Bosonization and Vertex Operator of Supersymmetry $U_q(\widehat{sl}(N|1))$ for Level k

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Abstract. We construct a bosonization of the quantum superalgebra $U_q(\hat{sl}(N|1))$ for an arbitrary level k. We construct the screening that commutes with the quantum superalgebra for an arbitrary level $k \neq -N+1$. We propose a bosonization of the vertex operator that gives the intertwiner among the Wakimoto realization and the typical representation.

1. Introduction

There have been many developments in exactly solvable models. Various methods were invented to solve models. The bosonization provides a powerful method to study exactly solvable models. We review recent developments in bosonizations of the $U_q(\widehat{sl}(N|1))$ [7, 8, 10]. The trace of our bosonizations of the vertex operators gives the correlation function of the higher level(spin) k and rank N generalization of the supersymmetric t-J model ($U_q(\widehat{sl}(2|1))$ for level $k=1$) [1].

2. Bosonization of $U_q(\widehat{sl}(N|1))$

We recall Drinfeld generators of the superalgebra $U_q(\widehat{sl}(N|1))$ [3]. We fix a complex number $q \neq 0, |q| < 1$. We use the notations $[x,y] = xy - yx,\ \{x,y\} = xy + yx,\ [a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}$. The Cartan matrix $(A_{i,j})_{0 \leq i,j \leq N}$ of the affine Lie algebra $\widehat{sl}(N|1)$ is given by

$$A_{i,j} = (\nu_i + \nu_{i+1})\delta_{i,j} - \nu_i \delta_{i,j+1} - \nu_{i+1}\delta_{i+1,j}.$$

Here we set $\nu_1 = \cdots = \nu_N = +, \nu_{N+1} = \nu_0 = -$. The Drinfeld generators of the quantum superalgebra $U_q(\widehat{sl}(N|1))$ are $x^{\pm}_{i,m},\ h_{i,m},\ c,\ (1 \leq i \leq N, m \in \mathbf{Z})$. Defining relations are

$$\begin{aligned}
&c : \text{central},\ [h_i, h_{j,m}] = 0,\\
&[a_{i,m}, h_{j,n}] = \frac{[A_{i,j}m]_q [cm]_q}{m} q^{-c|m|}\delta_{m+n,0} \quad (m,n \neq 0),\ [h_i, x^{\pm}_j(z)] = \pm A_{i,j} x^{\pm}_j(z),\\
&[h_{i,m}, x^{+}_j(z)] = \frac{[A_{i,j}m]_q}{m} q^{-c|m|} z^m x^{+}_j(z), \quad [h_{i,m}, x^{-}_j(z)] = -\frac{[A_{i,j}m]_q}{m} z^m x^{-}_j(z) \quad (m \neq 0),\\
&(z_1 - q^{\pm A_{i,j}} z_2) x^{\pm}_i(z_1) x^{\pm}_j(z_2) = (q^{\pm A_{j,i}} z_1 - z_2) x^{\pm}_j(z_2) x^{\pm}_i(z_1) \quad \text{for } |A_{i,j}| \neq 0,\\
&x^{\pm}_i(z_1) x^{\pm}_j(z_2) = x^{\pm}_j(z_2) x^{\pm}_i(z_1) \quad \text{for } |A_{i,j}| = 0, (i,j) \neq (N,N), \quad \{x^{\pm}_N(z_1), x^{\pm}_N(z_2)\} = 0,\\
&[x^{+}_i(z_1), x^{-}_j(z_2)] = \frac{\delta_{i,j}}{(q - q^{-1}) z_1 z_2} \left(\delta(q^{-c} z_1/z_2) \psi^{+}_i(q^{\frac{c}{2}} z_2) - \delta(q^{c} z_1/z_2) \psi^{-}_i(q^{-\frac{c}{2}} z_2) \right),
\end{aligned}$$

$$
\begin{aligned}
&\text{for} \quad (i,j) \neq (N,N), \\
&\{x_N^+(z_1), x_N^-(z_2)\} = \frac{1}{(q-q^{-1})z_1 z_2}\left(\delta(q^{-c}z_1/z_2)\psi_N^+(q^{\frac{c}{2}}z_2) - \delta(q^c z_1/z_2)\psi_N^-(q^{-\frac{c}{2}}z_2)\right), \\
&\Big(x_i^\pm(z_1)x_i^\pm(z_2)x_j^\pm(z) - (q+q^{-1})x_i^\pm(z_1)x_j^\pm(z)x_i^\pm(z_2) + x_j^\pm(z)x_i^\pm(z_1)x_i^\pm(z_2)\Big) \\
&+ (z_1 \leftrightarrow z_2) = 0 \quad \text{for } |A_{i,j}| = 1,\ i \neq N,
\end{aligned}
$$

where we have used $\delta(z) = \sum_{m\in\mathbf{Z}} z^m$. Here we have used the generating function $x_j^\pm(z) = \sum_{m\in\mathbf{Z}} x_{j,m}^\pm z^{-m-1}$, $\psi_i^\pm(q^{\pm\frac{c}{2}}z) = q^{\pm h_i} e^{\pm(q-q^{-1})\sum_{m>0} h_{i,\pm m} z^{\mp m}}$ and the abbreviation $h_i = h_{i,0}$.

We construct bosonizations of superalgebra $U_q(\hat{sl}(N|1))$ for an arbitrary level [7]. We fix the level $c = k \in \mathbf{C}$. We introduce the bosons and the zero-mode operators a_m^j, Q_a^j $(m \in \mathbf{Z}, 1 \leq j \leq N)$, $b_m^{i,j}, Q_b^{i,j}$ $(m \in \mathbf{Z}, 1 \leq i < j \leq N+1)$, $c_m^{i,j}, Q_c^{i,j}$ $(m \in \mathbf{Z}, 1 \leq i < j \leq N)$ which satisfy

$$
\begin{aligned}
&[a_m^i, a_n^j] = \frac{[(k+N-1)m]_q [A_{i,j}m]_q}{m}\delta_{m+n,0} \quad (m,n \neq 0), \quad [a_0^i, Q_a^j] = (k+N-1)A_{i,j}, \\
&[b_m^{i,j}, b_n^{i',j'}] = -\nu_i\nu_j \frac{[m]_q^2}{m}\delta_{i,i'}\delta_{j,j'}\delta_{m+n,0} \quad (m,n \neq 0), \quad [b_0^{i,j}, Q_b^{i',j'}] = -\nu_i\nu_j\delta_{i,i'}\delta_{j,j'}, \\
&[c_m^{i,j}, c_n^{i',j'}] = \frac{[m]_q^2}{m}\delta_{i,i'}\delta_{j,j'}\delta_{m+n,0} \quad (m,n \neq 0), \quad [c_0^{i,j}, Q_c^{i',j'}] = \delta_{i,i'}\delta_{j,j'}, \\
&[Q_b^{i,j}, Q_b^{i',j'}] = \delta_{j,N+1}\delta_{j',N+1}\pi\sqrt{-1} \qquad ((i,j) \neq (i',j')).
\end{aligned}
$$

Other commutation relations are zero. In what follows we use the standard symbol of the normal orderings :: and use the following abbreviations $b^{i,j}(z), c^{i,j}(z), b_\pm^{i,j}(z), a_\pm^j(z)$ and $\left(\frac{\gamma_1}{\beta_1}\frac{\gamma_2}{\beta_2}\cdots\frac{\gamma_r}{\beta_r}\ a^i\right)(z|\alpha)$ given by

$$
\begin{aligned}
&b^{i,j}(z) = -\sum_{m\neq 0}\frac{b_m^{i,j}}{[m]_q}z^{-m} + Q_b^{i,j} + b_0^{i,j}\log z,\ c^{i,j}(z) = -\sum_{m\neq 0}\frac{c_m^{i,j}}{[m]_q}z^{-m} + Q_c^{i,j} + c_0^{i,j}\log z, \\
&b_\pm^{i,j}(z) = \pm(q-q^{-1})\sum_{\pm m>0} b_m^{i,j}z^{-m} \pm b_0^{i,j}\log q,\ a_\pm^j(z) = \pm(q-q^{-1})\sum_{\pm m>0} a_m^j z^{-m} \pm a_0^j \log q, \\
&\left(\frac{\gamma_1}{\beta_1}\frac{\gamma_2}{\beta_2}\cdots\frac{\gamma_r}{\beta_r}\ a^i\right)(z|\alpha) = -\sum_{m\neq 0}\frac{[\gamma_1 m]_q\cdots[\gamma_r m]_q}{[\beta_1 m]_q\cdots[\beta_r m]_q}\frac{a_m^i}{[m]_q}q^{-\alpha|m|}z^{-m} + \frac{\gamma_1\cdots\gamma_r}{\beta_1\cdots\beta_r}(Q_a^i + a_0^i\log z).
\end{aligned}
$$

The generating functions $x_i^\pm(z)$, $\psi_i^\pm(z)$, $(1 \leq i \leq N)$ of $U_q(\widehat{sl}(N|1))$ for an arbitrary level k are realized by the bosonic operators as follows. This is main result of [7].

$$
\begin{aligned}
x_i^+(z) &= \frac{1}{(q-q^{-1})z} : \sum_{j=1}^{i} e^{(b+c)^{j,i}(q^{j-1}z) + \sum_{l=1}^{j-1}(b_+^{l,i+1}(q^{l-1}z) - b_+^{l,i}(q^l z))} \times \\
&\times \left\{e^{b_+^{j,i+1}(q^{j-1}z) - (b+c)^{j,i+1}(q^j z)} - e^{b_-^{j,i+1}(q^{j-1}z) - (b+c)^{j,i+1}(q^{j-2}z)}\right\} :, \\
x_N^+(z) &= : \sum_{j=1}^{N} e^{(b+c)^{j,N}(q^{j-1}z) + b^{j,N+1}(q^{j-1}z) - \sum_{l=1}^{j-1}(b_+^{l,N+1}(q^l z) + b_+^{l,N}(q^l z))} :, \\
x_i^-(z) &= q^{k+N-1} : e^{a_+^i(q^{\frac{k+N-1}{2}}z) - b^{i,N+1}(q^{k+N-1}z) - b_+^{i+1,N+1}(q^{k+N-1}z) + b^{i+1,N+1}(q^{k+N}z)} : \\
&+ \frac{1}{(q-q^{-1})z} : \left\{\sum_{j=1}^{i-1} e^{a_-^i(q^{-\frac{k+N-1}{2}}z) + (b+c)^{j,i+1}(q^{-k-j}z) + b_-^{i,n+1}(q^{-k-n}z) - b_-^{i+1,n+1}(q^{-k-n+1}z)}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times\ e^{\sum_{l=j+1}^{i}(b_-^{l,i+1}(q^{-k-l+1}z)-b_-^{l,i}(q^{-k-l}z))+\sum_{l=i+1}^{N}(b_-^{i,l}(q^{-k-l}z)-b_-^{i+1,l}(q^{-k-l+1}z))}\\
&\times\ \left(e^{-b_-^{j,i}(q^{-k-j}z)-(b+c)^{j,i}(q^{-k-j+1}z)}-e^{-b_+^{j,i}(q^{-k-j}z)-(b+c)^{j,i}(q^{-k-j-1}z)}\right)\\
&+\ e^{a_-^i(q^{-\frac{k+N-1}{2}}z)+(b+c)^{i,i+1}(q^{-k-i}z)+\sum_{l=i+1}^{N}(b_-^{i,l}(q^{-k-l}z)-b_-^{i+1,l}(q^{-k-l+1}z))+b_-^{i,N+1}(q^{-k-N}z)-b_-^{i+1,N+1}(q^{-k-N+1}z)}\\
&-\ e^{a_+^i(q^{\frac{k+N-1}{2}}z)+(b+c)^{i,i+1}(q^{k+i}z)+\sum_{l=i+1}^{N}(b_+^{i,l}(q^{k+l}z)-b_+^{i+1,l}(q^{k+l-1}z))+b_+^{i,N+1}(q^{k+N}z)-b_+^{i+1,N+1}(q^{k+N-1}z)}\\
&-\ \sum_{j=i+1}^{N-1} e^{a_+^i(q^{\frac{k+N-1}{2}}z)+(b+c)^{i,j+1}(q^{k+j}z)+b_+^{i,N+1}(q^{k+N}z)-b_+^{i+1,N+1}(q^{k+N-1}z)+\sum_{l=j+1}^{N}(b_+^{i,l}(q^{k+l}z)-b_+^{i+1,l}(q^{k+l-1}z))}\\
&\times\ \left(e^{b_+^{i+1,j+1}(q^{k+j}z)-(b+c)^{i+1,j+1}(q^{k+j+1}z)}-e^{b_-^{i+1,j+1}(q^{k+j}z)-(b+c)^{i+1,j+1}(q^{k+j-1}z)}\right)\Bigg\}:.
\end{aligned}
$$

$$
\begin{aligned}
x_N^-(z) &= \frac{1}{(q-q^{-1})z}:\Bigg\{\sum_{j=1}^{N-1} e^{a_-^N(q^{-\frac{k+N-1}{2}}z)-b_+^{j,N+1}(q^{-k-j}z)-b^{j,N+1}(q^{-k-j-1}z)-\sum_{l=j+1}^{N-1}(b_-^{l,N}(q^{-k-l}z)+b_-^{l,N+1}(q^{-k-l}z))}\\
&\times\ q^{j-1}\left(e^{-b_+^{j,N}(q^{-k-j}z)-(b+c)^{j,N}(q^{-k-j-1}z)}-e^{-b_-^{j,N}(q^{-k-j}z)-(b+c)^{j,N}(q^{-k-j+1}z)}\right)\Bigg\}:\\
&+\ q^{N-1}:\left(e^{a_+^N(q^{\frac{k+N-1}{2}}z)-b^{N,N+1}(q^{k+N-1}z)}-e^{a_-^N(q^{-\frac{k+N-1}{2}}z)-b^{N,N+1}(q^{-k-N+1}z)}\right):.
\end{aligned}
$$

$$
\begin{aligned}
\psi_i^{\pm}(q^{\pm\frac{k}{2}}z) &= e^{a_\pm^i(q^{\pm\frac{k+N-1}{2}}z)+\sum_{l=1}^{i}(b_\pm^{l,i+1}(q^{\pm(l+k-1)}z)-b_\pm^{l,i}(q^{\pm(l+k)}z)}\\
&\quad\times e^{\sum_{l=i+1}^{N}(b_\pm^{i,l}(q^{\pm(k+l)}z)-b_\pm^{i-1,l}(q^{\pm(k+l-1)}z)+b_\pm^{i,N+1}(q^{\pm(k+N)}z)-b_\pm^{i+1,N+1}(q^{\pm(k+N-1)}z)},\\
\psi_N^{\pm}(q^{\pm\frac{k}{2}}z) &= e^{a_\pm^N(q^{\pm\frac{k+N-1}{2}}z)-\sum_{l=1}^{N-1}(b_\pm^{l,N}(q^{\pm(k+l)}z)+b_\pm^{l,N+1}(q^{\pm(k+l)}z))}.
\end{aligned}
$$

3. Vertex Operator

We construct the screening $\mathcal{Q}_i$ that commutes with the quantum superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \neq -N+1$ [10]. The Jackson integral with parameter $p \in \mathbf{C}$ ($|p| < 1$) and $s \in \mathbf{C}^*$ is defined by $\int_0^{s\infty} f(z)d_pz = s(1-p)\sum_{m\in\mathbf{Z}} f(sp^m)p^m$. We introduce the screening operators $\mathcal{Q}_i$ ($1 \le i \le N$) by using the Jackson integral. This is one of main result of [10].

$$
\mathcal{Q}_i = \int_0^{s\infty} : e^{-\left(\frac{1}{k+N-1}a^i\right)\left(z|\frac{k+N-1}{2}\right)}\widetilde{S}_i(z) : d_pz, \quad (p = q^{2(k+N-1)}).
$$

Here we have set the bosonic operators $\widetilde{S}_i(z)$ ($1 \le i \le N$) by

$$
\begin{aligned}
\widetilde{S}_i(z) &= \frac{1}{(q-q^{-1})z}\sum_{j=i+1}^{N} : \left(e^{-b_-^{i,j}(q^{N-1-j}z)-(b+c)^{i,j}(q^{N-j}z)}-e^{-b_+^{i,j}(q^{N-1-j}z)-(b+c)^{i,j}(q^{N-j-2}z)}\right)\\
&\times\ e^{(b+c)^{i+1,j}(q^{N-1-j}z)+\sum_{l=j+1}^{N}(b_-^{i+1,l}(q^{N-l}z)-b_-^{i,l}(q^{N-l-1}z))+b_-^{i+1,N+1}(z)-b_-^{i,N+1}(q^{-1}z)} :\\
&+\ q : e^{b^{i,N+1}(z)+b_+^{i+1,N+1}(z)-b^{i+1,N+1}(qz)} : \qquad (1 \le i \le N-1),\\
\widetilde{S}_N(z) &= -q^{-1} : e^{b^{N,N+1}(z)} : .
\end{aligned}
$$

The screening $\mathcal{Q}_i$ commutes with the quantum superalgebra.

$$
[\mathcal{Q}_i, U_q(\widehat{sl}(N|1))] = 0 \quad (1 \le i \le N).
$$

For $p_a^i \in \mathbf{C}$ ($1 \le i \le N$), $p_b^{i,j} \in \mathbf{C}$ ($1 \le i < j \le N+1$), $p_c^{i,j} \in \mathbf{C}$ ($1 \le i < j \le N$), we set the vector $|p_a, p_b, p_c\rangle$ which satisfies the following conditions.

$$
\begin{aligned}
&a_m^i|p_a,p_b,p_c\rangle = b_m^{i,j}|p_a,p_b,p_c\rangle = c_m^{i,j}|p_a,p_b,p_c\rangle = 0 \quad (m > 0),\\
&a_0^i|p_a,p_b,p_c\rangle = p_a^i|p_a,p_b,p_c\rangle,\ b_0^{i,j}|p_a,p_b,p_c\rangle = p_b^{i,j}|p_a,p_b,p_c\rangle,\ c_0^{i,j}|p_a,p_b,p_c\rangle = p_c^{i,j}|p_a,p_b,p_c\rangle.
\end{aligned}
$$

The boson Fock space $F(p_a, p_b, p_c)$ is generated by the bosons $a^i_m, b^{i,j}_m, c^{i,j}_m$ on the vector $|p_a, p_b, p_c\rangle$. We set the space $F(p_a)$ by

$$F(p_a) = \bigoplus_{\substack{p_b^{i,j}=-p_c^{i,j}\in\mathbf{Z}\ (1\le i<j\le N)\\ p_b^{i,N+1}\in\mathbf{Z}\ (1\le i\le N)}} F(p_a, p_b, p_c).$$

We would like to construct the vertex operator $\Phi^*(z)$ [10] which gives the intertwiner among $F(p_a)$ and the typical representation [13].

$$\Phi^*(z) : F(p_a) \longrightarrow F(p_a + l_a + x_a) \otimes V_z^{*S}, \quad \Phi^*(z) = \sum_j \Phi^*_j(z) \otimes v^*_j.$$

Here V_z^{*S} is the dual evaluation representation of the typical representation with the weight $l_a = (l_a^1, l_a^2, \cdots, l_a^N)$ [13]. The basis of V^{*S} is given by $\{v^*_j\}$. The coefficients are linear maps : $\Phi^*_j(z) : F(p_a + l_a + x_a) \longrightarrow F(p_a)$. For $l_a = (l_a^1, l_a^2, \cdots, l_a^N)$ and $x_a = (x_a^1, x_a^2, \cdots, x_a^N)$, we set the highest element $\Phi^*_1(z)$ of the vertex operator by

$$\Phi^*_1(z) =: \prod_{j=1}^N \mathcal{Q}_j^{x_j} \quad e^{-\sum_{i,j=1}^N \left(\frac{l_a^i}{k+N-1}\frac{\mathrm{Min}(i,j)}{N-1}\frac{N-1-\mathrm{Max}(i,j)}{1}a^j\right)(q^k z|-\frac{k-N+1}{2})} :,$$

where $(x_1, x_2, \cdots, x_N)$ is related to x_a^i by $x_a^i = \sum_{j=1}^N A_{i,j}x_j$. Other elements $\Phi^*_j(z)$ $(j \neq 1)$ of the vertex operators are determined by the intertwining property. We conjecture that this bosonization of the vertex operator $\Phi^*(z)$ gives the intertwiner among our bosonization and the typical representation. Using the Gelfand-Zetlin basis [13], we have checked this conjecture in some cases for $N = 2, 3, 4$ [10]. We balance the "background charge" of the vertex operators by using the screening $\mathcal{Q}_i$. Sometimes we have to multiply nontrivial product of the screenings $\mathcal{Q}_i$ inside the vertex operator in order to have non-zero correlation functions (trace of vertex operators). At the end of this paper, we would like to give some comments on relating works [8, 12]. In [8] we study how to get the Wakimoto realization from our bosonization by ξ-η system. In [12] we study how to construct the elliptic deformed algebra $U_{q,p}(\widehat{sl}(M|N))$ from the quantum group $U_q(\widehat{sl}(M|N))$. Using deformation method developed in [12] we obtain a bosonization of the elliptic deformed algebra $U_{q,p}(\widehat{sl}(N|1))$ for an arbitrary level k.

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