}(\eta^j_0) = \xi^j_0\eta^j_0\mathcal{F}_{(\Lambda_i,\beta)} \text{ or } \xi^j_0\eta^j_0\mathcal{F}_{(\alpha,\beta)}. \tag{4.54}$$

We set the operators η_0 and ξ_0 by

$$\eta_0 = \prod_{j=1}^{N+1} \eta^j_0, \xi_0 = \prod_{j=1}^{N+1} \xi^j_0. \tag{4.55}$$

Following the conjectures in Ref. 16, we expect the following identifications:

$$L(\Lambda_i) = \mathrm{Coker}_{\eta_0} = \xi_0\eta_0\mathcal{F}_{(\Lambda_i,\beta)}(i = 1, 2, \cdots, M + N + 1), \tag{4.56}$$

$$L((1-\alpha)\Lambda_0 + \alpha\Lambda_{M+1}) = \begin{cases} \mathrm{Coker}(\eta_0) = \xi_0\eta_0\mathcal{F}_{(\alpha,\beta)} \quad (\alpha = 0, 1, 2, \cdots), \\ \mathrm{Ker}(\eta_0) = \eta_0\xi_0\mathcal{F}_{(\alpha,\beta)} \quad (\alpha = -1, -2, \cdots). \end{cases} \tag{4.57}$$

Here $L(\lambda)$ the irreducible highest weight module. Since the operators η^j_0 and ξ^j_0 commute with $U_q(\widehat{sl}(M + 1|N + 1))$ up to sign $\pm$, we can regard $\mathrm{Ker}(\eta_0)$ and $\mathrm{Coker}(\eta_0)$ as $U_q(\widehat{sl}(M + 1|N + 1))$-module. In what follows we will work on the space, that is expected to be the irreducible highest weight module $L(\Lambda_{M+1})$.

$$L(\Lambda_{M+1}) = \xi_0\eta_0\mathcal{F}_{(1,\beta)}.$$

D. Vertex operator

In this section we give the bosonization of the vertex operators $\Phi^*_j(z)$, and give the integral representations of them. We set the following combinations of the Drinfeld generators:

$$h^*_{i,m} = \sum_{j=1}^{M+N+1} \frac{[\alpha_{i,j}m]_q[\beta_{i,j}m]_q}{[(M-N)m]_q[m]_q} h_{j,m}, \tag{4.58}$$

$$Q_{h^*_i} = \sum_{j=1}^{M+N+1} \frac{\alpha_{i,j}\beta_{i,j}}{M-N} Q_{h_{j,0}}, h^*_{i,0} = \sum_{j=1}^{M+N+1} \frac{\alpha_{i,j}\beta_{i,j}}{M-N} h_{j,0}. \tag{4.59}$$

Here we have set

$$\alpha_{i,j} = \begin{cases} \mathrm{Min(i, j)} & (\mathrm{Min}(i,\ j) \leq M+1), \\ 2(M+1) - \mathrm{Min}(i,\ j) & (\mathrm{Min}(i,\ j) > M+1), \end{cases} \tag{4.60}$$

$$\beta_{i,j} = \begin{cases} M - N - \mathrm{Max}(i,\ j) & (\mathrm{Max}(i,\ j) \leq M+1), \\ -M - N - 2 + \mathrm{Max}(i,\ j) & (\mathrm{Max}(i,\ j) > M+1). \end{cases} \tag{4.61}$$

We have the following commutation relations:

$$[h^*_{i,m}, h_{j,n}] = \delta_{i,j}\delta_{m+n,0}\frac{[m]_q^2}{m}, [h^*_{i,m}, h^*_{j,n}] = \delta_{m+n,0}\frac{[\alpha_{i,j}m]_q[\beta_{i,j}m]_q[m]_q}{m[(M-N)m]_q}, \tag{4.62}$$

$$[h^*_{i,0}, Q_{h_j}] = \delta_{i,j}, [h^*_{i,0}, Q_{h^*_j}] = \frac{\alpha_{i,j}\beta_{i,j}}{(M-N)}. \tag{4.63}$$

Theorem 4.3 (Ref. 16): *The bosonic operator $\phi^*(z)$ given below satisfies the same commutation relations as the vertex operator $\Phi^*(z)$. In other words, the bosonizations of the vertex operator $\Phi^*(z)$ on the space $\mathcal{F}_{(\alpha,\beta)}$, $\mathcal{F}_{(\Lambda_i,\beta)}$, $\mathcal{F}_{(\Lambda_{M+1+j},\beta)}$ are given as follows:*

$$\phi^*(z) = \sum_{j=1}^{M+N+2} \phi^*_j(z) \otimes v^*_j. \tag{4.64}$$

Here the bosonic operators $\phi^*_j(z)$ $(j = 1, 2, \ldots, M + N + 2)$ are defined iteratively by

$$\phi^*_1(z) =: e^{h^{*1}(qz;-1/2)} : \prod_{k=1}^{M+1} e^{\pi\sqrt{-1}\frac{k-1}{M-N}a_0^k}, \tag{4.65}$$

$$\nu_j q^{\nu_j}\phi^*_{j+1}(z) = -[\phi^*_j(z), f_j]_{q^{\nu_j}} (j = 1, 2, \cdots, M+N+1). \tag{4.66}$$

The $\mathbf{Z}_2$-grading is given by $|\phi^*_j(z)| = \frac{\nu_j+1}{2}(j = 1, 2, \ldots, M + N + 2)$.

Corollary 4.4: The bosonizations of the vertex operators $\Phi^*_j(z)$ on the space $\xi_0\eta_0\mathcal{F}_{(\alpha,\beta)}$ $(\alpha = 0, 1, 2, \ldots)$ are given by the following projection:

$$\Phi^*_j(z) = \xi_0\eta_0 \cdot \phi^*_j(z) \cdot \xi_0\eta_0 (j = 1, 2, \cdots, M+N+2). \tag{4.67}$$

In what follows we call the bosonic operators $\phi^*_j(z)$ the vertex operators. We prepare the auxiliary operators $X^-_{M+j,\epsilon}(w)$ $(\epsilon = \pm)$ by

$$X^-_{M+j}(w) = \frac{1}{(q-q^{-1})w}(X^-_{M+j,+}(w) - X^-_{M+j,-}(w))(j = 1, \cdots, N+1). \tag{4.68}$$

In other words, we set

$$X^-_{M+1,\epsilon}(w) = : e^{-h^{M+1}(w)-c^1(q^\epsilon w)} : \prod_{j=1}^{M} e^{\pi\sqrt{-1}a_0^j}(\epsilon = \pm), \tag{4.69}$$

$$X^-_{M+1+j,\epsilon}(w) = - : e^{-h^{M+j+1}(w)+c^j(w)-c^{j+1}(q^\epsilon w)} : (\epsilon = \pm,\ j = 2, 3, \cdots, N+1). \tag{4.70}$$

Using the normal orderings in Appendix B we have the following normal orderings for $j = 1, 2, \ldots, M$:

$$
\begin{aligned}
&: \phi_1^*(z) X_1^-(qw_1) \cdots X_j^-(qw_j) : X_{j+1}^-(qw_{j+1}) \\
&= e^{\frac{\pi\sqrt{-1}}{M}} \frac{1}{qw_j(1 - qw_{j+1}/w_j)} : \phi_1^*(z) X_1^-(qw_1) \cdots X_j^-(qw_j) X_{j+1}^-(qw_{j+1}) :, \\
&X_{j+1}^-(qw_{j+1}) : \phi_1^*(z) X_1^-(qw_1) \cdots X_j^-(qw_j) : \\
&= -e^{\frac{\pi\sqrt{-1}}{M}} \frac{1}{qw_{j+1}(1 - qw_j/w_{j+1})} : X_{j+1}^-(qw_{j+1}) \phi_1^*(z) X_1^-(qw_1) \cdots X_j^-(qw_j) :, \\
&: \phi_1^*(z) X_1^-(qw_1) \cdots X_M^-(qw_M) : X_{M+1}^-(qw_{M+1}) \\
&= -e^{\frac{\pi\sqrt{-1}}{M}} \frac{1}{qw_M(1 - qw_{M+1}/w_M)} : \phi_1^*(z) X_1^-(qw_1) \cdots X_M^-(qw_M) X_{M+1}^-(qw_{M+1}) :, \\
&X_{M+1}^-(qw_{M+1}) : \phi_1^*(z) X_1^-(qw_1) \cdots X_M^-(qw_M) : \\
&= -e^{\frac{\pi\sqrt{-1}}{M}} \frac{1}{qw_{M+1}(1 - qw_M/w_{M+1})} : X_{M+1}^-(qw_{M+1}) \phi_1^*(z) X_1^-(qw_1) \cdots X_M^-(qw_M) : .
\end{aligned}
$$

For $\epsilon = \pm$ and $j = 1, 2, \ldots, N + 1$, we have

$$
\begin{aligned}
&qX_{M+j,+}^-(w_1) X_{M+j+1,\epsilon}^-(w_2) - X_{M+j+1,\epsilon}^-(w_2) X_{M+j,+}^-(w_1) \\
&= \frac{(q^2 - 1)}{(1 - qw_1/w_2)} : X_{M+j,+}^-(w_1) X_{M+j+1,\epsilon}^-(w_2) :, \\
&qX_{M+j,-}^-(w_1) X_{M+j+1,\epsilon}^-(w_2) - X_{M+j+1,\epsilon}^-(w_2) X_{M+j,-}^-(w_1) \\
&= \frac{(q^2 - 1)}{(1 - w_1/qw_2)} : X_{M+j,-}^-(w_1) X_{M+j+1,\epsilon}^-(w_2) : .
\end{aligned}
$$

Using these normal orderings and (4.66), we have the following integral representations.

Proposition 4.5: The vertex operators $\phi_j^*(z)$ ($j = 1, 2, \ldots, M + N + 2$) have the following integral representations:

$$
\phi_1^*(z) = : e^{h^{*1}(qz)} : \prod_{k=1}^{M+1} e^{\pi\sqrt{-1}\frac{k-1}{M} a_0^k}, \tag{4.71}
$$

$$
\begin{aligned}
\phi_i^*(z) = {} & e^{\frac{\pi\sqrt{-1}}{M}(i-1)} (q - q^{-1})^{i-1} \prod_{k=1}^{i-1} \int_{C_i} \frac{dw_k}{2\pi\sqrt{-1} w_k} \frac{z^{-1} w_{i-1}}{\prod_{k=0}^{i-2} (1 - qw_k/w_{k+1})(1 - qw_{k+1}/w_k)} \\
& \times : \phi_1^*(z) X_1^-(qw_1) \cdots X_{i-1}^-(qw_{i-1}) : (i = 2, \cdots, M + 1),
\end{aligned} \tag{4.72}
$$

$$
\begin{aligned}
\phi_{M+2}^*(z) = {} & e^{\frac{M\pi\sqrt{-1}}{M-N}} (q - q^{-1})^M (qz)^{-1} \sum_{\epsilon=\pm} \epsilon \prod_{k=1}^{M+1} \int_{C_{M+2}} \frac{dw_k}{2\pi\sqrt{-1} w_k} \frac{1}{\prod_{k=0}^{M} (1 - qw_k/w_{k+1})(1 - qw_{k+1}/w_k)} \\
& \times : \phi_1^*(z) X_1^-(qw_1) \cdots X_M^-(qw_M) X_{M+1,\epsilon}^-(qw_{M+1}) :,
\end{aligned} \tag{4.73}
$$

$$
\phi_{M+1+j}^*(z) = e^{\frac{M\pi\sqrt{-1}}{M-N}} q^{j-1} (q - q^{-1})^M (qz)^{-1} \sum_{\epsilon_1, \cdots, \epsilon_j = \pm} \prod_{k=1}^{j} \epsilon_k \prod_{k=1}^{M+j} \int_{C_{M+1+j}} \frac{dw_k}{2\pi\sqrt{-1} w_k}
$$

$$\times \frac{1}{\prod_{k=0}^{M}(1-qw_k/w_{k+1})(1-qw_{k+1}/w_k)\prod_{k=1}^{j-1}(1-q^{\epsilon_k}w_{M+k}/w_{M+k+1})}$$
$$\times :\phi_1^*(z)X_1^-(qw_1)\cdots X_M^-(qw_M)X_{M+1,\epsilon_1}^-(qw_{M+1})\cdots X_{M+j,\epsilon_j}^-(qw_{M+j}):,$$
$$(j=1,2,\cdots,N+1). \tag{4.74}$$

Here we have read $w_0 = z$. We take the integration contour C_i $(i = 1, 2, \ldots, M + N + 2)$ to be simple closed curve that encircles $w_l = 0, qw_{l-1}$ but not $q^{-1}w_{l-1}$ for $l = 1, 2, \ldots, i - 1$.

V. BOUNDARY STATE

In this section we give the bosonization of the boundary state ${}_{(i)}\langle B|$ (3.28). The construction of the boundary state is the main result of this paper. We give detailed proof of this bosonization of the boundary state.

A. Boundary state

In this section we give the bosonization of the boundary state ${}_{(i)}\langle B|$. We use the highest weight vector $v^*_{\Lambda_{M+1}} \in L^*(\Lambda_{M+1})$ given by

$$v^*_{\Lambda_{M+1}} = \langle 0|e^{-\beta\sum_{i=1}^{M+1}Q_{a^i}+(1-\beta)\sum_{j=1}^{N+1}Q_{b^j}+\sum_{j=1}^{N+1}Q_{c^j}}, \tag{5.1}$$

where $\langle 0|$ is the vacuum vector satisfying

$$\langle 0|a_n^i = \langle 0|b_n^j = \langle 0|c_n^j = 0 (n \geq 0, i = 1, 2, \cdots, M+1, j = 1, 2, \cdots, N+1). \tag{5.2}$$

We have

$$v^*_{\Lambda_{M+1}}h_i = \delta_{i,M+1}v^*_{\Lambda_{M+1}}, v^*_{\Lambda_{M+1}}c_0^j = -v^*_{\Lambda_{M+1}}, \tag{5.3}$$

for $i = 1, 2, \ldots, M + N + 1$ and $j = 1, 2, \ldots, N + 1$. In what follows we use the auxiliary function

$$\theta_m = \begin{cases} 1 & (m : \text{even}), \\ 0 & (m : \text{odd}). \end{cases} \tag{5.4}$$

Definition 5.1: We define the bosonic operators $G^{(i)}$ $(i = 1, 2, 3)$ by

$$G^{(i)} = -\frac{1}{2}\sum_{j=1}^{M+N+1}\sum_{m=1}^{\infty}\frac{mq^{-2m}}{[m]_q^2}h_{j,m}h^*_{j,m} - \frac{1}{2}\sum_{j=1}^{N+1}\sum_{m=1}^{\infty}\frac{mq^{-2m}}{[m]_q^2}c_m^jc_m^j$$
$$+\sum_{j=1}^{M+N+1}\sum_{m=1}^{\infty}\beta_{j,m}^{(i)}h^*_{j,m} + \sum_{j=1}^{N+1}\sum_{m=1}^{\infty}\gamma_{j,m}c_m^j. \tag{5.5}$$

Here we have set

$$\gamma_{j,m} = -\frac{q^{-m}}{[m]_q}\theta_m (j = 1, 2, \cdots, N+1). \tag{5.6}$$

Here we have set $\beta_{j,m}^{(i)}$ $(i = 1, 2, 3)$ as follows:

Condition 1: For $i = 1$ we have set

$$\beta^{(1)}_{j,m} = \begin{cases} \dfrac{q^{-3m/2} - q^{-m/2}}{[m]_q}\theta_m & (1 \le j \le M), \\ \dfrac{-2q^{-3m/2}}{[m]_q}\theta_m & (j = M+1), \\ \dfrac{q^{-3m/2} + q^{-m/2}}{[m]_q}\theta_m & (M+2 \le j \le M+N+1). \end{cases} \tag{5.7}$$

Condition 2: For $i = 2$ we have set

$$\beta^{(2)}_{j,m} = \beta^{(1)}_{j,m} - \frac{r^m q^{(\beta_1 - 3/2)m}}{[m]_q}\delta_{j,L} (j = 1, 2, \cdots, M+N+1). \tag{5.8}$$

Condition 2.1: For $L \le M + 1$ we have set

$$\beta_1 = -L. \tag{5.9}$$

Condition 2.2: For $M + 2 \le L \le M + N + 1$ we have set

$$\beta_1 = L - 2M - 2. \tag{5.10}$$

Condition 3: For $i = 3$ we have set

$$\beta^{(2)}_{j,m} = \beta^{(1)}_{j,m} - \frac{r^m q^{(\beta_1 - 3/2)m}}{[m]_q}\delta_{j,L} - \frac{q^{(\beta_2 - 3/2)m}/r^m}{[m]_q}\delta_{j,L+K} \quad (j = 1, 2, \cdots, M+N+1). \tag{5.11}$$

Condition 3.1: For $L + K \le M + 1$ we have set

$$(\beta_1, \beta_2) = (-L, L - K). \tag{5.12}$$

Condition 3.2: For $L \le M + 1 \le L + K - 1$ we have set

$$(\beta_1, \beta_2) = (-L, 3L + K - 2M - 2). \tag{5.13}$$

Condition 3.3: For $M + 2 \le L$ we have set

$$(\beta_1, \beta_2) = (L - 2M - 2, 2M + K - L + 2). \tag{5.14}$$

The following is **main theorem** of this paper.

Theorem 5.2: *The bosonization of the boundary state ${}_{(i)}\langle B|(i = 1, 2, 3)$ is given by*

$${}_{(i)}\langle B| = v^*_{\Lambda_{M+1}} \cdot e^{G^{(i)}}. \tag{5.15}$$

Here the bosonic operator $G^{(i)}$ $(i = 1, 2, 3)$ is given by (5.5). In other words the vector ${}_{(i)}\langle B|$ becomes the eigenvector of the commuting transfer matrix $T^{(i)}_B(z)$ with the eigenvalue 1,

$${}_{(i)}\langle B|T^{(i)}_B(z) = {}_{(i)}\langle B|. \tag{5.16}$$

In Sec. VI we shall give the proof of this theorem.

B. Excitation

In this section we give other eigenvectors of $T_B^{(i)}(z)$, that describes the excitations of the physical model.

Definition 5.3: We define the type-II vertex operators $\Psi(z)$ and $\Psi^*(z)$ as the intertwiners of $U_q(\widehat{sl}(M+1|N+1))$-module if they exist,

$$\Psi(z) : L(\lambda) \to V_z \otimes L(\mu),\ \Psi^*(z) : L(\mu) \to V_z^{*S} \otimes L(\lambda), \tag{5.17}$$

$$\Psi(z) \cdot x = \Delta(x) \cdot \Psi(z),\ \Psi^*(z) \cdot x = \Delta(x) \cdot \Psi^*(z), \tag{5.18}$$

for $x \in U_q(\widehat{sl}(M+1|N+1))$. The operators $\Psi(z)$ and $\Psi^*(z)$ depend on the weight λ and μ.

We expand the vertex operators $\Psi(z) = \sum_{k=1}^{M+N+2} v_k \otimes \Psi_k(z)$ and $\Psi^*(z) = \sum_{k=1}^{M+N+2} v_k^* \otimes \Psi_k^*(z)$. The type-II vertex operator $\Psi_k^*(\xi)$ and type-I vertex operators $\Phi_j(z)$, $\Phi_j^*(z)$ are expected to satisfy the following commutation relations:

$$\Psi_k^*(\xi)\Phi_j(z) = \tau(\xi/z)\Phi_j(z)\Psi_k^*(\xi)(-1)^{[v_k][v_j]}, \tag{5.19}$$

$$\Psi_k^*(\xi)\Phi_j^*(z) = \tau(\xi/z)\Phi_j^*(z)\Psi_k^*(\xi)(-1)^{[v_k][v_j]}. \tag{5.20}$$

Here we have set

$$\tau(z) = -z^{\frac{1-M+N}{M-N}} \exp\left(-\sum_{m=1}^{\infty} \frac{[(M-N-1)m]_q}{m[(M-N)m]_q}(z^m - z^{-m})\right). \tag{5.21}$$

Definition 5.4: We call the following vectors ${}_{k_1,k_2,\cdots,k_n(i)}\langle\xi_1, \xi_2, \cdots, \xi_n|$ the excitations. We set

$${}_{k_1,k_2,\cdots,k_n(i)}\langle k_1, k_2, \cdots, k_n| = {}_{(i)}\langle B|\Psi_{k_1}^*(\xi_1)\Psi_{k_2}^*(\xi_2)\cdots\Psi_{k_n}^*(\xi_n),$$

for $k_1, k_2, \ldots, k_n = 1, 2, \ldots, M+N+2$.

Corollary 5.5: The excitations become the eigenvector of the transfer matrix $T_B^{(i)}(z)$.

$${}_{k_1,k_2,\cdots,k_n(i)}\langle\xi_1, \xi_2, \cdots, \xi_n|T_B^{(i)}(z) = {}_{k_1,k_2,\cdots,k_n(i)}\langle\xi_1, \xi_2, \cdots, \xi_n|\prod_{s=1}^{n}\tau(\xi_s z)\tau(\xi_s/z). \tag{5.22}$$

We expect that the excitations (5.22) are the basis of the space of the physical state of the supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ chain with a boundary.

VI. PROOF OF MAIN THEOREM

In this section we give detailed proof of the main theorem. We would like to show

$${}_{(i)}\langle B|\phi_j^*(z^{-1})K^{(i)}(z)_j^j = {}_{(i)}\langle B|\phi_j^*(z^{-1})(j = 1, 2, \cdots, M+N+2). \tag{6.1}$$

It is convenient to use the following abbreviations:

$$h_+^i(z) = -\sum_{m=1}^{\infty}\frac{h_{i,m}}{[m]_q}q^{\frac{m}{2}}z^{-m},\ h_-^i(z) = \sum_{m=1}^{\infty}\frac{h_{i,-m}}{[m]_q}q^{\frac{m}{2}}z^{m}(i = 1, 2, \cdots, M+N+1), \tag{6.2}$$

$$h_+^{*1}(z) = -\sum_{m=1}^{\infty}\frac{h_{1,m}^*}{[m]_q}q^{\frac{m}{2}}z^{-m},\ h_-^{*1}(z) = \sum_{m=1}^{\infty}\frac{h_{1,-m}^*}{[m]_q}q^{\frac{m}{2}}z^{m}, \tag{6.3}$$

$$c_+^j(z) = -\sum_{m=1}^{\infty}\frac{c_m^j}{[m]_q}z^{-m},\ c_-^j(z) = \sum_{m=1}^{\infty}\frac{c_{-m}^j}{[m]_q}z^{m}(j = 1, 2, \cdots, N+1). \tag{6.4}$$

We set the function $D(z, w)$ by

$$D(z, w) = (1 - qzw)(1 - qz/w)(1 - qw/z)(1 - q/wz). \tag{6.5}$$

The function $D(z, w)$ is invariant under $(z, w) \to (1/z, w), (z, 1/w), (1/z, 1/w)$.

Proposition 6.1: The operator $G^{(i)}$ $(i = 1, 2, 3)$ given in (5.5) satisfies

$$[G^{(i)}, h_{j,-m}] = -q^{-2m} h_{j,m} + \frac{[m]_q^2}{m} \beta_{j,m}^{(i)} (m > 0,\ j = 1, 2, \cdots, M + N + 1), \tag{6.6}$$

$$[G^{(i)}, c_{-m}^{j}] = -q^{-2m} c_m^j + \frac{[m]_q^2}{m} \gamma_{j,m} (m > 0,\ j = 1, 2, \cdots, N + 1). \tag{6.7}$$

Proof. Using the commutation relations (4.3) and (4.62), we have

$$[G^{(i)}, h_{j,-m}] = -\frac{1}{2} \sum_{k=1}^{M+N+1} q^{-2m} \frac{[A_{k,j} m]_q}{[m]_q} h_{k,m}^* - \frac{1}{2} q^{-2m} h_{j,m} + \frac{[m]_q^2}{m} \beta_{j,m}. \tag{6.8}$$

Using definition of $h_{k,m}^*$ (4.58) and the fact that the inversion matrix of $\left(\frac{[A_{k,j} m]_q}{[m]_q}\right)_{1 \le k, j \le M+N+1}$ is given by $\left(\frac{[\alpha_{k,j} m]_q [\beta_{k,j} m]_q}{[(M-N)m]_q [m]_q}\right)_{1 \le k, j \le M+N+1}$, we have $[G^{(i)}, h_{j,-m}] = -q^{-2m} h_{j,m} + \frac{[m]_q^2}{m} \beta_{j,m}^{(i)}$. Proof of the relation $[G^{(i)}, c_{-m}^{j}] = -q^{-2m} c_m^j + \frac{[m]_q^2}{m} \gamma_{j,m}$ is given by similar way. □

Proposition 6.2: For the boundary conditions $i = 1, 2, 3$, we have

$${}_{(i)}\langle B | h_j = \delta_{j,M+1}\, {}_{(i)}\langle B | (j = 1, 2, \cdots, M + N + 1), \tag{6.9}$$

$${}_{(i)}\langle B | h_{1,0}^* = -\frac{N+1}{M-N}\, {}_{(i)}\langle B |, \tag{6.10}$$

$${}_{(i)}\langle B | c_0^j = -{}_{(i)}\langle B | (j = 1, 2, \cdots, N + 1). \tag{6.11}$$

We show the relation (6.1) for each boundary condition ${}_{(i)}\langle B|$ $(i = 1, 2, 3)$.

A. Boundary condition 1

In this section we show (6.1) for the boundary condition ${}_{(1)}\langle B|$. Very explicitly we would like to show

$$z^{\frac{M}{M-N}} \varphi^{(1)}(z^{-1})\, {}_{(1)}\langle B | \phi_j^*(z) = z^{-\frac{M}{M-N}} \varphi^{(1)}(z)\, {}_{(1)}\langle B | \phi_j^*(z^{-1}) (1 \le j \le M + N + 2). \tag{6.12}$$

We would like to comment that RHS (respectively, LHS) is obtained from LHS (respectively, RHS) under $z \to 1/z$. The proof for $1 \le j \le M + 1$ is similar as those of non-super $\widehat{sl}(N)$ case. The proof for $M + 2 \le j \le M + N + 2$ is different from those of non-super case. We prepare propositions.

Proposition 6.3: The actions of $h_+^i(w)$, $h_-^{*1}(w)$, and $c_-^j(w)$ on the boundary state ${}_{(1)}\langle B|$ are given as follows:

$${}_{(1)}\langle B | e^{-h_-^i(qw)} = g_i^{(1)}(w)\, {}_{(1)}\langle B | e^{-h_+^i(q/w)} (i = 1, 2, \cdots, M + N + 1), \tag{6.13}$$

$${}_{(1)}\langle B | e^{h_-^{*1}(qw)} = \varphi^{(1)}(w)\, {}_{(1)}\langle B | e^{h_+^{*1}(q/w)}, \tag{6.14}$$

$${}_{(1)}\langle B | e^{-c_-^j(qw)} = c_j^{(1)}(w)\, {}_{(1)}\langle B | e^{-c_+^j(q/w)} (j = 1, 2, \cdots, N + 1). \tag{6.15}$$

Here we have set

$$g_i^{(1)}(w) = \begin{cases} (1-w^2) & (1 \le i \le M), \\ (1+w^2) & (i = M+1), \\ 1 & (M+2 \le i \le M+N+1). \end{cases} \tag{6.16}$$

The function $\varphi^{(1)}(w)$ *is given in (2.20) and* $c_j^{(1)}(w) = 1$ *for* $j = 1, 2, \ldots, N+1$.

Proof. Using Proposition 6.1 and the commutation relation (4.62), we have

$$[G^{(1)}, [G^{(1)}, h_{j,-m}]] = 0. \tag{6.17}$$

Hence we have

$$e^{G^{(1)}} h_{j,-m} e^{-G^{(1)}} = \sum_{n=0}^{\infty} \frac{1}{n!} ad(G^{(1)})^n (h_{j,-m}) = h_{j,-m} + [G^{(1)}, h_{j,-m}] = h_{j,-m} - q^{-2m} h_{j,m} + \frac{[m]_q^2}{m} \beta_{j,m}^{(1)}. \tag{6.18}$$

Using this relation we have

$$e^{G^{(1)}} e^{-h_-^i(qw)} e^{-G^{(1)}} = \exp\left(-\sum_{m=1}^{\infty} \frac{[m]_q}{m} q^{\frac{3m}{2}} \beta_{i,m}^{(1)} w^m \right) e^{-h_-^i(qw) - h_+^i(q/w)}. \tag{6.19}$$

Using relation $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ we have

$$e^{G^{(1)}} e^{-h_-^i(qw)} e^{-G^{(1)}} = g_i^{(1)}(w) e^{-h_-^i(qw)} e^{-h_+^i(q/w)}. \tag{6.20}$$

Multiplying $v_{\Lambda_{M+1}}^*$ from the left and $e^{G^{(1)}}$ from the right, we have

$$v_{\Lambda_{M+1}}^* \cdot e^{G^{(1)}} e^{-h_-^i(qw)} = g_i^{(1)}(w) v_{\Lambda_{M+1}}^* \cdot e^{G^{(1)}} e^{-h_+^i(q/w)}. \tag{6.21}$$

The others are shown by similar way. □

Proposition 6.4: The following relation holds,

$$\sum_{\epsilon=\pm} \int_C \frac{dw_1}{w_1} \frac{\epsilon q^\epsilon (1 - q w_1 w_2) e^{-c_+^j(q^{1+\epsilon} w_1) - c_+^j(q^{1-\epsilon}/w_1)}}{(1 - q^\epsilon w_1 w_2)(1 - q^\epsilon w_1/w_2)} = \int_C \frac{dw_1}{w_1} \frac{q^{-1}(-1 + w_1^2) e^{-c_+^j(w_1) - c_+^j(q^2/w_1)}}{(1 - w_1 w_2/q)(1 - w_1/q w_2)}. \tag{6.22}$$

Here the integration contour C *encircles* $w_1 = 0, q w_2^{\pm 1}$ *but not* $w_1 = q^{-1} w_2^{\pm 1}$. *This integral is invariant under* $w_2 \to 1/w_2$.

Proof. We start from LHS:

$$\int_C \frac{dw_1}{w_1} \frac{q e^{-c_+^j(q^2 w_1) - c_+^j(1/w_1)}}{(1 - q w_1/w_2)} - \int_C \frac{dw_1}{w_1} \frac{q^{-1}(1 - q w_1 w_2) e^{-c_+^j(w_1) - c_+^j(q^2/w_1)}}{(1 - w_1 w_2/q)(1 - w_1/q w_2)}. \tag{6.23}$$

Changing the variable $w_1 \to 1/w_1$ in the first term and summing up the first and the second terms, we have RHS. □

Proof for boundary condition 1: We show main theorem for the boundary condition ${}_{(1)}\langle B|$. We show the relation (6.12).

• The case for $j = 1$: ${}_{(1)}\langle B|\phi_1^*(z)$.

Using the bosonization (4.71) and the relation (6.14), we get LHS of (6.12) as follows:

$$z^{\frac{M}{M-N}} \varphi^{(1)}(1/z) {}_{(1)}\langle B|\phi_1^*(z) = q^{-\frac{M}{M-N}} \varphi^{(1)}(z) \varphi^{(1)}(1/z) {}_{(1)}\langle B| e^{h_+^{*1}(qz) + h_+^{*1}(q/z)} e^{Q_{h_1^*}}. \tag{6.24}$$

This is invariant under $z \to 1/z$. Hence LHS and RHS of (6.12) coincide.

• The case for $j = 2$: ${}_{(1)}\langle B|\phi_2^*(z)$.

Using the bosonization (4.72), the relations (6.13), (6.14), and the normal orderings in Appendix B, we get LHS of (6.12) as follows:

$$\begin{aligned}
&z^{\frac{M}{M-N}}\varphi^{(1)}(1/z)_{(1)}\langle B|\phi_2^*(z)\\
&= q^{-\frac{M}{M-N}-1}(q-q^{-1})e^{\frac{\pi\sqrt{-1}}{M-N}}\varphi^{(1)}(z)\varphi^{(1)}(1/z)\\
&\times\int_{C_1}\frac{dw}{2\pi\sqrt{-1}w}\frac{(1-w^2)(1-q/zw)}{D(z,w)}{}_{(1)}\langle B|e^{Q_{h_1^*-h_1}}e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-h_+^1(qw)-h_+^1(q/w)}. \qquad (6.25)
\end{aligned}$$

We note that the integrand $\frac{\varphi^{(1)}(z)\varphi^{(1)}(1/z)}{D(z,w)}e^{h_+^{*1}(qz)+h_+^{*1}(q/z)}$ is invariant $z\to 1/z$. We have LHS − RHS of (6.12) as follows:

$$\begin{aligned}
&q^{-\frac{M}{M-N}}(q-q^{-1})e^{\frac{\pi\sqrt{-1}}{M-N}}\varphi^{(1)}(z)\varphi^{(1)}(1/z)(z-z^{-1})\\
&\times\int_{\widetilde{C}_1}\frac{dw}{2\pi\sqrt{-1}w}\frac{(w^{-1}-w)}{D(z,w)}{}_{(1)}\langle B|e^{Q_{h_1^*-h_1}}e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-h_+^1(qw)-h_+^1(q/w)}. \qquad (6.26)
\end{aligned}$$

Here the integration contour $\widetilde{C}_1$ encircles $w=0, qz^{\pm 1}$ but not $w=q^{-1}z^{\pm 1}$. The integration contour $\widetilde{C}_1$ is invariant under $w\to 1/w$. The integrand $\frac{(w^{-1}-w)}{D(z,w)}e^{-h_+^1(qw)-h_+^1(q/w)}$ creates just signature (-1) under $w\to 1/w$. Hence we have LHS − RHS=0.

• The case for $3\le j\le M+1$: ${}_{(1)}\langle B|\phi_j^*(z)$.

Using the bosonization (4.72), the relations (6.13), (6.14), and normal orderings in Appendix B, we have LHS − RHS of (6.12) as follows:

$$\begin{aligned}
&z^{\frac{M}{M-N}}\varphi^{(1)}(1/z)_{(1)}\langle B|\phi_j^*(z)-z^{-\frac{M}{M-N}}\varphi^{(1)}(z)_{(1)}\langle B|\phi_j^*(1/z)\\
&= q^{-\frac{M}{M-N}}(q-q^{-1})^{j-1}\varphi^{(1)}(z)\varphi^{(1)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}(j-1)}{M-N}}\\
&\times\prod_{k=1}^{j-1}\int_{\widetilde{C}_j}\frac{dw_k}{2\pi\sqrt{-1}w_k}\frac{\prod_{k=1}^{j-2}(1-q/w_kw_{k+1})}{\prod_{k=0}^{j-2}D(w_k,w_{k+1})}(w_1^{-1}-w_1)\prod_{k=2}^{j-1}(1-w_k^2)\\
&\times{}_{(1)}\langle B|e^{Q_{h_1^*-h_1\cdots-h_{j-1}}}e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{j-1}(h_+^k(qw_k)+h_+^k(q/w_k))}. \qquad (6.27)
\end{aligned}$$

Here the integration contour $\widetilde{C}_j$ encircles $w_k=0, qw_{k-1}^{\pm 1}$ but not $w_k=q^{-1}w_{k-1}^{\pm 1}$ for $1\le k\le j-1$. Let us study the changing of the variable $w_1\to 1/w_1$. We note that the integrand $\frac{1}{D(z,w_1)}e^{-h_+^1(qw_1)-h_+^1(q/w_1)}$ and the integration contour $\widetilde{C}_j$ are invariant under $w_1\to 1/w_1$. Taking into account of symmetrization $\oint_{|w_1|=1}\frac{dw_1}{w_1}f(w_1)=\frac{1}{2}\oint_{|w_1|=1}\frac{dw_1}{w_1}(f(w_1)+f(1/w_1))$ and relation $(1-q/w_1w_2)-(w_1\leftrightarrow 1/w_1)=(w_1^{-1}-w_1)(-q/w_2)$, we symmetrize the variables $w_1,w_2,\cdots,w_{j-2}$, iteratively, we have

$$\begin{aligned}
&(-q/2)^{j-2}q^{-\frac{M}{M-N}}(q-q^{-1})^{j-1}\varphi^{(1)}(z)\varphi^{(1)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}(j-1)}{M-N}}\\
&\times\prod_{k=1}^{j-1}\int_{\widetilde{C}_j}\frac{dw_k}{2\pi\sqrt{-1}w_k}\frac{\prod_{k=1}^{j-2}(w_k^{-1}-w_k)^2(w_{j-1}^{-1}-w_{j-1})}{\prod_{k=0}^{j-2}D(w_k,w_{k+1})}\\
&\times{}_{(i)}\langle B|e^{Q_{h_1^*-h_1\cdots-h_{j-1}}}e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{j-1}(h_+^k(qw_k)+h_+^k(q/w_k))}=0. \qquad (6.28)
\end{aligned}$$

Here we have used $\oint_{|w_{j-1}|=1}\frac{dw_{j-1}}{w_{j-1}}(w_{j-1}^{-1}-w_{j-1})e^{-h_+^{j-1}(qw_{j-1})-h_+^{j-1}(q/w_{j-1})}=0$.

• The case for $j=M+2$: ${}_{(1)}\langle B|\phi_{M+2}^*(z)$.

Using the bosonization (4.73) , the relations (6.13), (6.14), (6.15), and the normal orderings in Appendix B, we have LHS − RHS of (6.12) as follows:

$$
\begin{aligned}
& z^{\frac{M}{M-N}}\varphi^{(1)}(1/z)_{(1)}\langle B|\phi^*_{M+2}(z) - z^{-\frac{M}{M-N}}\varphi^{(1)}(z)_{(1)}\langle B|\phi^*_{M+2}(1/z) \\
&= q^{-\frac{M}{M-N}+1}(q-q^{-1})^M\varphi^{(1)}(z)\varphi^{(1)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}}{M-N}M} \\
&\times \sum_{\epsilon=\pm}\epsilon\prod_{k=1}^{M+1}\int_{\widetilde{C}_{M+2}}\frac{dw_k}{2\sqrt{-1}w_k}\frac{(w_1^{-1}-w_1)(1+w_{M+1}^2)}{\prod_{k=0}^{M}D(w_k,w_{k+1})}\prod_{k=1}^{M}(1-q/w_kw_{k+1})\prod_{k=2}^{M}(1-w_k^2) \qquad (6.29) \\
&\times {}_{(1)}\langle B|e^{Q_{h_1^*-h_1\cdots-h_{M+1}-c_1}}e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+1}(h_+^k(qw_k)+h_+^k(q/w_k))-c_+^1(q^{1+\epsilon}w_{M+1})-c_+^1(q^{1-\epsilon}/w_{M+1})}.
\end{aligned}
$$

Here the integration contour $\widetilde{C}_{M+2}$ encircles $w_k=0, qw_{k-1}^{\pm1}$ but not $w_k=q^{-1}w_{k-1}^{\pm1}$ for $1\le k\le M+1$. Taking into account of symmetrization $\oint_{|w|=1}\frac{dw}{w}f(w)=\frac{1}{2}\oint_{|w|=1}(f(w)+f(1/w))$ and relation $(1-q/w_1w_2)-(w_1\leftrightarrow w_1^{-1})=(-q/w_2)(w_1^{-1}-w_1)$, we symmetrize the variables $w_1, w_2, \cdots, w_M$ iteratively. Then we have

$$
\begin{aligned}
& (-q/2)^M q^{-\frac{M}{M-N}+1}(q-q^{-1})^M\varphi^{(1)}(z)\varphi^{(1)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}}{M-N}M} \\
&\times\prod_{k=1}^{M+1}\int_{\widetilde{C}_{M+2}}\frac{dw_k}{2\sqrt{-1}w_k}\frac{\prod_{k=1}^{M}(w_k^{-1}-w_k)^2(w_{M+1}^{-1}+w_{M+1})}{\prod_{k=0}^{M}D(w_k,w_{k+1})} \\
&\times {}_{(1)}\langle B|e^{Q_{h_1^*-h_1\cdots-h_{M+1}-c_1}}e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+1}(h_+^k(qw_k)+h_+^k(q/w_k))} \\
&\times\left(e^{-c_+^1(q^2w_{M+1})-c_+^1(1/w_{M+1})}-e^{-c_+^1(w_{M+1})-c_+^1(q^2/w_{M+1})}\right)=0. \qquad (6.30)
\end{aligned}
$$

Here we have used

$$
\oint_{|w_{M+1}|=1}\frac{dw_{M+1}}{w_{M+1}}\frac{(w_{M+1}+w_{M+1}^{-1})}{D(w_M,w_{M+1})}\left(e^{-c_+^1(q^2w_{M+1})-c_+^1(1/w_{M+1})}-e^{-c_+^1(w_{M+1})-c_+^1(q^2/w_{M+1})}\right)=0.
$$

- The case for $2\le j\le N+1$: ${}_{(1)}\langle B|\phi^*_{M+1+j}(z)$.

Using the bosonization (4.74), the relations (6.13), (6.14), (6.15), and the normal orderings in Appendix B, we have LHS − RHS of (6.12) as follows:

$$
\begin{aligned}
& z^{\frac{M}{M-N}}\varphi^{(1)}(1/z)_{(1)}\langle B|\phi^*_{M+j+1}(z) - z^{-\frac{M}{M-N}}\varphi^{(1)}(z)_{(1)}\langle B|\phi^*_{M+1+j}(1/z) \\
&= q^{-\frac{M}{M-N}+1}(q-q^{-1})^M\varphi^{(1)}(z)\varphi^{(1)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}M}{M-N}} \\
&\times\sum_{\epsilon_1,\cdots,\epsilon_j=\pm}\epsilon_j\prod_{k=1}^{M+j}\int_{\widetilde{C}_{M+1+j}}\frac{dw_k}{2\pi\sqrt{-1}w_k}\frac{(w_1^{-1}-w_1)(1+w_{M+1}^2)}{\prod_{k=0}^{M}D(w_k,w_{k+1})}\prod_{k=1}^{M}(1-q/w_kw_{k+1})\prod_{k=2}^{M}(1-w_k^2) \\
&\times\prod_{k=1}^{j-1}\frac{\epsilon_kq^{\epsilon_k}(1-qw_{M+k}w_{M+k+1})}{(1-q^{\epsilon_k}w_{M+k}w_{M+k+1})(1-q^{\epsilon_k}w_{M+k}/w_{M+k+1})}{}_{(1)}\langle B|e^{Q_{h_1^*-h_1\cdots-h_{M+j}-c_j}}e^{h_+^{*1}(qz)+h_+^{*1}(q/z)} \\
&\times e^{-\sum_{k=1}^{M+j}(h_+^k(qw_k)+h_+^k(q/w_k))+\sum_{k=1}^{j-1}(c_+^k(qw_{M+k+1})+c_+^k(q/w_{M+k+1}))-\sum_{k=1}^{j}(c_+^k(q^{1+\epsilon_k}w_{M+k})+c_+^k(q^{1-\epsilon_k}/w_{M+k}))}.
\end{aligned}
\qquad (6.31)
$$

Here the integration contour $\widetilde{C}_{M+1+j}$ encircles $w_k=0, qw_{k-1}^{\pm1}$ but not $w_k=q^{-1}w_{k-1}^{\pm1}$ for $1\le k\le M+j$. Using relation $(1-q/w_1w_2)-(w_1\leftrightarrow w_1^{-1})=(-q/w_2)(w_1^{-1}-w_1)$ we symmetrize the

variables $w_1, w_2, \cdots, w_M$ iteratively, we have

$$
\begin{aligned}
& q^{-\frac{M}{M-N}}(-q/2)^M(q-q^{-1})^M\varphi^{(1)}(z)\varphi^{(1)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}M}{M-N}} \\
& \times \sum_{\epsilon_1,\cdots,\epsilon_j=\pm}\epsilon_j\prod_{k=1}^{M+j}\int_{\widetilde{C}_{M+1+j}}\frac{dw_k}{2\pi\sqrt{-1}w_k}\frac{\prod_{k=1}^{M}(w_k^{-1}-w_k)^2(w_{M+1}^{-1}+w_{M+1})}{\prod_{k=0}^{M}D(w_k,w_{k+1})} \\
& \times \prod_{k=1}^{j-1}\frac{\epsilon_k q^{\epsilon_k}(1-qw_{M+k}w_{M+k+1})}{(1-q^{\epsilon_k}w_{M+k}w_{M+k+1})(1-q^{\epsilon_k}w_{M+k}/w_{M+k+1})}{}_{(1)}\langle B|e^{Q_{h_1^*-h_1\cdots-h_{M+j}-c_j}}e^{h_+^{*1}(qz)+h_+^{*1}(q/z)} \\
& \times e^{-\sum_{k=1}^{M+j}(h_+^k(qw_k)+h_+^k(q/w_k))+\sum_{k=1}^{j-1}(c_+^k(qw_{M+k+1})+c_+^k(q/w_{M+k+1}))-\sum_{k=1}^{j}(c_+^k(q^{1+\epsilon_k}w_{M+k})+c_+^k(q^{1-\epsilon_k}/w_{M+k}))}.
\end{aligned}
\tag{6.32}
$$

Using the relation (6.22) for the variables $w_{M+1}, \cdots, w_{M+j-1}$ iteratively, we have

$$
\begin{aligned}
& q^{-\frac{M}{M-N}}(-q/2)^M(q-q^{-1})^M\varphi^{(1)}(z)\varphi^{(1)}(1/z)(z-z^{-1})(-q)^{-j+1}e^{\frac{\pi\sqrt{-1}M}{M-N}} \\
& \times \prod_{k=1}^{M+j}\int_{\widetilde{C}_{M+1+j}}\frac{dw_k}{2\pi\sqrt{-1}w_k}\frac{\prod_{k=1}^{M}(w_k^{-1}-w_k)^2(w_{M+1}^{-1}+w_{M+1})}{\prod_{k=0}^{M}D(w_k,w_{k+1})} \\
& \times {}_{(1)}\langle B|e^{Q_{h_1^*-h_1\cdots-h_{M+j}-c_j}}\prod_{k=1}^{j-1}\frac{(-1+w_{M+k}^2)e^{-c_+^k(w_{M+k})-c_+^k(q^2/w_{M+k})}}{(1-w_{M+k}w_{M+k+1}/q)(1-w_{M+k}/qw_{M+k+1})} \\
& \times e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+j}(h_+^k(qw_k)+h_+^k(q/w_k))+\sum_{k=1}^{j-1}(c_+^k(qw_{M+k+1})+c_+^k(q/w_{M+k+1}))} \\
& \times \left(e^{-c_+^j(q^2w_{M+j})-c_+^j(1/w_{M+j})}-e^{-c_+^j(q^2/w_{M+j})-c_+^j(w_{M+j})}\right)=0.
\end{aligned}
\tag{6.33}
$$

Here we have used

$$
\oint_{|w|=1}\frac{dw}{w}e^{c_+^{j-1}(qw)+c_+^{j-1}(q/w)}\left(e^{-c_+^j(q^2w)-c_+^j(1/w)}-e^{-c_+^j(w)-c_+^j(q^2/w)}\right)f(w)=0,
$$

where $f(w)=f(1/w)$.

Now we have shown the relation (6.12) for every $j=1,2,\ldots,M+N+2$. □

B. Boundary condition 2

In this section we study (6.1) for the boundary condition ${}_{(2)}\langle B|$. Very explicitly we study

$$
z^{\frac{M}{M-N}}\varphi^{(2)}(z^{-1})_{(2)}\langle B|\phi_j^*(z)=z^{-\frac{M}{M-N}}\varphi^{(2)}(z)_{(2)}\langle B|\phi_j^*(z^{-1})(1\le j\le L),
\tag{6.34}
$$

$$
z^{\frac{M}{M-N}}(1-rz)\varphi^{(2)}(z^{-1})_{(2)}\langle B|\phi_j^*(z)=z^{-\frac{M}{M-N}}(1-r/z)_{(2)}\langle B|\phi_j^*(z^{-1})(L+1\le j\le M+N+2).
\tag{6.35}
$$

The structure of (6.34) is the same as those of (6.12) for the boundary condition ${}_{(1)}\langle B|$. In this section we focus our attention on the relation (6.35) that is new for the boundary condition ${}_{(2)}\langle B|$. We give proofs for following two conditions:

$$
\text{Condition 2.1}: L\le M+1,
$$

$$
\text{Condition 2.2}: M+2\le L\le M+N+1.
$$

Proposition 6.5: The actions of $h^i_+(w)$, $h^{*1}_-(w)$, and $c^j_-(w)$ on the boundary state ${}_{(2)}\langle B|$ are given as follows:

$$ {}_{(2)}\langle B|e^{-h^i_-(qw)} = g^{(2)}_i(w)\,{}_{(2)}\langle B|e^{-h^i_+(q/w)} (i = 1, 2, \cdots, M + N + 1), \tag{6.36}$$

$$ {}_{(2)}\langle B|e^{h^{*1}_-(qw)} = \varphi^{(2)}(w)\,{}_{(2)}\langle B|e^{h^{*1}_+(q/w)}, \tag{6.37}$$

$$ {}_{(2)}\langle B|e^{-c^j_-(qw)} = c^{(2)}_j(w)\,{}_{(2)}\langle B|e^{-c^j_+(q/w)} (j = 1, 2, \cdots, N + 1). \tag{6.38}$$

Here $\varphi^{(2)}(w)$ are given in (2.22) and (2.23). We have set $c^{(2)}_j(w) = 1$ ($j = 1, 2, \ldots, N + 1$). We have set $g^{(2)}_i(w)$ ($i = 1, 2, \ldots, M + N + 1$) by

$$ g^{(2)}_i(w) = \begin{cases} g^{(1)}_i(w) & (1 \le i \ne L \le M + N + 1), \\ \dfrac{1}{(1 - rq^{\beta_1} w)} g^{(1)}_L(w) & (i = L), \end{cases} \tag{6.39}$$

where $g^{(1)}_i(w)$ is given by (6.16). The parameter β_1 is given as follows:

Condition 2.1: For $1 \le L \le M + 1$ we have set

$$ \beta_1 = -L. \tag{6.40}$$

Condition 2.2: For $M + 2 \le L \le M + N + 2$ we have set

$$ \beta_1 = -2M - 2 + L. \tag{6.41}$$

Proposition 6.6: The following relation holds,

$$ \sum_{\epsilon=\pm} \int_C \frac{dw_1}{w_1} \frac{\epsilon q^\epsilon (1 - q^\alpha r w_1)(1 - q w_1 w_2) e^{-c^j_+(q^{1+\epsilon} w_1) - c^j_+(q^{1-\epsilon}/w_1)}}{(1 - q^\epsilon w_1 w_2)(1 - q^\epsilon w_1/w_2)}$$
$$ = q(1 - q^{\alpha+1} r w_2) \int_C \frac{dw_1}{w_1} \frac{(-1 + w_1^2) e^{-c^j_+(w_1) - c^j_+(q^2/w_1)}}{(1 - w_1 w_2/q)(1 - w_1/q w_2)}, \tag{6.42}$$

where the integration contour C encircles $w_1 = 0, q w_2^{\pm 1}$ but not $w_1 = q^{-1} w_2^{\pm 1}$.

The integral $\int_C \frac{dw_1}{w_1} \frac{(-1+w_1^2) e^{-c^j_+(w_1) - c^j_+(q^2/w_1)}}{(1-w_1 w_2/q)(1-w_1/q w_2)}$ in RHS is invariant under $w_2 \to w_2^{-1}$.

Proof for boundary condition 2.1: We show the main theorem for the boundary condition ${}_{(2)}\langle B|$ and $1 \le L \le M + 1$. Here we show the relation (6.35).

• The case for $L + 1 \le j \le M + 1$: ${}_{(2)}\langle B|\phi^*_j(z)$.

Using the bosonization (4.72), the relations (6.36), (6.37), and the normal ordering in Appendix B, we have LHS − RHS of (6.35) as follows:

$$ z^{\frac{M}{M-N}} (1 - rz) \varphi^{(2)}(1/z)\,{}_{(2)}\langle B|\phi^*_j(z) - z^{-\frac{M}{M-N}} (1 - r/z) \varphi^{(2)}(z)\,{}_{(2)}\langle B|\phi^*_j(1/z)$$
$$ = q^{-\frac{M}{M-N}} (q - q^{-1})^{j-1} \varphi^{(2)}(z) \varphi^{(2)}(1/z) (z - z^{-1}) e^{\frac{\pi\sqrt{-1}(j-1)}{M-N}}$$
$$ \times \prod_{k=1}^{j-1} \int_{\widetilde{C}_j} \frac{dw_k}{2\pi\sqrt{-1} w_k} \frac{\prod_{k=1}^{j-2} (1 - q/w_k w_{k+1})}{\prod_{k=0}^{j-2} D(w_k, w_{k+1})} \frac{(w_1^{-1} - w_1)(1 - rq^{-1} w_1)}{(1 - rq^{-L} w_L)} \prod_{k=2}^{j-1} (1 - w_k^2)$$
$$ \times {}_{(2)}\langle B|e^{Q_{h^*_1 - h_1 \cdots - h_{j-1}}} e^{h^{*1}_+(qz) + h^{*1}_+(q/z) - \sum_{k=1}^{j-1} (h^k_+(q w_k) + h^k_+(q/w_k))}. \tag{6.43}$$

Here the integration contour $\widetilde{C}_j$ encircles $w_k = 0, q w_{k-1}^{\pm 1}$ but not $w_k = q^{-1} w_{k-1}^{\pm 1}$ for $1 \le k \le j-1$. Taking into account of symmetrization $\oint_{|w|=1} \frac{dw}{w} f(w) = \frac{1}{2} \oint_{|w|=1} \frac{dw}{w} (f(w) + f(1/w))$ and relation $(1 - q/w_1 w_2)(1 - r q^{-\alpha} w_1) + (w_1 \leftrightarrow w_1^{-1}) = (w_1^{-1} - w_1)(-q/w_2)(1 - r q^{-\alpha-1} w_2)$, we symmetrize the variables $w_1, w_2, \cdots, w_{L-1}$, iteratively. Using relation $(1 - q/w_L w_{L+1}) - (w_L \leftrightarrow w_L^{-1}) = (-q/w_{L+1})(w_L^{-1} - w_L)$, we symmetrize the variables $w_L, w_{L+1}, \cdots, w_{j-1}$, iteratively, we have

$$
\begin{aligned}
& q^{-\frac{M}{M-N}} (q - q^{-1})^{j-1} (-q/2)^{j-1} \varphi^{(2)}(z) \varphi^{(2)}(1/z)(z - z^{-1}) e^{\frac{\pi\sqrt{-1}(j-1)}{M-N}} \\
& \times \prod_{k=1}^{j-1} \int_{\widetilde{C}_j} \frac{dw_k}{2\pi\sqrt{-1} w_k} \frac{\prod_{k=1}^{j-2} (w_k^{-1} - w_k)^2}{\prod_{k=0}^{j-2} D(w_k, w_{k+1})} (w_{j-1}^{-1} - w_{j-1}) \\
& \times {}_{(2)}\langle B | e^{Q_{h_1^* - h_1 \cdots - h_{j-1}}} e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^{j-1} (h_+^k(q w_k) + h_+^k(q/w_k))} = 0.
\end{aligned} \tag{6.44}
$$

Here we have used $\oint_{|w|=1} \frac{dw}{w} (w^{-1} - w) f(w) = 0$, where $f(w) = f(1/w)$.

• The case for $j = L + 1 = M + 2$: ${}_{(2)}\langle B | \phi_{M+2}^*(z)$.

Using the bosonization (4.73), relations (6.36), (6.37), (6.38), and the normal orderings in Appendix B, we have LHS − RHS of (6.35) as follows:

$$
\begin{aligned}
& z^{\frac{M}{M-N}} \varphi^{(2)}(1/z)(1 - rz) {}_{(2)}\langle B | \phi_{M+2}^*(z) - z^{-\frac{M}{M-N}} \varphi^{(2)}(z)(1 - r/z) {}_{(2)}\langle B | \phi_{M+2}^*(1/z) \\
& = q^{-\frac{M}{M-N}+1} (q - q^{-1})^M \varphi^{(2)}(z) \varphi^{(2)}(1/z)(z - z^{-1}) e^{\frac{\pi\sqrt{-1}}{M-N} M} \\
& \times \sum_{\epsilon=\pm} \epsilon \prod_{k=1}^{M+1} \int_{\widetilde{C}_{M+2}} \frac{dw_k}{2\pi\sqrt{-1} w_k} \frac{\prod_{k=1}^{M} (1 - q/w_k w_{k+1})}{\prod_{k=0}^{M} D(w_k, w_{k+1})} \frac{(w_1^{-1} - w_1)(1 - r q^{-1} w_1)(1 + w_{M+1}^2)}{(1 - r q^{-L} w_L)} \\
& \times \prod_{k=2}^{M} (1 - w_k^2) \times {}_{(2)}\langle B | e^{Q_{h_1^* - h_1 \cdots - h_L - c_1}} \\
& \times e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^{L} (h_+^k(q w_k) + h_+^k(q/w_k)) - c_+^1(q^{1+\epsilon} w_{M+1}) - c_+^1(q^{1-\epsilon}/w_{M+1})}.
\end{aligned} \tag{6.45}
$$

Here the integration contour $\widetilde{C}_{M+2}$ encircles $w_k = 0, q w_{k-1}^{\pm 1}$ but not $w_k = q^{-1} w_{k-1}^{\pm 1}$ for $1 \le k \le M + 1$. Using relation $(1 - r q^{\alpha} w_1)(1 - q/w_1 w_2) - (w_1 \leftrightarrow w_1^{-1}) = (w_1 - w_1^{-1})(1 - r q^{\alpha-1} w_2)$, we symmetrize the variables $w_1, w_2, \cdots, w_L = w_{M+1}$, iteratively, we have

$$
\begin{aligned}
& q^{-\frac{M}{M-N}+1} (q - q^{-1})^M (-q/2)^M \varphi^{(2)}(z) \varphi^{(2)}(1/z)(z - z^{-1}) e^{\frac{\pi\sqrt{-1}}{M-N} M} \\
& \times \int_{\widetilde{C}_{M+2}} \frac{dw_k}{2\pi\sqrt{-1} w_k} \frac{\prod_{k=1}^{M} (w_k^{-1} - w_k)^2}{\prod_{k=0}^{M} D(w_k, w_{k+1})} (w_{M+1}^{-1} + w_{M+1}) \times {}_{(2)}\langle B | e^{Q_{h_1^* - h_1 \cdots - h_L - c_1}} \\
& \times e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^{L} (h_+^k(q w_k) + h_+^k(q/w_k))} \left(e^{-c_+^1(q^2 w_{M+1}) - c_+^1(1/w_{M+1})} - e^{-c_+^1(w_{M+1}) - c_+^1(q^2/w_{M+1})} \right) = 0.
\end{aligned} \tag{6.46}
$$

Here we have used $\oint_{|w|=1} \frac{dw}{w} (e^{-c_+^1(q^2 w) - c_+^1(1/w)} - e^{-c_+^1(w) - c_+^1(q^2/w)}) f(w) = 0$, where $f(w) = f(1/w)$.

• The case for $M + 3 \le j \le M + N + 2$: ${}_{(2)}\langle B | \phi_j^*(z)$.

Using the bosonization (4.74), the relations (6.36), (6.37), (6.38), and the normal orderings in Appendix B, we have LHS − RHS of (6.12) for $1 \le i \le N+1$,

$$
\begin{aligned}
& z^{\frac{M}{M-N}}(1-rz)\varphi^{(2)}(1/z)\,{}_{(2)}\langle B|\phi^*_{M+1+i}(z) - z^{-\frac{M}{M-N}}(1-r/z)\varphi^{(2)}(z)\,{}_{(2)}\langle B|\phi^*_{M+1+i}(1/z) \\
&= q^{-\frac{M}{M-N}+1}(q-q^{-1})^M\varphi^{(2)}(z)\varphi^{(2)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}M}{M-N}} \\
&\times \sum_{\epsilon_1,\cdots,\epsilon_i=\pm}\epsilon_i\prod_{k=1}^{M+i}\int_{\widetilde{C}_{M+1+i}}\frac{dw_k}{2\pi\sqrt{-1}w_k}\frac{\prod_{k=1}^{M}(1-q/w_kw_{k+1})}{\prod_{k=0}^{M}D(w_k,w_{k+1})}\frac{(w_1^{-1}-w_1)(1-rq^{-1}w_1)(1+w_{M+1}^2)}{(1-rq^{-L}w_L)} \\
&\times \prod_{k=2}^{M}(1-w_k^2)\prod_{k=1}^{i-1}\frac{\epsilon_kq^{\epsilon_k}(1-qw_{M+k}w_{M+k+1})}{(1-q^{\epsilon_k}w_{M+k}w_{M+k+1})(1-q^{\epsilon_k}w_{M+k}/w_{M+k+1})} \\
&\times {}_{(2)}\langle B|e^{Q_{h_1^*-h_1\cdots-h_{M+i}-c_i}}e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+i}(h_+^k(qw_k)+h_+^k(q/w_k))} \\
&\times e^{\sum_{k=1}^{i-1}(c_+^k(qw_{M+k+1})+c_+^k(q/w_{M+k+1}))-\sum_{k=1}^{i}(c_+^k(q^{1+\epsilon_k}w_{M+k})+c_+^k(q^{1-\epsilon_k}/w_{M+k}))}.
\end{aligned}
\tag{6.47}
$$

Here the integration contour $\widetilde{C}_{M+1+i}$ encircles $w_k = 0, qw_{k-1}^{\pm1}$ but not $w_k = q^{-1}w_{k-1}^{\pm1}$ for $1 \le k \le M+i$. Using relation $(1-rq^{\alpha}w_1)(1-q/w_1w_2)-(w_1 \leftrightarrow w_1^{-1}) = (1-rq^{\alpha-1}w_2)(w_1-w_1^{-1})$, we symmetrize the variables $w_1, w_2, \cdots, w_{L-1}$, iteratively. Using relation $(1-q/w_Lw_{L+1}) - (w_L \leftrightarrow w_L^{-1}) = (-q/w_{L+1})(w_L^{-1}-w_L)$ we symmetrize the variables $w_L, \cdots, w_M$, iteratively. Then we have

$$
\begin{aligned}
& q^{-\frac{M}{M-N}+1}(q-q^{-1})^M(-q/2)^M\varphi^{(2)}(z)\varphi^{(2)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}M}{M-N}} \\
&\times \sum_{\epsilon_1,\cdots,\epsilon_i=\pm}\epsilon_i\prod_{k=1}^{M+i}\int_{\widetilde{C}_{M+1+i}}\frac{dw_k}{2\pi\sqrt{-1}w_k}\frac{\prod_{k=1}^{M}(w_k^{-1}-w_k)^2}{\prod_{k=0}^{M}D(w_k,w_{k+1})}(w_{M+1}^{-1}+w_{M+1}) \\
&\times \prod_{k=1}^{i-1}\frac{\epsilon_kq^{\epsilon_k}(1-qw_{M+k}w_{M+k+1})}{(1-q^{\epsilon_k}w_{M+k}w_{M+k+1})(1-q^{\epsilon_k}w_{M+k}/w_{M+k+1})}\,{}_{(2)}\langle B|e^{Q_{h_1^*-h_1\cdots-h_{M+i}-c_i}}e^{h_+^{*1}(qz)+h_+^{*1}(q/z)} \\
&\times e^{-\sum_{k=1}^{M+i}(h_+^k(qw_k)+h_+^k(q/w_k))+\sum_{k=1}^{i-1}(c_+^k(qw_{M+k+1})+c_+^k(q/w_{M+k+1}))-\sum_{k=1}^{i}(c_+^k(q^{1+\epsilon_k}w_{M+k})+c_+^k(q^{1-\epsilon_k}/w_{M+k}))}.
\end{aligned}
\tag{6.48}
$$

Using relation (6.22) for variables $w_{M+1}, \cdots, w_{i-1}$, iteratively, we have

$$
\begin{aligned}
& q^{-\frac{M}{M-N}+1}(q-q^{-1})^M(-q/2)^Mq^{-i-M-1}\varphi^{(2)}(z)\varphi^{(2)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}M}{M-N}} \\
&\times \prod_{k=1}^{M+i}\int_{\widetilde{C}_{M+1+i}}\frac{dw_k}{2\pi\sqrt{-1}w_k}\frac{\prod_{k=1}^{M}(w_k^{-1}-w_k)^2}{\prod_{k=0}^{M}D(w_k,w_{k+1})}(w_{M+1}^{-1}+w_{M+1})\times {}_{(2)}\langle B|e^{Q_{h_1^*-h_1\cdots-h_{M+i}-c_i}} \\
&\times \prod_{k=1}^{i-1}\frac{(-1+w_{M+k}^2)e^{-c_+^k(w_{M+k})-c_+^k(q^2/w_{M+k})}}{(1-w_{M+k}w_{M+k+1}/q)(1-w_{M+k}/qw_{M+k+1})} \\
&\times e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+i}(h_+^k(qw_k)+h_+^k(q/w_k))+\sum_{k=1}^{i-1}(c_+^k(qw_{M+k+1})+c_+^k(q/w_{M+k+1}))} \\
&\times \left(e^{-c_+^i(q^2w_{M+i})-c_+^i(1/w_{M+i})} - e^{-c_+^i(w_{M+i})-c_+^i(q^2/w_{M+i})}\right) = 0.
\end{aligned}
\tag{6.49}
$$

Here we have used

$$\oint_{|w|=1} \frac{dw}{w} e^{c_+^{i-1}(qw)+c_+^{i-1}(q/w)} \left(e^{-c_+^i(q^2w)-c_+^i(1/w)} - e^{-c_+^i(w)-c_+^i(q^2/w)} \right) f(w) = 0,$$

where $f(w) = f(1/w)$.

Now we have shown (6.35) for every $j = L + 1, \ldots, M + N + 2$. □

Proof for boundary condition 2.2: We show the main theorem for the boundary condition ${}_{(2)}\langle B|$ and $M + 2 \le L \le M + N + 1$. Here we show (6.35) that is new for the boundary condition ${}_{(2)}\langle B|$. We use the integral relations (6.22) and (6.42).

• The case for $L + 1 \le j \le M + N + 2$: ${}_{(2)}\langle B|\phi_j^*(z)$.
Using the bosonization (4.74), the relations (6.36), (6.37), (6.38), and the normal orderings in Appendix B, we have LHS − RHS of (6.12) for $1 \le i \le N + 1$.

$$\begin{aligned}
& z^{\frac{M}{M-N}}(1-rz)\varphi^{(2)}(1/z){}_{(2)}\langle B|\phi_{M+1+i}^*(z) - z^{-\frac{M}{M-N}}(1-r/z)\varphi^{(2)}(z){}_{(2)}\langle B|\phi_{M+1+i}^*(1/z) \\
&= q^{-\frac{M}{M-N}+1}(q-q^{-1})^M \varphi^{(2)}(z)\varphi^{(2)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}M}{M-N}} \\
&\times \sum_{\epsilon_1,\cdots,\epsilon_i=\pm} \epsilon_i \prod_{k=1}^{M+i} \int_{\widetilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^{M}(1-q/w_kw_{k+1})}{\prod_{k=0}^{M} D(w_k, w_{k+1})} \frac{(w_1^{-1}-w_1)(1-rq^{-1}w_1)(1+w_{M+1}^2)}{(1-rq^{-2M-2+L}w_L)} \\
&\times \prod_{k=2}^{M}(1-w_k^2) \prod_{k=1}^{i-1} \frac{\epsilon_k q^{\epsilon_k}(1-qw_{M+k}w_{M+k+1})}{(1-q^{\epsilon_k}w_{M+k}w_{M+k+1})(1-q^{\epsilon_k}w_{M+k}/w_{M+k+1})} \\
&\times {}_{(2)}\langle B|e^{Q_{h_1^*-h_1\cdots-h_{M+i}-c_i}} e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+i}(h_+^k(qw_k)+h_+^k(q/w_k))} \\
&\times e^{\sum_{k=1}^{i-1}(c_+^k(qw_{M+k+1})+c_+^k(q/w_{M+k+1}))-\sum_{k=1}^{i}(c_+^k(q^{1+\epsilon_k}w_{M+k})+c_+^k(q^{1-\epsilon_k}/w_{M+k}))}.
\end{aligned} \tag{6.50}$$

Here the integration contour $\widetilde{C}_{M+1+i}$ encircles $w_k = 0, qw_{k-1}^{\pm 1}$ but not $w_k = q^{-1}w_{k-1}^{\pm 1}$ for $1 \le k \le M + i$. Using relation $(1-q/w_1w_2)(1-rq^{-\alpha}w_1) - (w_1 \leftrightarrow w_1^{-1}) = (w_1 - w_1^{-1})(1-rq^{-\alpha-1}w_2)$, we symmetrize the variables $w_1, w_2, \cdots, w_M$ iteratively. We have

$$\begin{aligned}
& q^{-\frac{M}{M-N}+1}(q-q^{-1})^M(-q/2)^M\varphi^{(2)}(z)\varphi^{(2)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}M}{M-N}} \\
&\times \sum_{\epsilon_1,\cdots,\epsilon_i=\pm} \epsilon_i \prod_{k=1}^{M+i} \int_{\widetilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^{M}(w_k^{-1}-w_k)^2}{\prod_{k=0}^{M} D(w_k, w_{k+1})} \frac{(w_{M+1}^{-1}+w_{M+1})(1-rq^{-M-1}w_{M+1})}{(1-rq^{-2M-2+L}w_L)} \\
&\times \prod_{k=1}^{i-1} \frac{\epsilon_k q^{\epsilon_k}(1-qw_{M+k}w_{M+k+1})}{(1-q^{\epsilon_k}w_{M+k}w_{M+k+1})(1-q^{\epsilon_k}w_{M+k}/w_{M+k+1})} \\
&\times {}_{(2)}\langle B|e^{Q_{h_1^*-h_1\cdots-h_{M+i}-c_i}} e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+i}(h_+^k(qw_k)+h_+^k(q/w_k))} \\
&\times e^{\sum_{k=1}^{i-1}(c_+^k(qw_{M+k+1})+c_+^k(q/w_{M+k+1}))-\sum_{k=1}^{i}(c_+^k(q^{1+\epsilon_k}w_{M+k})+c_+^k(q^{1-\epsilon_k}/w_{M+k}))}.
\end{aligned} \tag{6.51}$$

We use the relation (6.42) for the variables $w_{M+1}, \cdots, w_{L-1}$ iteratively. We use the relation (6.22) for the variables $w_L, \cdots, w_{M+i-1}$ iteratively. Then we have

$$
\begin{aligned}
&q^{-\frac{M}{M-N}+1}(q-q^{-1})^M(-q/2)^M q^{2L-2M-i-1}\varphi^{(2)}(z)\varphi^{(2)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}M}{M-N}}\\
&\times\prod_{k=1}^{M+i}\int_{\widetilde{C}_{M+1+i}}\frac{dw_k}{2\pi\sqrt{-1}w_k}\frac{\prod_{k=1}^{M}(w_k^{-1}-w_k)^2}{\prod_{k=0}^{M}D(w_k,w_{k+1})}(w_{M+1}^{-1}+w_{M+1})\\
&\times {}_{(2)}\langle B|e^{Q_{h_1^*-h_1\cdots-h_{M+i}-c_i}}\prod_{k=1}^{i-1}\frac{(-1+w_{M+k}^2)e^{-c_+^k(w_{M+k})-c_+^k(q^2/w_{M+k})}}{(2-w_{M+k}/qw_{M+k+1})(1-w_{M+k}w_{M+k+1}/q)}\\
&\times e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+i}(h_+^k(qw_k)+h_+^k(q/w_k))+\sum_{k=1}^{i-1}(c_+^k(qw_{M+k+1})+c_+^k(q/w_{M+k+1}))}\\
&\times\left(e^{-c_+^i(q^2w_{M+i})-c_+^i(1/w_{M+i})}-e^{-c_+^i(w_{M+i})-c_+^i(q^2/w_{M+i})}\right)=0.
\end{aligned}
\tag{6.52}
$$

Here we have used

$$
\oint_{|w|=1}\frac{dw}{w}e^{c_+^{i-1}(qw)+c_+^{i-1}(q/w)}\left(e^{-c_+^i(q^2w)-c_+^i(1/w)}-e^{-c_+^i(w)-c_+^i(q^2/w)}\right)f(w)=0,
$$

where $f(w)=f(1/w)$. □

Now we have shown (6.35) for every $j=L+1,\ldots,M+N+2$.

C. Boundary condition 3

In this section we study (6.1) for the boundary condition ${}_{(3)}\langle B|$. Very explicitly we study

$$
z^{\frac{M}{M-N}}\varphi^{(3)}(z^{-1})_{(3)}\langle B|\phi_j^*(z)=z^{-\frac{M}{M-N}}\varphi^{(3)}(z)_{(3)}\langle B|\phi_j^*(z^{-1})(1\le j\le L),
\tag{6.53}
$$

$$
z^{\frac{M}{M-N}}(1-rz)\varphi^{(3)}(z^{-1})_{(3)}\langle B|\phi_j^*(z)=z^{-\frac{M}{M-N}}(1-r/z)\varphi^{(3)}(z)_{(3)}\langle B|\phi_j^*(z^{-1})(L+1\le j\le L+K),
\tag{6.54}
$$

$$
z^{\frac{M}{M-N}+1}\varphi^{(3)}(z^{-1})_{(3)}\langle B|\phi_j^*(z)=z^{-\frac{M}{M-N}-1}\varphi^{(3)}(z)_{(3)}\langle B|\phi_j^*(z^{-1})(L+K+1\le j\le M+N+2).
\tag{6.55}
$$

The structures of (6.53) and (6.54) are the same as those of (6.34) and (6.35) for the boundary condition ${}_{(2)}\langle B|$. In this section we focus our attention on the relation (6.55) that is new for the boundary condition ${}_{(3)}\langle B|$. We give proofs for following three conditions:

$$
\begin{aligned}
&\text{Condition 3.1}: L+K\le M+1,\\
&\text{Condition 3.2}: L\le M+1\le L+K-1,\\
&\text{Condition 3.3}: M+2\le L.
\end{aligned}
$$

Proposition 6.7: The actions of $h_+^i(w)$, $h_-^{*1}(w)$, and $c_-^j(w)$ on the boundary state ${}_{(3)}\langle B|$ are given as follows:

$$
{}_{(3)}\langle B|e^{-h_-^i(qw)}=g_i^{(3)}(w)_{(3)}\langle B|e^{-h_+^i(q/w)}(i=1,2,\cdots,M+N+1),
\tag{6.56}
$$

$$
{}_{(3)}\langle B|e^{h_-^{*1}(qw)}=\varphi^{(3)}(w)_{(3)}\langle B|e^{h_+^{*1}(q/w)},
\tag{6.57}
$$

$$
{}_{(3)}\langle B|e^{-c_-^j(qw)}=c_j^{(3)}(w)_{(3)}\langle B|e^{-c_+^j(q/w)}(j=1,2,\cdots,N+1).
\tag{6.58}
$$

Here $\varphi^{(3)}(w)$ are given in (2.25), (2.26), and (2.27). We have set $c_j^{(3)}(w) = 1$ $(j = 1, 2, \ldots, N + 1)$. We have set $g_i^{(3)}(w)$ $(i = 1, 2, \ldots, M + N + 1)$ by

$$g_i^{(3)}(w) = \begin{cases} g_i^{(1)}(w) & (1 \le i \ne L, L + K \le M + N + 1), \\ \dfrac{1}{(1 - rq^{\beta_1} w)} g_L^{(1)}(w) & (i = L), \\ \dfrac{1}{(1 - q^{\beta_2} w/r)} g_{L+K}^{(1)}(w) & (i = L + K), \end{cases} \tag{6.59}$$

where $g_i^{(1)}(w)$ is given by (6.16). The parameters (β_1, β_2) are given as follows:

Condition 3.1: For $L + K \le M + 1$ we have set

$$(\beta_1, \beta_2) = (-L, L - K). \tag{6.60}$$

Condition 3.2: For $L \le M + 1 \le L + K - 1$ we have set

$$(\beta_1, \beta_2) = (-L, 3L + K - 2M - 2), \tag{6.61}$$

Condition 3.3: For $M + 2 \le L$ we have set

$$(\beta_1, \beta_2) = (L - 2M - 2, 2M + K - L + 2). \tag{6.62}$$

Proposition 6.8: We have the following two relations:

$$\sum_{\epsilon=\pm} \int_C \frac{dw_1}{w_1} \frac{\epsilon q^\epsilon w_1 (1 - q w_1 w_2) e^{-c_+^j(q^{1+\epsilon} w_1) - c_+^j(q^{1-\epsilon}/w_1)}}{(1 - rq^\alpha w_1)(1 - q^\epsilon w_1 w_2)(1 - q^\epsilon w_1/w_2)} \tag{6.63}$$

$$= -rq^{\alpha-1}(1 - q^{-\alpha+1} w_2/r) \int_C \frac{dw_1}{w_1} \frac{(-1 + w_1^2) e^{-c_+^j(w_1) - c_+^j(q^2/w_1)}}{(1 - w_1 w_2/q)(1 - w_1/q w_2)(1 - rq^\alpha w_1)(1 - rq^\alpha/w_1)},$$

$$\sum_{\epsilon=\pm} \int_C \frac{dw_1}{w_1} \frac{\epsilon q^\epsilon w_1 (1 - q w_1 w_2) e^{-c_+^j(q^{1+\epsilon} w_1) - c_+^j(q^{1-\epsilon}/w_1)}}{(1 - q^\epsilon w_1 w_2)(1 - q^\epsilon w_1/w_2)}$$

$$= w_2 \int_C \frac{dw_1}{w_1} \frac{(-1 + w_1^2) e^{-c_+^j(w_1) - c_+^j(q^2/w_1)}}{(1 - w_1 w_2/q)(1 - w_1/q w_2)}. \tag{6.64}$$

Here the integration contour C encircles $w_1 = 0, q w_2^{\pm 1}$ but not $w_1 = q^{-1} w_2^{\pm 1}$. Here the integrals $\int_C \frac{dw_1}{w_1} \frac{(-1+w_1^2) e^{-c_+^j(w_1) - c_+^j(q^2/w_1)}}{(1-w_1 w_2/q)(1-w_1/q w_2)(1-rq^\alpha w_1)(1-rq^\alpha/w_1)}$ and $\int_C \frac{dw_1}{w_1} \frac{(-1+w_1^2) e^{-c_+^j(w_1) - c_+^j(q^2/w_1)}}{(1-w_1 w_2/q)(1-w_1/q w_2)}$ in RHS are invariant under $w_2 \to w_2^{-1}$.

Proof for boundary condition 3.1: We show the main theorem for the boundary condition ${}_{(3)}\langle B|$ and $L + K \le M + 1$. Here we show the relation (6.55).

• The case for $1 \le L + K + 1 \le j \le M + 1$: ${}_{(3)}\langle B|\phi_j^*(z)$.
Using the bosonization (4.72), the relations (6.56), (6.57), (6.58), and the normal orderings in Appendix B, we have LHS − RHS of (6.55) as follows:

$$z^{\frac{M}{M-N}+1} \varphi^{(3)}(1/z) {}_{(3)}\langle B|\phi_{M+1+i}^*(z) - z^{-\frac{M}{M-N}-1} \varphi^{(3)}(z) {}_{(3)}\langle B|\phi_{M+1+i}(1/z)$$

$$= q^{-\frac{M}{M-N}-1} \varphi^{(3)}(z) \varphi^{(3)}(1/z)(z - z^{-1}) e^{\frac{\pi\sqrt{-1}}{M-N}(j-1)}$$

$$\times \prod_{k=1}^{j-1} \int_{\widetilde{C}_j} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^{j-2}(1-q/w_k w_{k+1})}{\prod_{k=0}^{j-2} D(w_k, w_{k+1})} \frac{\prod_{k=1}^{j-1}(1-w_k^2)}{(1-q^{-L}rw_L)(q^{L-K}w_{L+K}/r)}$$

$$\times {}_{(3)}\langle B| e^{Q_{h_1^*-h_1\cdots-h_{j-1}}} e^{h_+^*(qz)+h_+^*(q/z)-\sum_{k=1}^{j-1}(h_+^k(qw_k)+h_+^k(q/w_k))}. \tag{6.65}$$

Here the integration contour $\widetilde{C}_j$ encircles $w_k = 0, qw_{k-1}^{\pm 1}$ but not $w_k = q^{-1}w_{k-1}^{\pm 1}$ for $1 \le k \le j$. Taking into account of symmetrization $\oint \frac{dw}{w} f(w) = \frac{1}{2}\oint (f(w) + f(1/w))$ and relation $w_1(1 - q/w_1w_2) - (w_1 \leftrightarrow w_1^{-1}) = (w_1 - w_1^{-1})$, we symmetrize the variables $w_1, w_2, \cdots, w_{L-1}$ iteratively. Using relation $\frac{w_L(1-q/w_Lw_{L+1})}{(1-rq^{-L}w_L)} - (w_L \leftrightarrow w_L^{-1}) = \frac{(w_L-w_L^{-1})(1-rq^{-L+1}/w_{L+1})}{(1-rq^{-L}w_L)(1-rq^{-L}/w_L)}$, we symmetrize the variable w_L. Using relation $(1 - rq^{\alpha}/w_{L+1})w_{L+1}(1 - q/w_{L+1}w_{L+2}) - (w_{L+1} \leftrightarrow w_{L+1}^{-1}) = (w_{L+1} - w_{L+1}^{-1})(1 - rq^{\alpha+1}/w_{L+2})$, we symmetrize the variable $w_{L+1}, \cdots, w_{L+K-1}$ iteratively. Using relation $(1 - q/w_{L+K+1}w_{L+K+2}) - (w_{L+K+1} \leftrightarrow w_{L+K+1}^{-1}) = (-q/w_{L+K+2})(w_{L+K+1}^{-1} - w_{L+K+1})$, we symmetrize the variables $w_{L+K+1}, \cdots, w_{j-2}$ iteratively. Then we have

$$q^{-\frac{M}{M-N}+j-2L-3}(-1/2)^{j-2}\varphi^{(3)}(z)\varphi^{(3)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}}{M-N}(j-1)}$$

$$\times \prod_{k=1}^{j-1} \int_{\widetilde{C}_j} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^{j-2}(w_k^{-1}-w_k)^2}{\prod_{k=0}^{j-2} D(w_k, w_{k+1})} \frac{(w_{j-1}^{-1}-w_{j-1})}{(1-q^{-L}rw_L)(q^{-L}r/w_L)}$$

$$\times {}_{(3)}\langle B| e^{Q_{h_1^*-h_1\cdots-h_{j-1}}} e^{h_+^*(qz)+h_+^*(q/z)-\sum_{k=1}^{j-1}(h_+^k(qw_k)+h_+^k(q/w_k))} = 0. \tag{6.66}$$

Here we have used $\oint_{|w|=1} \frac{dw}{w}(w - w^{-1})f(w) = 0$, where $f(w) = f(1/w)$.

- The case for $1 \le i \le N + 1$: ${}_{(3)}\langle B|\phi_{M+1+i}^*(z)$.

Using the bosonization (4.74), the relations (6.56), (6.57), (6.58), and normal orderings in Appendix B, we have LHS − RHS of (6.55) as follows:

$$z^{\frac{M}{M-N}+1}\varphi^{(3)}(1/z){}_{(3)}\langle B|\phi_{M+1+i}^*(z) - z^{-\frac{M}{M-N}-1}\varphi^{(3)}(z){}_{(3)}\langle B|\phi_{M+1+i}(1/z)$$

$$= q^{-\frac{M}{M-N}-1}(q-q^{-1})^M\varphi^{(3)}(z)\varphi^{(3)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}}{M-N}M}$$

$$\times \sum_{\epsilon_1,\cdots,\epsilon_i=\pm} \epsilon_i \prod_{k=1}^{M+i} \int_{\widetilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^{M}(1-q/w_kw_{k+1})}{\prod_{k=0}^{M} D(w_k, w_{k+1})} \prod_{k=1}^{M}(1-w_k^2)$$

$$\times \frac{(1+w_{M+1}^2)}{(1-rq^{-L}w_L)(1-q^{L-K}w_{L+K}/r)} \prod_{k=1}^{i-1} \frac{\epsilon_k q^{\epsilon_k}(1-qw_{M+k}w_{M+k+1})}{(1-q^{\epsilon_k}w_{M+k}w_{M+k+1})(1-q^{\epsilon_k}w_{M+k}/w_{M+k+1})}$$

$$\times {}_{(3)}\langle B| e^{Q_{h_1^*-h_1\cdots-h_{M+i}-c_i}} e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+i}(h_+^k(qw_k)+h_+^k(q/w_k))}$$

$$\times e^{\sum_{k=1}^{i-1}(c_+^k(qw_{M+k})+c_+^k(q/w_{M+k}))-\sum_{k=1}^{i}(c_+^k(q^{1+\epsilon_k}w_{M+k})+c_+^k(q^{1-\epsilon_k}/w_{M+k}))}. \tag{6.67}$$

Here the integration contour $\widetilde{C}_{M+1+i}$ encircles $w_k = 0, qw_{k-1}^{\pm 1}$ but not $w_k = q^{-1}w_{k-1}^{\pm 1}$ for $1 \le k \le M + i$. Using relation $w_1(1 - q/w_1w_2) - (w_1 \leftrightarrow w_1^{-1}) = (w_1 - w_1^{-1})$, we symmetrize the variables $w_1, w_2, \cdots, w_{L-1}$ iteratively. Using relation $\frac{w_L(1-q/w_Lw_{L+1})}{(1-rq^{-L}w_L)} - (w_L \leftrightarrow w_L^{-1}) = \frac{(w_L-w_L^{-1})(1-rq^{-L+1}/w_{L+1})}{(1-rq^{-L}w_L)(1-rq^{-L}/w_L)}$, we symmetrize the variable w_L. Using relation $(1 - rq^s/w_{L+1})w_{L+1}$

$(1-q/w_{L+1}w_{L+2})-(w_{L+1}\leftrightarrow w_{L+1}^{-1})=(w_{L+1}-w_{L+1}^{-1})(1-rq^{s+1}/w_{L+2})$, we symmetrize the variable $w_{L+1},\cdots,w_{L+K-1}$ iteratively. Using relation $(1-q/w_{L+K+1}w_{L+K+2})-(w_{L+K+1}\leftrightarrow w_{L+K+1}^{-1})=(-q/w_{L+K+2})(w_{L+K+1}^{-1}-w_{L+K+1})$, we symmetrize the variables $w_{L+K+1},\cdots,w_{M-1}$ iteratively. Then we have

$$
\begin{aligned}
&q^{-\frac{M}{M-N}-M-2L-2}(q-q^{-1})^M(-1/2)^M\varphi^{(3)}(z)\varphi^{(3)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}}{M-N}M}\\
&\times\sum_{\epsilon_1,\cdots,\epsilon_i=\pm}\epsilon_i\prod_{k=1}^{M+i}\int_{\widetilde{C}_{M+1+i}}\frac{dw_k}{2\pi\sqrt{-1}w_k}\frac{\prod_{k=1}^{M}(w_k^{-1}-w_k)^2}{\prod_{k=0}^{M}D(w_k,w_{k+1})}\frac{(w_{M+1}^{-1}+w_{M+1})}{(1-rq^{-L}w_L)(1-rq^{-L}/w_L)}\\
&\times\prod_{k=1}^{i-1}\frac{\epsilon_k q^{\epsilon_k}(1-qw_{M+k}w_{M+k+1})}{(1-q^{\epsilon_k}w_{M+k}w_{M+k+1})(1-q^{\epsilon_k}w_{M+k}/w_{M+k+1})}\\
&\times{}_{(3)}\langle B|e^{Q_{h_1^*-h_1\cdots-h_{M+i}-c_i}}e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+i}(h_+^k(qw_k)+h_+^k(q/w_k))}\\
&\times e^{\sum_{k=1}^{i-1}(c_+^k(qw_{M+k})+c_+^k(q/w_{M+k}))-\sum_{k=1}^{i}(c_+^k(q^{1+\epsilon_k}w_{M+k})+c_+^k(q^{1-\epsilon_k}/w_{M+k}))}.
\end{aligned}
\tag{6.68}
$$

Using the relation (6.22) for the variables $w_{M+1},\cdots,w_{M+i-1}$, iteratively, we have

$$
\begin{aligned}
&q^{-\frac{M}{M-N}+M-2L-i-1}(q-q^{-1})^M(-1/2)^{M-1}\varphi^{(3)}(z)\varphi^{(3)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}}{M-N}M}\\
&\times\prod_{k=1}^{M+i}\int_{\widetilde{C}_{M+1+i}}\frac{dw_k}{2\pi\sqrt{-1}w_k}\frac{\prod_{k=1}^{M}(w_k^{-1}-w_k)^2}{\prod_{k=0}^{M}D(w_k,w_{k+1})}\frac{(w_{M+1}^{-1}+w_{M+1})}{(1-rq^{-L}w_L)(1-q^{L-K}w_{L+K}/r)}\\
&\times{}_{(3)}\langle B|e^{Q_{h_1^*-h_1\cdots-h_{M+i}-c_i}}e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+i}(h_+^k(qw_k)+h_+^k(q/w_k))+\sum_{k=1}^{i-1}(c_+^k(qw_{M+k})+c_+^k(q/w_{M+k}))}\\
&\times\frac{\prod_{k=1}^{i-1}(-1+w_{M+k}^2)e^{-c_+^k(w_{M+k})-c_+^k(q^2/w_{M+k})}}{(1-q^{\epsilon_k}w_{M+k}w_{M+k+1})(1-q^{\epsilon_k}w_{M+k}/w_{M+k+1})}\\
&\times\left(e^{-c_+^i(q^2w_{M+i})-c_+^i(q/w_{M+i})}-e^{-c_+^i(w_{M+i})-c_+^i(q^2/w_{M+i})}\right)=0.
\end{aligned}
\tag{6.69}
$$

Here we have used $\oint_{|w|=1}\frac{dw}{w}e^{c_+^{i-1}(qw)+c_+^{i-1}(1/w)}(e^{-c_+^i(q^2w)-c_+^i(1/w)}-e^{-c_+^i(w)-c_+^i(q^2/w)})f(w)=0$, where $f(w)=f(1/w)$. □

Proof for boundary condition 3.2: We show the main theorem for the boundary condition ${}_{(3)}\langle B|$ and $L\le M+1\le L+K-1$. Here we show the relation (6.55).

• The case for $L+K-M\le i\le N+1$: ${}_{(3)}\langle B|\phi^*_{M+1+i}(z)$.

Using the bosonization (4.74), the relations (6.56), (6.57), (6.58), and normal orderings in Appendix B, we have LHS − RHS of (6.55) as follows:

$$
\begin{aligned}
&z^{\frac{M}{M-N}+1}\varphi^{(3)}(1/z){}_{(3)}\langle B|\phi^*_{M+1+i}(z)-z^{\frac{M}{M-N}-1}\varphi^{(3)}(z){}_{(3)}\langle B|\phi^*_{M+1+i}(1/z)\\
&=q^{-\frac{M}{M-N}-1}(q-q^{-1})^M\varphi^{(3)}(z)\varphi^{(3)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}}{M-N}M}
\end{aligned}
$$

$$\times \sum_{\epsilon_1,\cdots,\epsilon_i=\pm} \epsilon_i \prod_{k=1}^{M+i} \int_{\widetilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^{M}(1-q/w_k w_{k+1})}{\prod_{k=0}^{M} D(w_k, w_{k+1})} \prod_{k=1}^{M}(1-w_k^2)$$

$$\times \frac{(1+w_{M+1}^2)}{(1-rq^{-L}w_L)(1-q^{3L+K-2M+K-2}w_{L+K}/r)} \prod_{k=1}^{i-1} \frac{\epsilon_k q^{\epsilon_k}(1-qw_{M+k}w_{M+k+1})}{(1-q^{\epsilon_k}w_{M+k}w_{M+k+1})(1-q^{\epsilon_k}w_{M+k}/w_{M+k+1})}$$

$$\times {}_{(3)}\langle B| e^{Q_{h_1^*-h_1\cdots-h_{M+i}-c_i}} e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+i}(h_+^k(qw_k)+h_+^k(q/w_k))}$$

$$\times e^{\sum_{k=1}^{i-1}(c_+^k(qw_{M+k})+c_+^k(q/w_{M+k}))-\sum_{k=1}^{i}(c_+^k(q^{1+\epsilon_k}w_{M+k})+c_+^k(q^{1-\epsilon_k}/w_{M+k}))}. \tag{6.70}$$

Here the integration contour $\widetilde{C}_{M+1+i}$ encircles $w_k = 0, qw_{k-1}^{\pm 1}$ but not $w_k = q^{-1}w_{k-1}^{\pm 1}$ for $1 \le k \le M + i$. Using relation $w_1(1-q/w_1w_2) - (w_1 \leftrightarrow w_1^{-1}) = (w_1 - w_1^{-1})$, we symmetrize the variables $w_1, w_2, \cdots, w_{L-1}$, iteratively. Using relation $\frac{w_L(1-q/w_Lw_{L+1})}{(1-rq^{-L}w_L)} - (w_L \leftrightarrow w_L^{-1}) = \frac{(w_L^{-1}-w_L)(1-rq^{-L+1}/w_{L+1})}{(1-rq^{-L}w_L)(1-rq^{-L}/w_L)}$, we symmetrize the variables w_L. Using relation $w_{L+1}(1-rq^{-\alpha}/w_{L+1})(1-q/w_{L+1}w_{L+2}) - (w_{L+1} \leftrightarrow w_{L+1}^{-1}) = (w_{L+1} - w_{L+1}^{-1})(1-rq^{-\alpha+1}/w_{L+2})$, we symmetrize the variables $w_{L+1}, \cdots, w_M$, iteratively. Then we have

$$q^{-\frac{M}{M-N}-1}(-1/2)^M(q-q^{-1})^M\varphi^{(3)}(z)\varphi^{(3)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}}{M-N}M}$$

$$\times \sum_{\epsilon_1,\cdots,\epsilon_i=\pm} \epsilon_i \prod_{k=1}^{M+i} \int_{\widetilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^{M}(w_k^{-1}-w_k)^2}{\prod_{k=0}^{M} D(w_k, w_{k+1})} \frac{(1-rq^{M-2L+1}/w_{M+1})}{(1-rq^{-L}w_L)(1-rq^{-L}/w_L)}$$

$$\times \frac{(1+w_{M+1}^2)}{(1-q^{3L+K-2M-2}w_{L+K}/r)} \prod_{k=1}^{i-1} \frac{\epsilon_k q^{\epsilon_k}(1-qw_{M+k}w_{M+k+1})}{(1-q^{\epsilon_k}w_{M+k}w_{M+k+1})(1-q^{\epsilon_k}w_{M+k}/w_{M+k+1})}$$

$$\times {}_{(3)}\langle B| e^{Q_{h_1^*-h_1\cdots-h_{M+i}-c_i}} e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+i}(h_+^k(qw_k)+h_+^k(q/w_k))}$$

$$\times e^{\sum_{k=1}^{i-1}(c_+^k(qw_{M+k})+c_+^k(q/w_{M+k}))-\sum_{k=1}^{i}(c_+^k(q^{1+\epsilon_k}w_{M+k})+c_+^k(q^{1-\epsilon_k}/w_{M+k}))}. \tag{6.71}$$

Using the relation (6.42), we symmetrize the variables $w_{M+1}, \cdots, w_{L+K-1}$ iteratively. Using the relation (6.22), we symmetrize the variables $w_{L+K}, \cdots, w_{M+i}$ iteratively. Then we have

$$(-r)q^{-\frac{M}{M-N}-M+2K-i-2}(-1/2)^M(q-q^{-1})^M\varphi^{(3)}(z)\varphi^{(3)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}}{M-N}M}$$

$$\times \sum_{\epsilon_1,\cdots,\epsilon_i=\pm} \epsilon_i \prod_{k=1}^{M+i} \int_{\widetilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^{M}(w_k^{-1}-w_k)^2}{\prod_{k=0}^{M} D(w_k, w_{k+1})} \frac{(w_{M+1}^{-1}+w_{M+1})}{(1-rq^{-L}w_L)(1-rq^{-L}/w_L)}$$

$$\times {}_{(3)}\langle B| e^{Q_{h_1^*-h_1\cdots-h_{M+i}-c_i}} e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+i}(h_+^k(qw_k)+h_+^k(q/w_k))+\sum_{k=1}^{i-1}(c_+^k(qw_{M+k})+c_+^k(q/w_{M+k}))}$$

$$\times \prod_{k=1}^{i-1} \frac{(-1-qw_{M+k}^2)e^{-c_+^k(w_{M+k})-c_+^k(q^2/w_{M+k})}}{(1-w_{M+k}w_{M+k+1}/q)(1-w_{M+k}/qw_{M+k+1})}$$

$$\times \left(e^{-c_+^i(q^2w_{M+i})-c_+^i(1/w_{M+i})} - e^{-c_+^i(w_{M+i})-c_+^i(q^2/w_{M+i})}\right) = 0. \tag{6.72}$$

Here we have used

$$\oint_{|w|=1} \frac{dw}{w} e^{c_+^{i-1}(qw)+c_+^{i-1}(q/w)} \left(e^{-c_+^i(q^2w)-c_+^i(1/w)} - e^{-c_+^i(w)-c_+^i(q^2/w)} \right) f(w) = 0,$$

where $f(w) = f(1/w)$. □

Proof for boundary condition 3.3: We show the main theorem for the boundary condition ${}_{(3)}\langle B|$ and $1 \le M + 1 \le L - 1$. Here we show the relation (6.55).

• The case for $L + K - M \le i \le N + 1$: ${}_{(3)}\langle B|\phi^*_{M+1+i}(z)$.
Using the bosonization (4.74) and the relations (6.56), (6.57), (6.58), and normal orderings in Appendix B, we have LHS − RHS of (6.55) as follows:

$$\begin{aligned}
& z^{\frac{M}{M-N}+1}\varphi^{(3)}(1/z)_{(3)}\langle B|\phi^*_{M+1+i}(z) - z^{-\frac{M}{M-N}-1}\varphi^{(3)}(z)_{(3)}\langle B|\phi_{M+1+i}(1/z) \\
&= q^{-\frac{M}{M-N}-1}(q-q^{-1})^M \varphi^{(3)}(z)\varphi^{(3)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}}{M-N}M} \\
&\times \sum_{\epsilon_1,\cdots,\epsilon_i=\pm} \epsilon_i \prod_{k=1}^{M+i} \int_{\widetilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^{M}(1-q/w_kw_{k+1})}{\prod_{k=0}^{M} D(w_k, w_{k+1})} \prod_{k=1}^{M}(1-w_k^2) \\
&\times \frac{(1+w_{M+1}^2)}{(1-rq^{L-2M-2}w_L)(1-q^{2M+K-L+2}w_{L+K}/r)} \prod_{k=1}^{i-1} \frac{\epsilon_k q^{\epsilon_k}(1-qw_{M+k}w_{M+k+1})}{(1-q^{\epsilon_k}w_{M+k}w_{M+k+1})(1-q^{\epsilon_k}w_{M+k}/w_{M+k+1})} \\
&\times {}_{(3)}\langle B|e^{Q_{h_1^*-h_1\cdots-h_{M+i}-c_i}} e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+i}(h_+^k(qw_k)+h_+^k(q/w_k))} \\
&\times e^{\sum_{k=1}^{i-1}(c_+^k(qw_{M+k})+c_+^k(q/w_{M+k}))-\sum_{k=1}^{i}(c_+^k(q^{1+\epsilon_k}w_{M+k})+c_+^k(q^{1-\epsilon_k}/w_{M+k}))}.
\end{aligned} \tag{6.73}$$

Here the integration contour $\widetilde{C}_{M+1+i}$ encircles $w_k = 0, qw_{k-1}^{\pm 1}$ but not $w_k = q^{-1}w_{k-1}^{\pm 1}$ for $1 \le k \le M + i$. Using relation $w_1(1-q/w_1w_2) - (w_1 \leftrightarrow w_1^{-1}) = (w_1 - w_1^{-1})$, we symmetrize the variables $w_1, w_2, \cdots, w_M$, iteratively. We have

$$\begin{aligned}
& q^{-\frac{M}{M-N}-1}(q-q^{-1})^M(-1/2)^M\varphi^{(3)}(z)\varphi^{(3)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}}{M-N}M} \\
&\times \sum_{\epsilon_1,\cdots,\epsilon_i=\pm} \epsilon_i \prod_{k=1}^{M+i} \int_{\widetilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^{M}(w_k^{-1}-w_k)^2}{\prod_{k=0}^{M} D(w_k, w_{k+1})} \frac{(w_{M+1}^{-1}+w_{M+1})}{(1-rq^{L-2M-2}w_L)(1-q^{2M+K-L+2}w_{L+K}/r)} \\
&\times {}_{(3)}\langle B|e^{Q_{h_1^*-h_1\cdots-h_{M+i}-c_i}} e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+i}(h_+^k(qw_k)+h_+^k(q/w_k))+\sum_{k=1}^{i-1}(c_+^k(qw_{M+k})+c_+^k(q/w_{M+k}))} \\
&\times w_{M+1} \prod_{k=1}^{i-1} \frac{\epsilon_k q^{\epsilon_k}(1-qw_{M+k}w_{M+k+1})e^{-c_+^k(q^{1+\epsilon_k}w_{M+k})-c_+^k(q^{1-\epsilon_k}/w_{M+k})}}{(1-q^{\epsilon_k}w_{M+k}w_{M+k+1})(1-q^{\epsilon_k}w_{M+k}/w_{M+k+1})} e^{-c_+^i(q^{1+\epsilon_i}w_{M+i})-c_+^i(q^{1-\epsilon_i}/w_{M+i})}.
\end{aligned} \tag{6.74}$$

We use the relation (6.64) for the variables $w_{M+1}, \cdots, w_{L-1}$, iteratively. We use the relation (6.64) for the variable w_L. Using the relation (6.42), we symmetrize the variables $w_{L+1}, \cdots, w_{L+K-1}$, iteratively. Using the relation (6.22), we symmetrize the variables $w_{L+K}, \cdots, w_{M+i-1}$, iteratively.

Then we have

$$
\begin{aligned}
&(-r)q^{-\frac{M}{M-N}+K+L-2M-5}(q-q^{-1})^M(-1/2)^M\varphi^{(3)}(z)\varphi^{(3)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}}{M-N}M}\\
&\times\prod_{k=1}^{M+i}\int_{\widetilde{C}_{M+1+i}}\frac{dw_k}{2\pi\sqrt{-1}w_k}\frac{\prod_{k=1}^{M}(w_k^{-1}-w_k)^2}{\prod_{k=0}^{M}D(w_k,w_{k+1})}\frac{(w_{M+1}^{-1}+w_{M+1})}{(1-rq^{L-2M-2}w_L)(1-rq^{L-2M-2}/w_L)}\\
&\times{}_{(3)}\langle B|e^{Q_{h_1^*-h_1\cdots-h_{M+i}-c_i}}e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+i}(h_+^k(qw_k)+h_+^k(q/w_k))+\sum_{k=1}^{i-1}(c_+^k(qw_{M+k})+c_+^k(q/w_{M+k}))}\\
&\times\prod_{k=1}^{i-1}\frac{(-1+w_{M+k}^2)e^{-c_+^k(w_{M+k})-c_+^k(q^2/w_{M+k})}}{(1-w_{M+k}w_{M+k+1}/q)(1-qw_{M+k}/w_{M+k+1})}\\
&\times\left(e^{-c_+^i(q^2w_{M+i})-c_+^i(1/w_{M+i})}-e^{-c_+^i(w_{M+i})-c_+^i(q^2/w_{M+i})}\right)=0.
\end{aligned}
\tag{6.75}
$$

Here we have used

$$
\oint_{|w|=1}\frac{dw}{w}e^{c_+^{i-1}(qw)+c_+^{i-1}(q/w)}\left(e^{-c_+^i(q^2w)-c_+^i(1/w)}-e^{-c_+^i(w)-c_+^i(q^2/w)}\right)f(w)=0,
$$

where $f(w)=f(1/w)$. □

Now we have shown (6.55) for every $j=L+K+1,\ldots,M+N+2$. Now we have shown the relation (3.28) for every $i=1,2,3$.

D. Conclusion

We have studied the supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ Hamiltonian on the semi-infinite chain (2.36). We started from the transfer matrix $T_B^{(i)}(z)$ (3.25) defined by the vertex operators $\Phi_j^*(z)$ for the supersymmetry $U_q(\widehat{sl}(M+1|N+1))$. In order to have more stable foundation of this mathematical interpretation, we should study up the CTM argument for $U_q(\widehat{sl}(M+1|N+1))$ in the future. We constructed the bosonization of the boundary state ${}_{(i)}\langle B|\in L^*(\Lambda_{M+1})$ satisfying the condition (3.28), by using the bosonizations of the vertex operators. We gave the detailed proof of this bosonization. As a corollary, we gave the space of the state of the Hamiltonian on the semi-infinite chain. It is surprising that the bosonization of the boundary state ${}_{(i)}\langle B|$ is realized by "monomial," even though the bosonization of the vertex operator $\Phi_j^*(z)$ for the supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ is given by "sum." The boundary state is given by ${}_{(i)}\langle B|=v_{\Lambda_{M+1}}^*\cdot e^{G^{(i)}}$, where $G^{(i)}$ is quadratic in the bosonic operator (see (5.5)). By now we have constructed the bosonizations of the boundary states for the $U_q(g)$ Hamiltonian on the semi-infinite chain for the cases $g=\widehat{sl}(N+1)$, $A_2^{(2)}$, and $\widehat{sl}(M+1|N+1)$.[24,25,29] All of them are realized by "monomial."

ACKNOWLEDGMENTS

This work is supported by the Grant-in-Aid for Scientific Research **C** (21540228) from Japan Society for Promotion of Science.

APPENDIX A: *K*-MATRIX

In this appendix, we classify diagonal solutions of the boundary Yang-Baxter equation associated with $U_q(\widehat{sl}(M+1|N+1))$. Let us set the vector space $V=\oplus_{j=1}^{M+N+2}\mathbf{C}v_j$. Let us consider the R-matrix $\bar{R}(z)\in\mathrm{End}(V\otimes V)$ introduced in (2.6), (2.7), (2.8), and (2.9). Non-zero elements of the R-matrix are restricted to the following:

$$
\bar{R}(z)_{i,j}^{i,j}\neq 0,\ \bar{R}(z)_{i,j}^{j,i}\neq 0,\ (1\le i,\ j\le M+N+2).
\tag{A1}
$$

Let us study the K-matrix $\bar{K}(z) \in \mathrm{End}(V)$ defined as follows:

$$\bar{K}(z) \in \mathrm{End}(V),\ \bar{K}(z)v_j = \sum_{k=1}^{M+N+2} v_k \bar{K}(z)_k^j, \tag{A2}$$

where we assume the diagonal matrix,

$$\bar{K}(z)_k^j = \delta_{j,k}\bar{K}(z)_j^j. \tag{A3}$$

The graded boundary Yang-Baxter equation

$$\bar{K}_2(z_2)\bar{R}_{21}(z_1z_2)\bar{K}_1(z_1)\bar{R}_{12}(z_1/z_2) = \bar{R}_{21}(z_1/z_2)\bar{K}_1(z_1)\bar{R}_{12}(z_1z_2)\bar{K}_2(z_2), \tag{A4}$$

is equivalent to the following two relations for $1 \le j < k \le M + N + 2$:

$$\bar{R}(z)_{j,k}^{j,k} = \bar{R}(z)_{k,j}^{k,j}, \tag{A5}$$

$$\begin{aligned}&\bar{R}(z_1/z_2)_{j,k}^{j,k}\left(\bar{R}(z_1z_2)_{k,j}^{j,k}\bar{K}(z_1)_k^k\bar{K}(z_2)_k^k - \bar{R}(z_1z_2)_{j,k}^{k,j}\bar{K}(z_1)_j^j\bar{K}(z_2)_j^j\right)\\ &+ \bar{R}(z_1z_2)_{j,k}^{j,k}\left(\bar{R}(z_1/z_2)_{j,k}^{k,j}\bar{K}(z_2)_k^k\bar{K}(z_1)_j^j - \bar{R}(z_1/z_2)_{k,j}^{j,k}\bar{K}(z_1)_k^k\bar{K}(z_2)_j^j\right) = 0.\end{aligned} \tag{A6}$$

The first condition (A5) holds for (2.7), (2.8), (2.9). The second condition (A6) is written as follows:

$$\left(1 - \frac{z_1}{z_2}\right)\left(z_1z_2 - \frac{\bar{K}(z_1)_j^j}{\bar{K}(z_1)_k^k}\frac{\bar{K}(z_2)_j^j}{\bar{K}(z_2)_k^k}\right) + (1 - z_1z_2)\left(\frac{\bar{K}(z_1)_j^j}{\bar{K}(z_1)_k^k} - \frac{z_1}{z_2}\frac{\bar{K}(z_2)_j^j}{\bar{K}(z_2)_k^k}\right) = 0. \tag{A7}$$

Differentiating partially (A7), at $(z_1, z_2) = (z, 1)$, with respect to z_2, we have the following necessary condition:

$$\frac{\bar{K}(z)_j^j}{\bar{K}(z)_k^k} = \frac{1 - \beta z}{1 - \beta/z}(\beta \in \mathbf{C}). \tag{A8}$$

This satisfies (A7) for all $\beta \in \mathbf{C}$. Taking into account of simultaneous compatibility for $1 \le j < k \le M + N + 2$, we have the following three kinds of general diagonal solutions of the boundary Yang-Baxter equation associated with $U_q(\widehat{sl}(M + 1|N + 1))$.

Case 1: One diagonal element. Dirichlet boundary condition,

$$\bar{K}(z)_k^j = \delta_{j,k}. \tag{A9}$$

Case 2: Two different diagonal elements. We assume $1 \le L \le M + N + 1$ and $r \in \mathbf{C}$,

$$\bar{K}(z)_k^j = \begin{cases} 1 & (1 \le j = k \le L), \\ \dfrac{1 - r/z}{1 - rz} & (L + 1 \le j = k \le M + N + 2), \\ 0 & (1 \le j \ne k \le M + N + 2). \end{cases} \tag{A10}$$

Case 3: Three different diagonal elements. We assume $1 \le L, 1 \le K, L + K \le M + N + 1$ and $r \in \mathbf{C}$,

$$\bar{K}(z)_k^j = \begin{cases} 1 & (1 \le j = k \le L), \\ \dfrac{1 - r/z}{1 - rz} & (L + 1 \le j = k \le L + K), \\ z^{-2} & (L + K + 1 \le j = k \le M + N + 2), \\ 0 & (1 \le j \ne k \le M + N + 2). \end{cases} \tag{A11}$$

Note. In the earlier studies,[11,12] *Case I and Case II have been studied. However, Case III is missing in the earlier studies.*[11,12] *For instance, we have new solution for* $U_q(\widehat{sl}(2|1))$, *which has*

three different diagonal elements.

$$\bar{K}(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1-r/z}{1-rz} & 0 \\ 0 & 0 & z^{-2} \end{pmatrix}. \tag{A12}$$

APPENDIX B: NORMAL ORDERING

In this appendix we summarize the normal orderings. The following normal orderings are convenient to calculations in a proof of main theorem:

$$e^{h_+^{*1}(z)}e^{-h_-^1(w)} = \frac{1}{(1-qw/z)} : e^{h_+^{*1}(z)}e^{-h_-^1(w)} :, \tag{B1}$$

$$e^{-h_+^1(w)}e^{h_-^{*1}(z)} = \frac{1}{(1-qz/w)} : e^{-h_+^1(w)}e^{h_-^{*1}(z)} :, \tag{B2}$$

$$e^{h_+^{*1}(z)}e^{-h_-^j(w)} = 1 : e^{h_+^{*1}(z)}e^{-h_-^j(w)} : (2 \le j \le M+N+1), \tag{B3}$$

$$e^{-h_+^j(w)}e^{h_-^{*1}(z)} = 1 : e^{-h_+^j(w)}e^{h_-^{*1}(z)} : (2 \le j \le M+N+1), \tag{B4}$$

$$e^{-h_+^j(w_1)}e^{-h_-^{j+1}(w_2)} = \frac{1}{(1-qw_2/w_1)} : e^{-h_+^j(w_1)}e^{-h_-^{j+1}(w_2)} : (1 \le j \le M), \tag{B5}$$

$$e^{-h_+^{j+1}(w_1)}e^{-h_-^j(w_2)} = \frac{1}{(1-qw_2/w_1)} : e^{-h_+^{j+1}(w_1)}e^{-h_-^j(w_2)} : (1 \le j \le M), \tag{B6}$$

$$e^{-h_+^{M+j}(w_1)}e^{-h_-^{M+j+1}(w_2)} = (1-qw_2/w_1) : e^{-h_+^{M+j}(w_1)}e^{-h_-^{M+j+1}(w_2)} : (1 \le j \le M), \tag{B7}$$

$$e^{-h_+^{M+j+1}(w_1)}e^{-h_-^{M+j}(w_2)} = (1-qw_2/w_1) : e^{-h_+^{M+j+1}(w_1)}e^{-h_-^{M+j}(w_2)} : (1 \le j \le N). \tag{B8}$$

The following normal orderings are convenient to get the integral representations of the vertex operators:

$$\phi_1^*(z)X_1^-(qw) = e^{\frac{\pi\sqrt{-1}}{M}}\frac{1}{qz(1-qw/z)} : \phi_1^*(z)X_1^-(qw) :, \tag{B9}$$

$$X_1^-(qw)\phi_1^*(z) = -e^{\frac{\pi\sqrt{-1}}{M}}\frac{1}{qw(1-qz/w)} : X_1^-(qw)\phi_1^*(z) :, \tag{B10}$$

$$\phi_1^*(z)X_j^-(w) = 1 : \phi_1^*(z)X_j^-(w) : (2 \le j \le M), \tag{B11}$$

$$X_j^-(w)\phi_1^*(z) = 1 : X_j^-(w)\phi_1^*(z) : (2 \le j \le M), \tag{B12}$$

$$\phi_1^*(z)X_{M+1,\epsilon}^-(w) = -1 : \phi_1^*(z)X_{M+1,\epsilon}^-(w) : (\epsilon = \pm), \tag{B13}$$

$$X_{M+1,\epsilon}^-(w)\phi_1^*(z) = 1 : X_{M+1,\epsilon}^-(w)\phi_1^*(z) : (\epsilon = \pm), \tag{B14}$$

$$X_j^-(qw_1)X_{j+1}^-(qw_2) = \frac{1}{qw_1(1-qw_2/w_1)} : X_j^-(qw_1)X_{j+1}^-(qw_2) : (1 \le j \le M), \tag{B15}$$

$$X_{j+1}^-(qw_1)X_j^-(qw_2) = \frac{-1}{qw_1(1-qw_2/w_1)} : X_{j+1}^-(qw_1)X_j^-(qw_2) : (1 \le j \le M), \tag{B16}$$

$$X_j^-(qw_1)X_k^-(qw_2) = 1 : X_j^-(qw_1)X_k^-(qw_2) : (|j-k| \ge 2), \tag{B17}$$

$$X_{M+j,+}^-(w_1)X_{M+j+1,\epsilon}^-(w_2) = \frac{(1-qw_2/w_1)}{q(1-w_2/qw_1)} : X_{M+j,+}^-(w_1)X_{M+j+1,\epsilon}^-(w_2) : (\epsilon = \pm), \tag{B18}$$

$$X_{M+1+j,+}^-(w_1)X_{M+j,\epsilon}^-(w_2) = 1 : X_{M+1+j,+}^-(w_1)X_{M+j,\epsilon}^-(w_2) : (\epsilon = \pm), \tag{B19}$$

$$X_{M+j,-}^-(w_1)X_{M+j+1,\epsilon}^-(w_2) = q : X_{M+j,-}^-(w_1)X_{M+j+1,\epsilon}^-(w_2) : (\epsilon = \pm), \tag{B20}$$

$$X_{M+j+1,-}^-(w_1)X_{M+j,\epsilon}^-(w_2) = \frac{(1-qw_1/w_2)}{(1-w_1/qw_2)} : X_{M+j+1,-}^-(w_1)X_{M+j,\epsilon}^-(w_2) : (\epsilon = \pm). \tag{B21}$$

[1] M. Jimbo and T. Miwa, *Algebraic Analysis of Solvable Lattice Model*, CBMS Regional Conference Series in Mathematics (AMS, 1984), Vol. 85.
[2] B. Davies, O. Foda, M. Jimbo, T. Miwa, and A. Nakayashiki, "Diagonalization of the XXZ Hamiltonian by vertex operators," Commun. Math. Phys. **151**, 89–153 (1993).
[3] M. Gaudin, *La fonction d'onde de Bethe* (Masson, Paris, 1983).
[4] V. Korepin, N. Bogoliubov, and A. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 1993).
[5] V. Kac, "Lie superalgebras," Adv. Math. **26**(1), 8–96 (1977).
[6] V. Drinfeld, "A new realization of Yangians and quantized affine algebras," Sov. Math. Dokl. **36**, 212–216 (1988).
[7] H. Yamane, "On defining relations of the affine Lie superalgebras and their quantized universal enveloping superalgebras," Publ. Res. Inst. Math. Sci. **35**, 321–390 (1999).
[8] V. Bazhanov and A. Shadrikov, "Trigonometric solutions of the triangle equations, and simple Lie superalgebra," Theor. Math. Phys. **73**(3), 1302–1312 (1987).
[9] J. Perk and C. Shultz, "New families of commuting transfer matrices in q-state vertex models," Phys. Lett. A **84**, 407–410 (1981).
[10] H. J. de Vega and A. González Ruiz, "Boundary K-matrix for the six vertex and $n(2n-1)A_{n-1}$ vertex models," J. Phys. A **26**, L519–L524 (1993).
[11] G. Li, T. Yue, and B. Hou, "Nested Bethe ansatz for Perk-Schultz model with open boundary condition," Nucl. Phys. B **586**, 711–729 (2000).
[12] S. Belliard and E. Ragoucy, "Nested Bethe ansatz for 'all' open spin chains with diagonal boundary conditions," J. Phys. A **42**(20), 205203 (2009).
[13] F. Essler and V. Korepin, "Higher conservation laws and algebraic Bethe ansatz for the supersymmetric t-J model," Phys. Rev. B **46**, 9147–9162 (1992).
[14] S. Zhao, W. Yang, and Y. Zhang, "Determinant representations of correlation functions for the supersymmetric t-J model," Commun. Math. Phys. **268**(2), 505–541 (2006).
[15] H. Fan, M. Wadati, and X. Wang, "Exact diagonalization of the generalized supersymmetric t-J model with boundaries," Phys. Rev. B **61**, 3450–3469 (2000).
[16] K. Kimura, J. Shiraishi, and J. Uchiyama, "A level-one representation of the quantum affine superalgebra $U_q(\widehat{sl}(M+1|N+1))$," Commun. Math. Phys. **188**(2), 367–378 (1997).
[17] W. Yang and Y. Zhang, "Highest weight representations of $U_q(\widehat{sl}(2|1))$ and correlation functions of the q-deformed supersymmetric t-J model," Nucl. Phys. B **547**(3), 599–622 (1999).
[18] W. Yang and Y. Zhang, "Level-one highest weight representations of $U_q(\widehat{sl}(N|1))$ and bosonization of the multi-component super t-J model," J. Math. Phys. **41**(8), 5849–5869 (2000).
[19] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).
[20] I. Cherednik, "Factorizing particles on a half line, and root systems," Teor. Mat. Fiz. **61**(1), 35–44 (1984).
[21] E. K. Sklyanin, "Boundary conditions for integrable quantum systems," J. Phys. A **21**, 2375–2389 (1988).
[22] L. Mezincescu and R. Nepomechie, "Integrable open spin chains with nonsymmetric R-matrices," J. Phys. A **24**(1), L17–L23 (1991).
[23] S. Ghoshal and A. Zamolodchikov, "Boundary S matrix and boundary state in two-dimensional integrable quantum field theory," Int. J. Mod. Phys. A **9**(21), 3841–3885 (1994).

[24] M. Jimbo, R. Kedem, T. Kojima, H. Konno, and T. Miwa, "XXZ chain with a boundary," Nucl. Phys. B **441**, 437–470 (1995).
[25] H. Furutsu and T. Kojima, "The $U_q(\widehat{sl_n})$ analogue of the XXZ chain with a boundary," J. Math. Phys. **41**, 4413–4436 (2000).
[26] T. Miwa and R. Weston, "Boundary ABF models," Nucl. Phys. B **486**, 517–545 (1997).
[27] T. Kojima, "Diagonalization of infinite transfer matrix of boundary $U_{q,p}(A_{N-1}^{(1)})$ face model," J. Math. Phys. **52**, 013501 (2011).
[28] W. Yang and Y. Zhang, "Izergin-Korepin model with a boundary," Nucl. Phys. B **596**(3), 495–512 (2001).
[29] T. Kojima, "A remark on ground state of boundary Izergin-Korepin model," Int. J. Mod. Phys. A **26**(12), 1973–1989 (2011).
[30] B. Hou, W. Yang, Y. Zhang, and Y. Zhen, "The q-deformed supersymmetric t-J model with a boundary," J. Phys. A **35**(11), 2593–2608 (2002).
[31] R. M. Gade, "The $U_q(\widehat{sl}(2|1))_1$ module $V(\Lambda_2)$ and a corner transfer matrix at $q = 0$," Nucl. Phys. B **659**(3), 387–423 (2003).
[32] R. Baxter, "Variational approximations for square lattice models in statistical mechanics," J. Stat. Phys. **19**, 461–478 (1978).
[33] M. Jimbo, T. Miwa, and A. Nakayashiki, "Difference equations for the correlation functions of the eight-vertex model," J. Phys. A **26**, 2199–2209 (1993).
[34] M. Jimbo, R. Kedem, H. Konno, T. Miwa, and R. Weston, "Difference equations in spin chains with a boundary," Nucl. Phys. B **448**, 429–456 (1995).