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Screenings and vertex operators of quantum superalgebra $U_q(\widehat{sl}(N|1))$

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We construct the screening currents of the quantum superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \neq -N + 1$. We show that these screening currents commute with the superalgebra modulo total difference. We propose bosonizations of the vertex operators by using the screening currents. We check that these vertex operators are the intertwiners among the Fock-Wakimoto representation and the typical representation for rank $N \leq 4$. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4742125]

I. INTRODUCTION

Bosonizations are known to be a powerful method to construct correlation functions not only in conformal field theory,[1] but also in exactly solvable lattice model.[2] In the previous paper[3] we constructed a bosonization of the quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \in \mathbf{C}$. Bosonizations for an arbitrary level $k \in \mathbf{C}$ (Refs. 3, 5–11) are completely different from those of level $k = 1$.[12–21] This paper is a continuation of the paper.[3] In this paper we focus our attention on the screening currents that play an important role in level k bosonizations.[3] We construct the screening currents that commute with the quantum superalgebra $U_q(\widehat{sl}(N|1))$ modulo total difference, for an arbitrary level $k \neq -N + 1$. Using the screening currents, we construct the screening operators that commute with the quantum superalgebra. The screening currents are useful to study level k bosonizations that is not irreducible representation. For instance, (1) the screening currents balance the "background charge" of the vertex operators,[7,11,23–25] and (2) the irreducible representation is constructed from the Felder complex by the screening currents.[24,26–28,30] In this paper we focus our attention on the background charge problem. We propose bosonizations of the vertex operators[31] that are the intertwiners among the Fock-Wakimoto module and the typical representation, by using the screening operators. We check the intertwining property of these bosonizations of the vertex operators for rank $N \leq 4$. The screening currents and the vertex operators have been constructed only for $U_q(\widehat{sl}(N))$, $U_q(\widehat{sl}(2|1))$ (Refs. 7–9, and 11) by now. This paper gives a higher rank generalization of the screenings and the vertex operators in $U_q(\widehat{sl}(2|1))$ paper.[11] The representation theories of the superalgebra are much more complicated than non-superalgebra and have rich structures.[32–34,37]

This paper is organized as follows. In Sec. 2 we recall the Chevalley realization and the Drinfeld realization of the quantum affine superalgebra $U_q(\widehat{sl}(N|1))$. In Sec. 3 we give the bosonization of the quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level k. We propose the Fock-Wakimoto module by the ξ-η system. In Sec. 4 we introduce the screening currents that commute with the superalgebra modulo total difference, for an arbitrary level $k \neq -N + 1$. In Sec. 5 we propose bosonizations of the vertex operators of $U_q(\widehat{sl}(N|1))$. We give the level-zero representation of the Drinfeld generators for $U_q(\widehat{sl}(3|1))$ in this section (respectively, $U_q(\widehat{sl}(4|1))$ in Appendix B). We check that the vertex operators are the intertwiners among the Fock-Wakimoto realization and the typical representation of the quantum superalgebra $U_q(\widehat{sl}(N|1))$ for small rank $N \leq 4$. We show non-vanishing property of the correlation functions. In Appendix A we summarize useful formulae

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of the normal orderings. In Appendix B we summarize the level-zero representation of the Drinfeld generators for $U_q(\widehat{sl}(4|1))$.

II. QUANTUM AFFINE SUPERALGEBRA $U_q(\widehat{sl}(N|1))$

In this section we recall the definition of the quantum superalgebra $U_q(\widehat{sl}(N|1))$. Throughout this paper we fix a complex number $0 < |q| < 1$.

A. Chevalley generator

We recall the definition of the quantum superalgebra $U_q(\widehat{sl}(N|1))$ ($N = 2, 3, \ldots$) in terms of the Chevalley generators.[35] The Cartan matrix of the affine superalgebra $\widehat{sl}(N|1)$ is given by

$$(A_{i,j})_{0\le i,j\le N} = \begin{pmatrix} 0 & -1 & 0 & \cdots & \cdots & 0 & 1 \\ -1 & 2 & -1 & \cdots & \cdots\cdots & & 0 \\ 0 & -1 & 2 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 2 & -1 & 0 \\ 0 & \cdots & \cdots & \cdots & -1 & 2 & -1 \\ 1 & 0 & \cdots & \cdots & 0 & -1 & 0 \end{pmatrix}. \quad (2.1)$$

We introduce the orthonormal basis $\{\epsilon_i | i = 1, 2, \ldots, N+1\}$ with the bilinear form, $(\epsilon_i|\epsilon_j) = \nu_i \delta_{i,j}$, where $\nu_j = +(j = 1, 2, \ldots, N)$ and $\nu_{N+1} = -$. Define $\bar{\epsilon}_i = \epsilon_i - \frac{\nu_i}{N-1}\sum_{j=1}^{N+1}\epsilon_j$. Note that $\sum_{j=1}^{N}\bar{\epsilon}_j = 0$. The classical simple roots $\bar{\alpha}_i$ and the classical fundamental weights $\bar{\Lambda}_i$ are defined by $\bar{\alpha}_i = \nu_i\epsilon_i - \nu_{i+1}\epsilon_{i+1}$, $\bar{\Lambda}_i = \sum_{j=1}^{i}\bar{\epsilon}_j (1 \le i \le N)$. Introduce the affine weight Λ_0 and the null root δ satisfying $(\Lambda_0|\Lambda_0) = (\delta|\delta) = 0$, $(\Lambda_0|\delta) = 1$, $(\Lambda_0|\epsilon_i) = 0$, $(\delta|\epsilon_i) = 0$, $(1 \le i \le N)$. The other affine weights and the affine roots are given by $\alpha_0 = \delta - \sum_{j=1}^{N}\bar{\alpha}_j$, $\alpha_i = \bar{\alpha}_i$, $\Lambda_i = \bar{\Lambda}_i + \Lambda_0$, $(1 \le i \le N)$. Let $P = \oplus_{j=1}^{N}\mathbf{Z}\Lambda_j \oplus \mathbf{Z}\delta$ and $P^* = \oplus_{j=1}^{N}\mathbf{Z}h_j \oplus \mathbf{Z}d$ the affine $\widehat{sl}(N|1)$ weight lattice and its dual lattice, respectively.

Definition 2.1 (Ref. 35): The quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ is generated by the Chevalley generators $h_i, e_i, f_i (1 \le i \le N)$. The $\mathbf{Z}_2$-grading of the generators is $|e_0| = |f_0| = |e_N| = |f_N| = 1$ and zero otherwise. The defining relations are given by the Cartan-Kac relations and the Serre relations.

The Cartan-Kac relations: *For $N \ge 2$, $0 \le i, j \le N$, the generators subject to the following relations.*

$$[h_i, h_j] = 0, \quad [h_i, e_j] = A_{i,j}e_j, \quad [h_i, f_j] = -A_{i,j}f_j, \quad [e_i, f_j] = \delta_{i,j}\frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}. \quad (2.2)$$

The Serre relations: *For $N \ge 2$, the generators subject to the following relations for $1 \le i \le N-1$, $0 \le j \le N$ such that $|A_{i,j}| = 1$.*

$$[e_i, [e_i, e_j]_{q^{-1}}]_q = 0, \quad [f_i, [f_i, f_j]_{q^{-1}}]_q = 0. \quad (2.3)$$

For $N \ge 2$, the generators subject to the following relations for $0 \le i, j \le N$ such that $|A_{i,j}| = 0$.

$$[e_i, e_j] = 0, \quad [f_i, f_j] = 0. \quad (2.4)$$

For $N \geq 3$, the Serre relations of fourth degree hold.

$$[e_N,[e_0,[e_N,e_{N-1}]_{q^{-1}}]_q]=0, \qquad [e_0,[e_1,[e_0,e_N]_q]_{q^{-1}}]=0,$$
$$[f_N,[f_0,[f_N,f_{N-1}]_{q^{-1}}]_q]=0, \qquad [f_0,[f_1,[f_0,f_N]_q]_{q^{-1}}]=0. \tag{2.5}$$

For $N=2$, the extra Serre relations of fifth degree hold.

$$[e_2,[e_0,[e_2,[e_0,e_1]_q]]]_{q^{-1}}=[e_0,[e_2,[e_0,[e_2,e_1]_q]]]_{q^{-1}},$$
$$[f_2,[f_0,[f_2,[f_0,f_1]_q]]]_{q^{-1}}=[f_0,[f_2,[f_0,[f_2,f_1]_q]]]_{q^{-1}}. \tag{2.6}$$

Here and throughout this paper, we use the notations

$$[X,Y]_\xi = XY-(-1)^{|X||Y|}\xi YX. \tag{2.7}$$

We write $[X, Y]_1$ as $[X, Y]$ for simplicity. The quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ has the $\mathbf{Z}_2$-graded Hopf-algebra structure. We take the following coproduct

$$\Delta(e_i)=e_i\otimes 1+q^{h_i}\otimes e_i, \quad \Delta(f_i)=f_i\otimes q^{-h_i}+1\otimes f_i, \quad \Delta(h_i)=h_i\otimes 1+1\otimes h_i, \tag{2.8}$$

and the antipode

$$S(e_i)=-q^{-h_i}e_i, \quad S(f_i)=-f_iq^{h_i}, \quad S(h_i)=-h_i. \tag{2.9}$$

The coproduct Δ satisfies an algebra automorphism $\Delta(XY)=\Delta(X)\Delta(Y)$ and the antipode S satisfies a $\mathbf{Z}_2$-graded algebra anti-automorphism $S(XY)=(-1)^{|X||Y|}S(Y)S(X)$. The multiplication rule for the tensor product is $\mathbf{Z}_2$-graded and is defined for homogeneous elements $X, Y, X', Y' \in U_q(\widehat{sl}(N|1))$ and $v \in V, w \in W$ by $X\otimes Y\cdot X'\otimes Y'=(-1)^{|Y||X'|}XX'\otimes YY'$ and $X\otimes Y\cdot v\otimes w=(-1)^{|Y||v|}Xv\otimes Yw$, which extends to inhomogeneous elements through linearity.

We sometimes use the anti-commutator $\{X, Y\} = XY + YX = [X, Y]_1$ for $|X| = |Y| = 1$.

B. Drinfeld realization

We recall the Drinfeld's second realization of the quantum affine superalgebra $U_q(\widehat{sl}(N|1))$.[35,36] The Drinfeld realization is convenient for constructions of bosonizations. We use the standard symbol of q-integer

$$[a]=\frac{q^a-q^{-a}}{q-q^{-1}}. \tag{2.10}$$

Definition 2.2 (Ref. 35): The Drinfeld generators of the quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ are $X_{i,m}^{\pm}, h_{i,m}, c$ $(1\leq i\leq N, m\in\mathbf{Z})$. The $\mathbf{Z}_2$-grading of the Drinfeld generators are: $|X_{N,m}^{\pm}|=1$ ($m \in \mathbf{Z}$) and zero otherwise. Defining relations are

$$c:\text{central}, [h_i,h_{j,m}]=0, \tag{2.11}$$

$$[h_{i,m},h_{j,n}]=\frac{[A_{i,j}m][cm]}{m}\delta_{m+n,0}(m,n\neq 0), \tag{2.12}$$

$$[h_i,X_j^{\pm}(z)]=\pm A_{i,j}X_j^{\pm}(z), \tag{2.13}$$

$$[h_{i,m},X_j^{+}(z)]=\frac{[A_{i,j}m]}{m}q^{-\frac{c}{2}|m|}z^mX_j^{+}(z)(m\neq 0), \tag{2.14}$$

$$[h_{i,m},X_j^{-}(z)]=-\frac{[A_{i,j}m]}{m}q^{\frac{c}{2}|m|}z^mX_j^{-}(z)(m\neq 0), \tag{2.15}$$

$$(z_1 - q^{\pm A_{i,j}} z_2) X_i^{\pm}(z_1) X_j^{\pm}(z_2) = (q^{\pm A_{j,i}} z_1 - z_2) X_j^{\pm}(z_2) X_i^{\pm}(z_1) \text{for} |A_{i,j}| \neq 0, \tag{2.16}$$

$$[X_i^{\pm}(z_1), X_j^{\pm}(z_2)] = 0 \text{for} |A_{i,j}| = 0, \tag{2.17}$$

$$[X_i^{+}(z_1), X_j^{-}(z_2)] = \frac{\delta_{i,j}}{(q-q^{-1}) z_1 z_2} \left(\delta(q^{-c} z_1/z_2) \Psi_i^{+}(q^{\frac{c}{2}} z_2) - \delta(q^{c} z_1/z_2) \Psi_i^{-}(q^{-\frac{c}{2}} z_2) \right), \tag{2.18}$$

$$\left[X_i^{\pm}(z_1), \left[X_i^{\pm}(z_2), X_j^{\pm}(z) \right]_{q^{-1}} \right]_q + (z_1 \leftrightarrow z_2) = 0 \text{ for } |A_{i,j}| = 1, i \neq N, \tag{2.19}$$

where we have used $\delta(z) = \sum_{m \in \mathbf{Z}} z^m$. *Here we have used the abbreviation* $h_i = h_{i,0}$. *We have used the generating function*

$$X_j^{\pm}(z) = \sum_{m \in \mathbf{Z}} X_{j,m}^{\pm} z^{-m-1}, \tag{2.20}$$

$$\Psi_i^{+}(z) = q^{h_i} \exp\left((q-q^{-1}) \sum_{m>0} h_{i,m} z^{-m} \right), \tag{2.21}$$

$$\Psi_i^{-}(z) = q^{-h_i} \exp\left(-(q-q^{-1}) \sum_{m>0} h_{i,-m} z^{m} \right). \tag{2.22}$$

The relations between the Chevalley generators and the Drinfeld realization are given by

$$h_i = h_{i,0}, e_i = X_{i,0}^{+}, f_i = X_{i,0}^{-} \text{ for } 1 \le i \le N, \tag{2.23}$$

$$h_0 = c - (h_{1,0} + \cdots + h_{N,0}), \tag{2.24}$$

$$e_0 = (-1)[X_{N,0}^{-} \cdots, [X_{3,0}^{-}, [X_{2,0}^{-}, X_{1,1}^{-}]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-1}} q^{-h_{1,0}-h_{2,0}-\cdots-h_{N,0}}, \tag{2.25}$$

$$f_0 = q^{h_{1,0}+h_{2,0}+\cdots+h_{N,0}} [\cdots [[X_{1,-1}^{+}, X_{2,0}^{+}]_q, X_{3,0}^{+}]_q, \cdots X_{N,0}^{+}]_q. \tag{2.26}$$

III. BOSONIZATION OF $U_q(\widehat{sl}(N|1))$

In this section we recall the bosonization of $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \in \mathbf{C}$.[3]

A. Boson

We introduce the bosons and the zero-mode operators a_m^j, Q_a^j ($m \in \mathbf{Z}$, $1 \le j \le N$), $b_m^{i,j}$, $Q_b^{i,j}$, $c_m^{i,j}$, $Q_c^{i,j}$($m \in \mathbf{Z}$, $1 \le i < j \le N+1$). The bosons $a_m^i, b_m^{i,j}, c_m^{i,j}$, ($m \in \mathbf{Z}_{\neq 0}$) and the zero-mode operators a_0^i, Q_a^i, $b_0^{i,j}$, $Q_b^{i,j}$, $c_0^{i,j}$, $Q_c^{i,j}$ that satisfy

$$[a_m^i, a_n^j] = \frac{[(k+N-1)m][A_{i,j}m]}{m} \delta_{m+n,0}, [a_0^i, Q_a^j] = (k+N-1) A_{i,j}, \tag{3.1}$$

$$[b_m^{i,j}, b_n^{i',j'}] = -\nu_i \nu_j \frac{[m]^2}{m} \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, [b_0^{i,j}, Q_b^{i',j'}] = -\nu_i \nu_j \delta_{i,i'} \delta_{j,j'}, \tag{3.2}$$

$$[c_m^{i,j}, c_n^{i',j'}] = \nu_i \nu_j \frac{[m]^2}{m} \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, [c_0^{i,j}, Q_c^{i',j'}] = \nu_i \nu_j \delta_{i,i'} \delta_{j,j'}, \tag{3.3}$$

and other commutators vanish. We impose the cocycle condition on the zero-mode operator $Q_b^{i,j}$ ($1 \le i < j \le N+1$) by

$$[Q_b^{i,j}, Q_b^{i',j'}] = \delta_{j,N+1}\delta_{j',N+1}\pi\sqrt{-1} \quad \text{for } (i,j) \neq (i',j'). \tag{3.4}$$

We have the following (anti)commutation relations

$$\left[\exp\left(Q_b^{i,j}\right), \exp\left(Q_b^{i',j'}\right)\right] = 0 \quad (1 \le i < j \le N, 1 \le i' < j' \le N), \tag{3.5}$$

$$\left\{\exp\left(Q_b^{i,N+1}\right), \exp\left(Q_b^{j,N+1}\right)\right\} = 0 \quad (1 \le i \neq j \le N). \tag{3.6}$$

In what follows we use the standard normal ordering symbol ::. We set $b^{i,j}(z)$, $c^{i,j}(z)$, $b_\pm^{i,j}(z)$, $a_\pm^j(z)$, and $\left(\frac{\gamma_1}{\beta_1}\frac{\gamma_2}{\beta_2}\cdots\frac{\gamma_r}{\beta_r}a^i\right)(z|\alpha)$ by

$$a_\pm^j(z) = \pm(q-q^{-1})\sum_{\pm m>0} a_m^j z^{-m} \pm a_0^j \log q, \tag{3.7}$$

$$b^{i,j}(z) = -\sum_{m\neq 0}\frac{b_m^{i,j}}{[m]}z^{-m} + Q_b^{i,j} + b_0^{i,j}\log z, \tag{3.8}$$

$$b_\pm^{i,j}(z) = \pm(q-q^{-1})\sum_{\pm m>0} b_m^{i,j} z^{-m} \pm b_0^{i,j}\log q, \tag{3.9}$$

$$c^{i,j}(z) = -\sum_{m\neq 0}\frac{c_m^{i,j}}{[m]}z^{-m} + Q_c^{i,j} + c_0^{i,j}\log z, \tag{3.10}$$

$$\left(\frac{\gamma_1}{\beta_1}\frac{\gamma_2}{\beta_2}\cdots\frac{\gamma_r}{\beta_r}a^i\right)(z|\alpha) = -\sum_{m\neq 0}\frac{[\gamma_1 m]\cdots[\gamma_r m]}{[\beta_1 m]\cdots[\beta_r m]}\frac{a_m^i}{[m]}q^{-\alpha|m|}z^{-m} + \frac{\gamma_1\cdots\gamma_r}{\beta_1\cdots\beta_r}(Q_a^i + a_0^i\log z). \tag{3.11}$$

B. Bosonization of $U_q(\widehat{sl}(N|1))$

We recall the bosonizations of the quantum superalgebra $U_q(\widehat{sl}(N|1))$.

Theorem 3.1 (Ref. 3): *A bosonization of the quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \in \mathbf{C}$ is given as follows. For $1 \le i \le N-1$ we set*

$$X_i^+(z) = \frac{1}{(q-q^{-1})z}\sum_{j=1}^{i}(X_i^{+(j,1)}(z) - X_i^{+(j,2)}(z)), \tag{3.12}$$

$$X_N^+(z) = q^{N-2}\sum_{j=1}^{N} X_N^{+(j,0)}(z), \tag{3.13}$$

$$X_i^-(z) = \frac{1}{(q-q^{-1})z}\left(\sum_{j=1}^{i-1}(X_i^{-(j,1)}(z) - X_i^{-(j,2)}(z)) + (X_i^{-(i,1)}(z) - X_i^{-(i,2)}(z)) - \sum_{j=i+1}^{N-1}(X_i^{-(j,1)}(z) - X_i^{-(j,2)}(z))\right) + q^{k+N-1}X_i^{-(N,0)}(z), \tag{3.14}$$

$$X_N^-(z) = \frac{1}{(q-q^{-1})z}\sum_{j=1}^{N} q^{-N+j+1}\left(-X_N^{-(j,1)}(z) + X_N^{-(j,2)}(z)\right), \tag{3.15}$$

$$\Psi_i^{\pm}(q^{\pm\frac{k}{2}}z) = \exp\Bigg(a_{\pm}^i(q^{\pm\frac{k+N-1}{2}}z) + \sum_{l=1}^{i}(b_{\pm}^{l,i+1}(q^{\pm(l+k-1)}z) - b_{\pm}^{l,i}(q^{\pm(l+k)}z)) + \sum_{l=i+1}^{N}(b_{\pm}^{i,l}(q^{\pm(k+l)}z) - b_{\pm}^{i-1,l}(q^{\pm(k+l-1)}z)) + b_{\pm}^{i,N+1}(q^{\pm(k+N)}z) - b_{\pm}^{i+1,N+1}(q^{\pm(k+N-1)}z)\Bigg), \quad (3.16)$$

$$\Psi_N^{\pm}(q^{\pm\frac{k}{2}}z) = \exp\Bigg(a_{\pm}^N(q^{\pm\frac{k+N-1}{2}}z) - \sum_{l=1}^{N-1}(b_{\pm}^{l,N}(q^{\pm(k+l)}z) + b_{\pm}^{l,N+1}(q^{\pm(k+l)}z))\Bigg). \quad (3.17)$$

Here we have used the auxiliary bosonic operators $X_i^{\pm(j,s)}(z)$ *as follows.*

For $1 \le i \le N-1$ *and* $1 \le j \le i$ *we set*

$$X_i^{+(j,1)}(z) = :\exp\Bigg((b+c)^{j,i}(q^{j-1}z) + b_+^{j,i+1}(q^{j-1}z) - (b+c)^{j,i+1}(q^jz) + \sum_{l=1}^{j-1}(b_+^{l,i+1}(q^{l-1}z) - b_+^{l,i}(q^lz))\Bigg):, \quad (3.18)$$

$$X_i^{+(j,2)}(z) = :\exp\Bigg((b+c)^{j,i}(q^{j-1}z) + b_-^{j,i+1}(q^{j-1}z) - (b+c)^{j,i+1}(q^{j-2}z) + \sum_{l=1}^{j-1}(b_+^{l,i+1}(q^{l-1}z) - b_+^{l,i}(q^lz))\Bigg):. \quad (3.19)$$

For $1 \le j \le N$ *we set*

$$X_N^{+(j,0)}(z) = :\exp\Bigg((b+c)^{j,N}(q^{j-1}z) + b^{j,N+1}(q^{j-1}z) - \sum_{l=1}^{j-1}(b_+^{l,N+1}(q^lz) + b_+^{l,N}(q^lz))\Bigg):. \quad (3.20)$$

For $1 \le i \le N-1$ *and* $1 \le j \le i-1$ *we set*

$$X_i^{-(j,1)}(z) = :\exp\Bigg(a_-^i(q^{-\frac{k+N-1}{2}}z) + (b+c)^{j,i+1}(q^{-k-j}z) - b_-^{j,i}(q^{-k-j}z) - (b+c)^{j,i}(q^{-k-j+1}z) + \sum_{l=j+1}^{i}(b_-^{l,i+1}(q^{-k-l+1}z) - b_-^{l,i}(q^{-k-l}z)) + \sum_{l=i+1}^{N}(b_-^{i,l}(q^{-k-l}z) - b_-^{i+1,l}(q^{-k-l+1}z)) + b_-^{i,N+1}(q^{-k-N}z) - b_-^{i+1,N+1}(q^{-k-N+1}z)\Bigg):, \quad (3.21)$$

$$X_i^{-(j,2)}(z) = :\exp\Bigg(a_-^i(q^{-\frac{k+N-1}{2}}z) + (b+c)^{j,i+1}(q^{-k-j}z) - b_+^{j,i}(q^{-k-j}z) - (b+c)^{j,i}(q^{-k-j-1}z) + \sum_{l=j+1}^{i}(b_-^{l,i+1}(q^{-k-l+1}z) - b_-^{l,i}(q^{-k-l}z)) + \sum_{l=i+1}^{N}(b_-^{i,l}(q^{-k-l}z) - b_-^{i+1,l}(q^{-k-l+1}z)) + b_-^{i,N+1}(q^{-k-N}z) - b_-^{i+1,N+1}(q^{-k-N+1}z)\Bigg):. \quad (3.22)$$

For $1 \le i \le N-1$ *we set*

$$X_i^{-(i,1)}(z) = :\exp\Bigg(a_-^i(q^{-\frac{k+N-1}{2}}z) + (b+c)^{i,i+1}(q^{-k-i}z) + \sum_{l=i+1}^{N}(b_-^{i,l}(q^{-k-l}z) - b_-^{i+1,l}(q^{-k-l+1}z)) + b_-^{i,N+1}(q^{-k-N}z) - b_-^{i+1,N+1}(q^{-k-N+1}z)\Bigg):, \quad (3.23)$$

$$X_i^{-(i,2)}(z) = :\exp\left(a_+^i(q^{\frac{k+N-1}{2}}z) + (b+c)^{i,i+1}(q^{k+i}z) + \sum_{l=i+1}^{N}(b_+^{i,l}(q^{k+l}z) - b_+^{i+1,l}(q^{k+l-1}z))\right.$$
$$\left. + b_+^{i,N+1}(q^{k+N}z) - b_+^{i+1,N+1}(q^{k+N-1}z)\right):. \tag{3.24}$$

For $1 \le i \le N-1$ *and* $i+1 \le j \le N-1$ *we set*

$$X_i^{-(j,1)}(z) = :\exp\left(a_+^i(q^{\frac{k+N-1}{2}}z) + (b+c)^{i,j+1}(q^{k+j}z) + b_+^{i+1,j+1}(q^{k+j}z) - (b+c)^{i+1,j+1}(q^{k+j+1}z)\right.$$
$$\left. + \sum_{l=j+1}^{N}(b_+^{i,l}(q^{k+l}z) - b_+^{i+1,l}(q^{k+l-1}z)) + b_+^{i,N+1}(q^{k+N}z) - b_+^{i+1,N+1}(q^{k+N-1}z)\right):, \tag{3.25}$$

$$X_i^{-(j,2)}(z) = :\exp\left(a_+^i(q^{\frac{k+N-1}{2}}z) + (b+c)^{i,j+1}(q^{k+j}z) + b_-^{i+1,j+1}(q^{k+j}z) - (b+c)^{i+1,j+1}(q^{k+j-1}z)\right.$$
$$\left. + \sum_{l=j+1}^{N}(b_+^{i,l}(q^{k+l}z) - b_+^{i+1,l}(q^{k+l-1}z)) + b_+^{i,N+1}(q^{k+N}z) - b_+^{i+1,N+1}(q^{k+N-1}z)\right):. \tag{3.26}$$

For $1 \le i \le N-1$ *we set*

$$X_i^{-(N,0)}(z) = :\exp\left(a_+^i(q^{\frac{k+N-1}{2}}z) - b^{i,N+1}(q^{k+N-1}z) - b_+^{i+1,N+1}(q^{k+N-1}z) + b^{i+1,N+1}(q^{k+N}z)\right):. \tag{3.27}$$

For $1 \le j \le N-1$ *we set*

$$X_N^{-(j,1)}(z) = :\exp\left(a_-^N(q^{-\frac{k+N-1}{2}}z) - b_-^{j,N}(q^{-k-j}z) - (b+c)^{j,N}(q^{-k-j+1}z)\right. \tag{3.28}$$

$$\left. - b_-^{j,N+1}(q^{-k-j}z) - b^{j,N+1}(q^{-k-j+1}z) - \sum_{l=j+1}^{N-1}(b_-^{l,N}(q^{-k-l}z) + b_-^{l,N+1}(q^{-k-l}z))\right):,$$

$$X_N^{-(j,2)}(z) = :\exp\left(a_-^N(q^{-\frac{k+N-1}{2}}z) - b_+^{j,N}(q^{-k-j}z) - (b+c)^{j,N}(q^{-k-j-1}z)\right. \tag{3.29}$$

$$\left. - b_+^{j,N+1}(q^{-k-j}z) - b^{j,N+1}(q^{-k-j-1}z) - \sum_{l=j+1}^{N-1}(b_-^{l,N}(q^{-k-l}z) + b_-^{l,N+1}(q^{-k-l}z))\right):,$$

$$X_N^{-(N,1)}(z) = :\exp\left(a_-^N(q^{-\frac{k+N-1}{2}}z) - b^{N,N+1}(q^{-k-N+1}z)\right):, \tag{3.30}$$

$$X_N^{-(N,2)}(z) = :\exp\left(a_+^N(q^{\frac{k+N-1}{2}}z) - b^{N,N+1}(q^{k+N-1}z)\right):. \tag{3.31}$$

The $\mathbf{Z}_2$*-grading is :* $|X_N^{\pm(j,s)}(z)| = 1$ *and zero otherwise.*

Very explicitly, we have

$$h_{i,m} = q^{-\frac{N-1}{2}|m|}a_m^i + \sum_{l=1}^{i}(q^{-(\frac{k}{2}+l-1)|m|}b_m^{l,i+1} - q^{-(\frac{k}{2}+l)|m|}b_m^{l,i})$$
$$+ \sum_{l=i+1}^{N}(q^{-(\frac{k}{2}+l)|m|}b_m^{i,l} - q^{-(\frac{k}{2}+l-1)|m|}b_m^{i+1,l})$$
$$+q^{-(\frac{k}{2}+N)|m|}b_m^{i,N+1} - q^{-(\frac{k}{2}+N-1)|m|}b_m^{i+1,N+1}(1 \le i \le N-1), \quad (3.32)$$

$$h_{N,m} = q^{-\frac{N-1}{2}|m|}a_m^N - \sum_{l=1}^{N-1}(q^{-(\frac{k}{2}+l)|m|}b_m^{l,N} + q^{-(\frac{k}{2}+l)|m|}b_m^{l,N+1}). \quad (3.33)$$

C. Fock-Wakimoto module

We introduce the vacuum state $|0\rangle$ of the boson Fock space by

$$a_m^i|0\rangle = b_m^{i,j}|0\rangle = c_m^{i,j}|0\rangle = 0 (m \ge 0). \quad (3.34)$$

For $p_a^i \in \mathbf{C}$ $(1 \le i \le N)$, $p_b^{i,j} \in \mathbf{C}$ $(1 \le i < j \le N+1)$, $p_c^{i,j} \in \mathbf{C}$ $(1 \le i < j \le N)$, we set

$$|p_a, p_b, p_c\rangle \quad (3.35)$$
$$= \exp\left(\sum_{i,j=1}^{N}\frac{\mathrm{Min}(i,j)(N-1-\mathrm{Max}(i,j))}{(N-1)(k+N-1)}p_a^iQ_a^j - \sum_{1\le i<j\le N+1}p_b^{i,j}Q_b^{i,j} + \sum_{1\le i<j\le N}p_c^{i,j}Q_c^{i,j}\right)|0\rangle.$$

It satisfies

$$a_0^i|p_a, p_b, p_c\rangle = p_a^i|p_a, p_b, p_c\rangle, b_0^{i,j}|p_a, p_b, p_c\rangle = p_b^{i,j}|p_a, p_b, p_c\rangle, c_0^{i,j}|p_a, p_b, p_c\rangle$$
$$= p_c^{i,j}|p_a, p_b, p_c\rangle. \quad (3.36)$$

The boson Fock space $F(p_a, p_b, p_c)$ is generated by the bosons a_m^i, $b_m^{i,j}$, $c_m^{i,j}$ on the vector $|p_a, p_b, p_c\rangle$. We set the space $F(p_a)$ by

$$F(p_a) = \bigoplus_{\substack{p_b^{i,j}=-p_c^{i,j}\in\mathbf{Z}(1\le i<j\le N)\\ p_b^{i,N+1}\in\mathbf{Z}(1\le i\le N)}} F(p_a, p_b, p_c). \quad (3.37)$$

We impose the restriction $p_b^{i,j} = -p_c^{i,j} \in \mathbf{Z}$ $(1 \le i < j \le N)$, because the $X_{i,m}^{\pm}$ change $Q_b^{i,j} + Q_c^{i,j}$. The $F(p_a)$ is $U_q(\widehat{sl}(N|1))$-module. We set the vector $|\lambda\rangle = |p_a, 0, 0\rangle$ upon the specialization $p_b^{i,j} = 0 (1 \le i < j \le N+1)$ and $p_c^{i,j} = 0 (1 \le i < j \le N)$.

Proposition 3.2: The $|\lambda\rangle = |p_a, 0, 0\rangle$ is the highest weight vector of the highest weight whose classical part is $\bar{\lambda} = \sum_{j=1}^{N} p_a^j \bar{\Lambda}_j$.

$$h_{i,m}|\lambda\rangle = 0, \quad X_{i,m}^{\pm}|\lambda\rangle = 0, \quad (m > 0), \quad (3.38)$$

$$X_{i,0}^{+}|\lambda\rangle = 0, \quad h_{i,0}|\lambda\rangle = p_a^i|\lambda\rangle. \quad (3.39)$$

Using the highest weight vector $|\lambda\rangle$, we have the highest weight module $V(\lambda)$ of $U_q(\widehat{sl}(N|1))$.

$$V(\lambda) \subset F(p_a). \quad (3.40)$$

The module $F(p_a)$ is not irreducible. In Ref. 27 the irreducible highest weight module $L(\lambda)$ for the affine algebra $\widehat{sl}(2)$ was constructed from the Fock-Wakimoto module on the boson Fock space[5] by the Felder complex. We recall the realizations of $\widehat{sl}(2)$ in Ref. 29 and $U_q(\widehat{sl}(2))$ in Refs. 7, 8, and 24. The realizations of Refs. 7, 8, 24, and 29 are different from those of Ref. 27 and are called the bosonic ghost system. In the bosonic ghost system, the irreducible highest weight module $L(\lambda)$ was constructed from the similar space as $F(p_a)$ by two steps; the first step is the construction of the submodule by the ξ-η system, and the second step is the resolution by the Felder complex. The submodule induced by the ξ-η system plays similar role as the Fock-Wakimoto module in Ref. 27. We call this submodule of the bosonic ghost system the Fock-Wakimoto module. Our bosonization[3] is a bosonic ghost system. In this paper we study the ξ-η system and propose the Fock-Wakimoto module for $U_q(\widehat{sl}(N|1))$.

Definition 3.3: We introduce the operators $\xi_m^{i,j}$ and $\eta_m^{i,j}$ $(1 \le i < j \le N, m \in \mathbf{Z})$ by

$$\eta^{i,j}(z) = \sum_{m\in\mathbf{Z}} \eta_m^{i,j} z^{-m-1} =: e^{c^{i,j}(z)} :, \quad \xi^{i,j}(z) = \sum_{m\in\mathbf{Z}} \xi_m^{i,j} z^{-m} =: e^{-c^{i,j}(z)} : . \tag{3.41}$$

The Fourier components $\eta_m^{i,j} = \oint \frac{dz}{2\pi\sqrt{-1}} z^m \eta^{i,j}(z)$, $\xi_m^{i,j} = \oint \frac{dz}{2\pi\sqrt{-1}} z^{m-1}\xi^{i,j}(z)$ $(m \in \mathbf{Z})$ are well defined on the space $F(p_a)$. They satisfy the anti-commutation relations.

$$\{\eta_m^{i,j}, \xi_n^{i,j}\} = \delta_{m+n,0}, \{\eta_m^{i,j}, \eta_n^{i,j}\} = \{\xi_m^{i,j}, \xi_n^{i,j}\} = 0 \quad (1 \le i < j \le N). \tag{3.42}$$

They commute with each other

$$[\eta_m^{i,j}, \xi_n^{i',j'}] = [\eta_m^{i,j}, \eta_n^{i',j'}] = [\xi_m^{i,j}, \xi_n^{i',j'}] = 0 \quad (i, j) \neq (i', j'). \tag{3.43}$$

We focus our attention on the operators $\eta_0^{i,j}, \xi_0^{i,j}$ satisfying $(\eta_0^{i,j})^2 = 0, (\xi_0^{i,j})^2 = 0$. They satisfy

$$\mathrm{Im}(\eta_0^{i,j}) = \mathrm{Ker}(\eta_0^{i,j}), \quad \mathrm{Im}(\xi_0^{i,j}) = \mathrm{Ker}(\xi_0^{i,j}). \tag{3.44}$$

The products $\eta_0^{i,j}\xi_0^{i,j}$ and $\xi_0^{i,j}\eta_0^{i,j}$ are the projection operators, which satisfy

$$\eta_0^{i,j}\xi_0^{i,j} + \xi_0^{i,j}\eta_0^{i,j} = 1, \tag{3.45}$$

and

$$(\eta_0^{i,j}\xi_0^{i,j})^2 = \eta_0^{i,j}\xi_0^{i,j}, (\xi_0^{i,j}\eta_0^{i,j})^2 = \xi_0^{i,j}\eta_0^{i,j}, (\xi_0^{i,j}\eta_0^{i,j})(\eta_0^{i,j}\xi_0^{i,j}) = 0, (\eta_0^{i,j}\xi_0^{i,j})(\xi_0^{i,j}\eta_0^{i,j}) = 0. \tag{3.46}$$

Hence we have a direct sum decomposition.

$$F(p_a) = \eta_0^{i,j}\xi_0^{i,j}F(p_a) \oplus \xi_0^{i,j}\eta_0^{i,j}F(p_a), \tag{3.47}$$

and

$$\mathrm{Ker}(\eta_0^{i,j}) = \eta_0^{i,j}\xi_0^{i,j}F(p_a), \quad \mathrm{Coker}(\eta_0^{i,j}) = \xi_0^{i,j}\eta_0^{i,j}F(p_a) = F_\Lambda/(\eta_0^{i,j}\xi_0^{i,j})F_\Lambda. \tag{3.48}$$

We set

$$\eta_0 = \prod_{1\le i<j\le N} \eta_0^{i,j}, \quad \xi_0 = \prod_{1\le i<j\le N} \xi_0^{i,j}. \tag{3.49}$$

Definition 3.4: We introduce the subspace $\mathcal{F}(p_a)$ by

$$\mathcal{F}(p_a) = \eta_0\xi_0 F(p_a). \tag{3.50}$$

The operators $\eta_0^{i,j}, \xi_0^{i,j}$ commute with the operators $X_{i',j'}^{\pm}(z), \Psi_{i'}^{\pm}(z)$ up to sign $\pm$. When we set the operators $\widetilde{X}_i^{\pm}(z), \widetilde{\Psi}_i^{\pm}(z)$ by the conditions $\widetilde{X}_i^{\pm}(z)\eta_0^{i',j'} = \eta_0^{i',j'}X_i^{\pm}(z)$, $\widetilde{\Psi}_i^{\pm}(z)\eta_0^{i',j'} = \eta_0^{i',j'}\Psi_i^{\pm}(z)$, the bosonic operators $\widetilde{X}_i^{\pm}(z), \widetilde{\Psi}_i^{\pm}(z)$ give a bosonization of $U_q(\widehat{sl}(N|1))$ again.

Proposition 3.5 (Ref. 4): The subspace $\mathcal{F}(p_a)$ is the $U_q(\widehat{sl}(N|1))$ module

We call the submodule $\mathcal{F}(p_a)$ the Fock-Wakimoto module. It is expected that we have the irreducible highest weight module $L(\lambda)$ with the highest weight λ, whose classical part $\bar{\lambda} = \sum_{j=1}^N p_a^i \bar{\Lambda}_i$, by the Felder complex. The construction of the Felder complex is open problem even for non-superalgebra $U_q(\widehat{sl}(3))$. We would like to report the Felder complex of $U_q(\widehat{sl}(N))$ and $U_q(\widehat{sl}(N|1))$ in the future publications.

IV. SCREENING CURRENT

In this section we introduce the screening operators S_i ($i = 1, 2, \dots, N$), which commute with $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \neq -N + 1$. We need the screening operators to construct the vertex operators.

A. Screening current

We set the q-difference operators with a parameter α by

$$({}_\alpha\partial_z f)(z) = \frac{f(q^\alpha z) - f(q^{-\alpha} z)}{(q - q^{-1})z}. \tag{4.1}$$

The Jackson integral with parameter $p \in \mathbf{C}$ ($|p| < 1$) and $s \in \mathbf{C}^*$ is defined by

$$\int_0^{s\infty} f(z) d_p z = s(1-p) \sum_{m \in \mathbf{Z}} f(sp^m) p^m. \tag{4.2}$$

The Jackson integral satisfies

$$\int_0^{s\infty} ({}_\alpha\partial_z f)(z) d_p z = 0 \quad (p = q^{2\alpha}). \tag{4.3}$$

For $r \in \mathbf{C}$ ($\mathrm{Re}(r) > 0$) we introduce the Jacobi elliptic theta function

$$[u]_r = q^{\frac{u^2}{r} - u} \frac{\Theta_{q^{2r}}(q^{2u})}{(q^{2r}; q^{2r})_\infty^3}, \tag{4.4}$$

where we have used

$$\Theta_p(z) = (z; p)_\infty (pz^{-1}; p)_\infty (p; p)_\infty, \quad (z; p)_\infty = \prod_{m=0}^{\infty} (1 - p^m z). \tag{4.5}$$

The Jacobi elliptic theta function satisfies the quasi-periodicity property

$$[u + r]_r = -[u]_r, \quad [u + r\tau]_r = -e^{-\pi i \tau - \frac{2\pi i}{r} u} [u]_r, \tag{4.6}$$

where τ such that $\mathrm{Im}(\tau) > 0$ is given by $q^{2r} = e^{-\frac{2\pi i}{\tau}}$.

Definition 4.1: We introduce the bosonic operators $S_i(z)$ ($i = 1, 2, \dots, N$) that we call the screening current as follows.

$$S_i(z) = \frac{1}{(q - q^{-1})z} \sum_{j=i+1}^{N} (S_i^{(j,1)}(z) - S_i^{(j,2)}(z)) + q S_i^{(N+1,0)}(z) (1 \leq i \leq N-1), \tag{4.7}$$

$$S_N(z) = -q^{-1} S_N^{(N+1,0)}(z). \tag{4.8}$$

For $2i + 1 \leq j \leq 2N + 1$ we have set

$$S_i^{(j,s)}(z) =: \exp\left(-\left(\frac{1}{k+N-1} a^i \right) \left(z \left| \frac{k+N-1}{2} \right.\right) \right) \widetilde{S}_i^{(j,s)}(z) : . \tag{4.9}$$

Here, for $1 \le i \le N-1$ *and* $i+1 \le j \le N$, *we have set*

$$\widetilde{S}_i^{(j,1)}(z) = : \exp\Bigg(-b_-^{i,j}(q^{N-1-j}z) - (b+c)^{i,j}(q^{N-j}z) + (b+c)^{i+1,j}(q^{N-1-j}z) + \sum_{l=j+1}^{N}(b_-^{i+1,l}(q^{N-l}z) - b_-^{i,l}(q^{N-l-1}z)) + b_-^{i+1,N+1}(z) - b_-^{i,N+1}(q^{-1}z)\Bigg) :, \quad (4.10)$$

$$\widetilde{S}_i^{(j,2)}(z) = : \exp\Bigg(-b_+^{i,j}(q^{N-1-j}z) - (b+c)^{i,j}(q^{N-j-2}z) + (b+c)^{i+1,j}(q^{N-1-j}z) + \sum_{l=j+1}^{N}(b_-^{i+1,l}(q^{N-l}z) - b_-^{i,l}(q^{N-l-1}z)) + b_-^{i+1,N+1}(z) - b_-^{i,N+1}(q^{-1}z)\Bigg) : . \quad (4.11)$$

For $1 \le i \le N-1$ *we have set*

$$\widetilde{S}_i^{(N+1,0)}(z) = : \exp\Big(b^{i,N+1}(z) + b_+^{i+1,N+1}(z) - b^{i+1,N+1}(qz)\Big) :, \quad (4.12)$$

$$\widetilde{S}_N^{(N+1,0)}(z) = : \exp\big(b^{N,N+1}(z)\big) : . \quad (4.13)$$

The $\mathbf{Z}_2$*-grading of the screening currents is :* $|S_N^{(N+1,0)}(z)| = 1$ *and zero otherwise.*

Theorem 4.2: *The screening currents* $S_i(z)$ $(i = 1, 2, \dots, N)$ *commute (or anti-commute) with* $U_q(\widehat{sl}(N|1))$ *modulo total difference.*

$$[h_{i,m}, S_j(z)] = 0, \quad (4.14)$$

$$[X_i^+(z_1), S_j(z_2)] = 0, \quad (4.15)$$

$$[X_i^-(z_1), S_j(z_2)] = \frac{\delta_{i,j}}{(q-q^{-1})z_1^2}\,({}_{k+N-1}\partial_z\delta)(z_2/z_1) \times : \exp\left(-\left(\frac{1}{k+N-1}a^j\right)\left(z_1\left|-\frac{k+N-1}{2}\right.\right)\right) : . \quad (4.16)$$

The screening currents $S_i(z)$ $(i = 1, 2, \dots, N)$ *satisfy*

$$\left[u_1 - u_2 + \frac{A_{i,j}}{2}\right]_{k+N-1} S_i(z_1)S_j(z_2) = \left[u_2 - u_1 + \frac{A_{i,j}}{2}\right]_{k+N-1} S_j(z_1)S_i(z_2). \quad (4.17)$$

The symbol $[u]_{k+N-1}$ *represents the Jacobi elliptic theta function. Here we have used* $z_j = q^{2u_j}$.

Definition 4.3: We introduce the screening operators Q_i $(i = 1, 2, \dots, N)$ *by the Jackson integral.*

$$Q_i = \int_0^{s\infty} S_i(z)d_pz, \quad (p = q^{2(k+N-1)}). \quad (4.18)$$

The screening operators Q_i *are convergent on the Fock space.*

Corollary 4.4: The screening operators Q_i $(i = 1, 2, \dots, N)$ *commute with the quantum superalgebra* $U_q(\widehat{sl}(N|1))$.

Proposition 4.5: The screening operators Q_i $(i = 1, 2, \dots, N)$ *commute with the projection operator* $\eta_0\xi_0$ *of the* ξ-η *system. Hence the screening operators* Q_i *act on the Fock-Wakimoto module* $\mathcal{F}(p_a)$.

B. Proof

Here we give proof of Theorem 4.2. Direct calculations of the normal orderings show Theorem 4.2.

• Proof of (4.16) for $1 \leq i = j \leq N$.

First we show (4.16) for $1 \leq i = j \leq N-1$. The commutators vanish, $[X_i^{-(l,s)}(z_1), S_i^{(m,t)}(z_2)] = 0$, for the following condition.

$$[(l,s),(m,t)] \neq \begin{cases} [(i,1),(i+1,1)], [(i,2),(i+1,2)] & (1 \leq i \leq N-1) \\ [(l,1),(l+1,2)], [(l,2),(l+1,1)] & (1 \leq i \leq N-1, i+1 \leq l \leq N-1) \\ [(N,0),(N+1,0)] & (1 \leq i \leq N-1) \end{cases}. \tag{4.19}$$

Hence we have the following relations for $1 \leq i \leq N-1$.

$$\begin{aligned}[X_i^-(z_1), S_i(z_2)] = q^{k+N}[X_i^{-(N,0)}(z_1), S_i^{(N+1,0)}(z_2)] \\ + \frac{1}{(q-q^{-1})^2 z_1 z_2}\Big\{[X_i^{-(i,1)}(z_1), S_i^{(i+1,1)}(z_2)] + [X_i^{-(i,2)}(z_1), S_i^{(i+1,2)}(z_2)] \\ + \sum_{l=i+1}^{N-1}\Big([X_i^{-(l,1)}(z_1), S_i^{(l+1,2)}(z_2)] + [X_i^{-(l,2)}(z_1), S_i^{(l+1,1)}(z_2)]\Big)\Big\}.\end{aligned} \tag{4.20}$$

Using the relations (A1)–(A6) in Appendix A, we have

$$\begin{aligned}&[X_i^-(z_1), S_i(z_2)](q-q^{-1})z_1 z_2 \\ &= \delta\left(\frac{q^{-N-k-1}z_2}{z_1}\right)(: X_i^{-(N,0)}(z_1)S_i^{(N+1,0)}(z_2) : - : X_i^{-(N-1,1)}(z_1)S_i^{(N,2)}(z_2) :) \\ &+ \delta\left(\frac{q^{N-3-k-2i}z_2}{z_1}\right)(: X_i^{-(i+1,2)}(z_1)S_i^{(i+2,1)}(z_2) : - : X_i^{-(i,2)}(z_1)S_i^{(i+1,2)}(z_2) :) \\ &+ \sum_{l=i+1}^{N-2}\delta\left(\frac{q^{N-3-k-2l}z_2}{z_1}\right)(: X_i^{-(l+1,2)}(z_1)S_i^{(l+2,1)}(z_2) : - : X_i^{-(l+1,1)}(z_1)S_i^{(l+2,2)}(z_2) :) \\ &+ \delta\left(\frac{q^{N+k-1}z_2}{z_1}\right) : X_i^{-(i,1)}(z_1)S_i^{(i+1,1)}(z_2) : -\delta\left(\frac{q^{-N-k+1}z_2}{z_1}\right) : X_i^{-(N,0)}(z_1)S_i^{(N+1,0)}(z_2) : .\end{aligned} \tag{4.21}$$

Using specializations (A29), (A31)–(A33) in Appendix A, we conclude (4.16) for $1 \leq i = j \leq N-1$. Next we show (4.16) for $i = j = N$. The commutators vanish, $[X_N^{-(l,s)}(z_1), S_N^{(m,t)}(z_2)] = 0$, for $[(l,s),(m,t)] \neq [(N,1),(N+1,0)], [(N,2),(N+1,0)]$. Hence, using the relations (A7) and (A8) in Appendix A, we have

$$\begin{aligned}&[X_N^-(z_1), S_N(z_2)](q-q^{-1})z_1 z_2 \\ &= \delta\left(\frac{q^{N+k-1}z_2}{z_1}\right) : X_N^{-(N,1)}(z_1)S_N^{(N+1,0)}(z_2) : -\delta\left(\frac{q^{N+k-1}z_2}{z_1}\right) : X_N^{-(N,2)}(z_1)S_N^{(N+1,0)}(z_2) : .\end{aligned} \tag{4.22}$$

Using the relation (A30) in Appendix A, we have (4.16) for $i = N$. Now we have shown (4.16) for $1 \leq i = j \leq N$.

• Proof of (4.16) for $1 \leq i \neq j \leq N$.

First we show (4.16) for $i+1 < j$. In this case the commutators vanish, $[X_i^{-(l,s)}(z_1), S_j^{(m,t)}(z_2)] = 0$, for every $[(l,s),(m,t)]$. Hence we conclude $[X_i^-(z_1), S_j(z_2)] = 0$. Next we show (4.16) for $j = i+1$. The commutators vanish, $[X_i^{-(l,s)}(z_1), S_j^{(m,t)}(z_2)] = 0$, for the following condition.

$$[(l,s),(m,t)] \neq [(l,1),(l+1,2)], [(l,2),(l+1,1)] \quad (1 \leq i \leq N-2, i+1 \leq l \leq N-1). \tag{4.23}$$

Hence, using the relations (A9) and (A10), we have the following relation for $1 \le i \le N-1$.

$$[X_i^-(z_1), S_{i+1}(z_2)](q-q^{-1})z_1z_2$$
$$= \sum_{l=i+1}^{N-1} \delta\left(\frac{q^{N-k-2l-2}z_2}{z_1}\right)\left(-:X_i^{-(l,1)}(z_1)S_{i+1}^{(l+1,2)}(z_2): + :X_i^{-(l,2)}(z_1)S_{i+1}^{(l+1,1)}(z_2):\right). \quad (4.24)$$

Using the specialization (A34), we conclude (4.16) for $j = i+1$. Next we show (4.16) for $1 \le j < i \le N-1$. The commutators vanish, $[X_i^{-(l,s)}(z_1), S_j^{(m,t)}(z_2)] = 0$, for the following condition.

$$[(l,s),(m,t)] \neq \begin{cases} [(j,1),(i+1,1)],[(j+1,1),(i,1)] \\ [(j,2),(i+1,1)],[(j+1,1),(i,2)] \\ [(j,2),(i,1)],[(j,1),(i,2)] \end{cases} (1 \le j < i \le N-1). \quad (4.25)$$

Hence, using the relations (A11)–(A16), we have the following relation for $1 \le j < i \le N-1$.

$$[X_i^-(z_1), S_j(z_2)](q-q^{-1})z_1z_2$$
$$= \delta\left(\frac{q^{N+k-i+j-1}z_2}{z_1}\right)\left(:X_i^{-(j,1)}(z_1)S_j^{(i+1,1)}(z_2): - :X_i^{-(j+1,1)}(z_1)S_j^{(i,1)}(z_2):\right.$$
$$- :X_i^{-(j,2)}(z_1)S_j^{(i+1,1)}(z_2): + :X_i^{-(j+1,1)}(z_1)S_j^{(i,2)}(z_2):$$
$$\left. -q^{-1}:X_i^{-(j,2)}(z_1)S_j^{(i,1)}(z_2): + q^{-1}:X_i^{-(j,1)}(z_1)S_j^{(i,2)}(z_2):\right). \quad (4.26)$$

Using the specializations (A35)–(A37), we have $[X_i^-(z_1), S_j(z_2)] = 0$ for $1 \le j < i \le N-1$. Next we show (4.16) for $1 \le j \le N-1$ and $i = N$. The commutators vanish, $[X_i^{-(l,s)}(z_1), S_j^{(m,t)}(z_2)] = 0$, for the following condition.

$$[(l,s),(m,t)] \neq \begin{cases} [(j,1),(N,2)],[(j,2),(N,1)] \\ [(j,1),(N+1,0)],[(j+1,1),(N,1)] \\ [(j,2),(N+1,0)],[(j+1,1),(N,2)] \end{cases} (1 \le j \le N-1). \quad (4.27)$$

Hence, using the relations (A12), (A14)–(A18) we have the following relation for $1 \le j \le N-1$.

$$[X_N^-(z_1), S_j(z_2)](q-q^{-1})q^{N-j-2}z_1z_2$$
$$= \delta\left(\frac{q^{k+j-1}z_2}{z_1}\right)\left(q^{-2}:X_N^{-(j,1)}(z_1)S_j^{(N,2)}(z_2): - q^{-2}:X_N^{-(j,2)}(z_1)S_j^{(N,1)}(z_2):\right.$$
$$- :X_N^{-(j,1)}(z_1)S_j^{(N+1,0)}(z_2): + :X_N^{-(j+1,1)}(z_1)S_j^{(N,1)}(z_2):$$
$$\left. \pm :X_N^{-(j,2)}(z_1)S_j^{(N+1,0)}(z_2): + :X_N^{-(j+1,1)}(z_1)S_j^{(N,2)}(z_2):\right). \quad (4.28)$$

Using the specializations (A37)–(A39), we have $[X_N^-(z_1), S_j(z_2)] = 0$ for $1 \le j \le N-1$. Now we have shown $[X_i^-(z_1), S_j(z_2)] = 0$ for $1 \le i \neq j \le N$.

- Proof of (4.15) for $1 \le i = j \le N$.

The commutators vanish, $[X_i^{+(l,s)}(z_1), S_i^{(m,t)}(z_2)] = 0$, for the condition

$$[(l,s),(m,t)] \neq [(i,1),(i+1,2)],[(i,2),(i+1,1)](1 \le i \le N-1). \quad (4.29)$$

We have $[X_N^+(z_1), S_N(z_2)] = 0$. For $1 \le i \le N-1$, using the relations (A19) and (A20), we have

$$[X_i^+(z_1), S_i(z_2)](q-q^{-1})z_1z_2$$
$$= \delta\left(\frac{q^{N-2i-1}z_2}{z_1}\right)\left(-:X_i^{+(i,1)}(z_1)S_i^{(i+1,2)}(z_2): + :X_i^{+(i,2)}(z_1)S_i^{(i+1,1)}(z_2):\right). \quad (4.30)$$

Using the specialization (A40), we conclude $[X_i^+(z_1), S_i(z_2)] = 0$ for $1 \le i \le N-1$.

- Proof of (4.15) for $1 \le i \neq j \le N$.

First we show (4.15) for $1 \leq i \leq N-1$, $1 \leq j \leq i-1$. The commutators vanish, $[X_i^{+(l,s)}(z_1), S_j^{(m,t)}(z_2)]=0$, for the following condition.

$$[(l,s),(m,t)] \neq \begin{cases} [(j,1),(i,2)],[(j+1,2),(i+1,1)] \\ [(j,1),(i+1,2)],[(j,2),(i+1,1)] \\ [(j,2),(i,2)],[(j+1,2),(i+1,2)] \end{cases} \quad (1 \leq i \leq N-1, 1 \leq j \leq i-1)\ . \tag{4.31}$$

Hence, using the relations (A21)–(A28), we have the following relation for $1 \leq i \leq N-1$ and $1 \leq j \leq i-1$.

$$\begin{aligned} &[X_i^+(z_1), S_j(z_2)](q-q^{-1})z_1z_2 \\ &= \delta\left(\frac{q^{N-i-j-1}z_2}{z_1}\right)\Big(q :X_i^{+(j,1)}(z_1)S_j^{(i,2)}(z_2): -q :X_i^{+(j+1,2)}(z_1)S_j^{(i+1,1)}(z_2): \\ &- :X_i^{+(j,1)}(z_1)S_j^{(i+1,2)}(z_2): + :X_i^{+(j,2)}(z_1)S_j^{(i+1,1)}(z_2): \\ &-q :X_i^{+(j,2)}(z_1)S_j^{(i,2)}(z_2): +q :X_i^{+(j+1,2)}(z_1)S_j^{(i+1,2)}(z_2):\Big). \end{aligned} \tag{4.32}$$

Using the specializations (A41), (A43), and (A44), we conclude $[X_i^+(z_1), S_j(z_2)]=0$ for $1 \leq i \leq N-1$, $1 \leq j \leq i-1$. Next we show (4.15) for $i=N$ and $1 \leq j \leq N-1$. The commutators vanish, $[X_i^{+(l,s)}(z_1), S_j^{(m,t)}(z_2)]=0$, for the following condition.

$$[(l,s),(m,t)] \neq [(j,0),(N,2)],[(j+1,0),(N+1,0)](1 \leq j \leq N-1). \tag{4.33}$$

Hence, using the relations (A23) and (A24), we have

$$\begin{aligned} &[X_N^+(z_1), S_j(z_2)]q^{-N+1}z_2 \\ &= \delta\left(\frac{q^{-j-1}z_2}{z_1}\right)\Big(:X_N^{+(j,0)}(z_1)S_j^{(N,2)}(z_2): - :X_N^{+(j+1,0)}(z_1)S_j^{(N+1,0)}(z_2):\Big). \end{aligned} \tag{4.34}$$

Using the specialization (A42), we conclude $[X_N^+(z_1), S_j(z_2)]=0$ for $1 \leq j \leq N-1$. Now we have shown the commutation relation (4.15).

- Proof of (4.14). This is a direct consequence of the relation,

$$[\Psi_i^{\pm}(z_1), S_j^{(l,s)}(z_2)]=0, \tag{4.35}$$

for every i, j, and (l,s).

- Proof of (4.17). This is a direct consequence of the relation,

$$\left[u_1-u_2+\frac{A_{i,j}}{2}\right]_{k+N-1} S_i^{(l,s)}(z_1)S_j^{(m,t)}(z_2)=\left[u_2-u_1+\frac{A_{i,j}}{2}\right]_{k+N-1} S_j^{(m,t)}(z_2)S_i^{(l,s)}(z_1), \tag{4.36}$$

for every (i,j) and $[(l,s),(m,t)]$.

V. VERTEX OPERATOR

In this section we propose bosonizations of the vertex operators for the quantum superalgebra $U_q(\widehat{sl}(N|1))$.[31] We check that the vertex operators are the intertwiners among the Fock-Wakimoto module and the typical representation for small rank $N \leq 4$.

A. Level-zero representation

We discuss level-zero representation of $U_q(\widehat{sl}(3|1))$ in this section (respectively, $U_q(\widehat{sl}(4|1))$ in Appendix B), that we will use for the investigation of the vertex operator. Let V_α be the one parameter family of the 2^N-dimensional typical representation of $U_q(sl(N|1))$.[37,38] In the case of $U_q(sl(3|1))$, we choose the basis $\{v_j\}_{1\le j\le 8}$ of V_α and assign them the $\mathbf{Z}_2$-gradings as following.

$$|v_1|=|v_5|=|v_6|=|v_7|=0, \quad |v_2|=|v_3|=|v_4|=|v_8|=1. \tag{5.1}$$

In the homogeneous gradation, the evaluation representation $V_{\alpha,z}$ of $U_q(\widehat{sl}(3|1))$ is given by

$$h_1=E_{3,3}-E_{4,4}+E_{5,5}-E_{6,6}, \tag{5.2}$$

$$h_2=E_{2,2}-E_{3,3}+E_{6,6}-E_{7,7}, \tag{5.3}$$

$$h_3=\alpha(E_{1,1}+E_{2,2})+(\alpha+1)(E_{3,3}+E_{4,4}+E_{5,5}+E_{6,6})+(\alpha+2)(E_{7,7}+E_{8,8}), \tag{5.4}$$

$$e_1=E_{3,4}+E_{5,6}, \tag{5.5}$$

$$e_2=E_{2,3}+E_{6,7}, \tag{5.6}$$

$$e_3=\sqrt{[\alpha]}E_{1,2}-\sqrt{[\alpha+1]}(E_{3,5}+E_{4,6})+\sqrt{[\alpha+2]}E_{7,8}, \tag{5.7}$$

$$f_1=E_{4,3}+E_{6,5}, \tag{5.8}$$

$$f_2=E_{3,2}+E_{7,6}, \tag{5.9}$$

$$f_3=\sqrt{[\alpha]}E_{2,1}-\sqrt{[\alpha+1]}(E_{5,3}+E_{6,4})+\sqrt{[\alpha+2]}E_{8,7}, \tag{5.10}$$

$$h_0=-\alpha(E_{1,1}+E_{4,4})-(\alpha+1)(E_{2,2}+E_{3,3}+E_{6,6}+E_{7,7})-(\alpha+2)(E_{5,5}+E_{8,8}), \tag{5.11}$$

$$e_0=-z(\sqrt{[\alpha]}E_{4,1}-\sqrt{[\alpha+1]}(E_{6,2}+E_{7,3})+\sqrt{[\alpha+2]}E_{8,5}), \tag{5.12}$$

$$f_0=z^{-1}(\sqrt{[\alpha]}E_{1,4}-\sqrt{[\alpha+1]}(E_{2,6}+E_{3,7})+\sqrt{[\alpha+2]}E_{5,8}). \tag{5.13}$$

We set the dual representation $V_{\alpha,z}^{*S}$ of $U_q(\widehat{sl}(3|1))$ by

$$\pi_{V_{\alpha,z}^{*S}}(a)=\left(\pi_{V_{\alpha,z}}(S(a))\right)^{st} \quad \text{for } a\in U_q(\widehat{sl}(3|1)), \tag{5.14}$$

where we have used the antipode S and have introduced the supertransposition "st" by

$$(E_{i,j})^{st}=(-1)^{|v_i|(|v_i|+|v_j|)}E_{j,i}. \tag{5.15}$$

We have chosen the dual basis $\{v_j^*\}_{1\le j\le 8}$ of V_α^{*S} and assign them the $\mathbf{Z}_2$-gradings as following.

$$|v_1^*|=|v_5^*|=|v_6^*|=|v_7^*|=0, \quad |v_2^*|=|v_3^*|=|v_4^*|=|v_8^*|=1. \tag{5.16}$$

In the homogeneous gradation, the evaluation representation $V_{\alpha,z}^{*S}$ of $U_q(\widehat{sl}(3|1))$ is given by

$$h_1=-E_{3,3}+E_{4,4}-E_{5,5}+E_{6,6}, \tag{5.17}$$

$$h_2=-E_{2,2}+E_{3,3}-E_{6,6}+E_{7,7}, \tag{5.18}$$

$$h_3=-\alpha(E_{1,1}+E_{2,2})-(\alpha+1)(E_{3,3}+E_{4,4}+E_{5,5}+E_{6,6})-(\alpha+2)(E_{7,7}+E_{8,8}), \tag{5.19}$$

$$e_1=-q^{-1}(E_{4,3}+E_{6,5}), \tag{5.20}$$

$$e_2=-q^{-1}(E_{3,2}+E_{7,6}), \tag{5.21}$$

$$e_3 = -(\sqrt{[\alpha]}q^{-\alpha}E_{2,1} + \sqrt{[\alpha+1]}q^{-\alpha-1}(E_{5,3}+E_{6,4}) + \sqrt{[\alpha+2]}q^{-\alpha-2}E_{8,7}), \tag{5.22}$$

$$f_1 = -q(E_{3,4}+E_{5,6}), \tag{5.23}$$

$$f_2 = -q(E_{2,3}+E_{6,7}), \tag{5.24}$$

$$f_3 = \sqrt{[\alpha]}q^{\alpha}E_{1,2} + \sqrt{[\alpha+1]}q^{\alpha+1}(E_{3,5}+E_{4,6}) + \sqrt{[\alpha+2]}q^{\alpha+2}E_{7,8}, \tag{5.25}$$

$$h_0 = \alpha(E_{1,1}+E_{4,4}) + (\alpha+1)(E_{2,2}+E_{3,3}+E_{6,6}+E_{7,7}) + (\alpha+2)(E_{5,5}+E_{8,8}), \tag{5.26}$$

$$e_0 = -z(\sqrt{[\alpha]}q^{\alpha}E_{1,4} + \sqrt{[\alpha+1]}q^{\alpha+1}(E_{2,6}+E_{3,7}) + \sqrt{[\alpha+2]}q^{\alpha+2}E_{5,8}), \tag{5.27}$$

$$f_0 = -z^{-1}(\sqrt{[\alpha]}q^{-\alpha}E_{4,1} + \sqrt{[\alpha+1]}q^{-\alpha-1}(E_{6,2}+E_{7,3}) + \sqrt{[\alpha+2]}q^{-\alpha-2}E_{8,5}). \tag{5.28}$$

We give the level-zero realization of the Drinfeld generators.

Proposition 5.1: On $V_{\alpha,z}$, the Drinfeld generators of $U_q(\widehat{sl}(3|1))$ are realized by

$$h_{1,m} = \frac{[m]}{m}(q^{\alpha+2}z)^m(q^{-m}E_{3,3} - q^mE_{4,4} + q^{-m}E_{5,5} - q^mE_{6,6}), \tag{5.29}$$

$$h_{2,m} = \frac{[m]}{m}(q^{\alpha+2}z)^m(q^{-2m}E_{2,2} - E_{3,3} + E_{6,6} - q^{2m}E_{7,7}), \tag{5.30}$$

$$h_{3,m} = \frac{1}{m}z^m([\alpha m](E_{1,1}+E_{2,2}) + [(\alpha+1)m]q^m(E_{3,3}+E_{4,4}+E_{5,5}+E_{6,6}) + [(\alpha+2)m]q^{2m}(E_{7,7}+E_{8,8})), \tag{5.31}$$

$$x^+_{1,n} = (q^{\alpha+2}z)^n(E_{3,4}+E_{5,6}), \tag{5.32}$$

$$x^+_{2,n} = (q^{\alpha+2}z)^n(q^{-n}E_{2,3}+q^nE_{6,7}), \tag{5.33}$$

$$x^+_{3,n} = (q^{\alpha+2}z)^n(\sqrt{[\alpha]}q^{-2n}E_{1,2} - \sqrt{[\alpha+1]}(E_{3,5}+E_{4,6}) + \sqrt{[\alpha+2]}q^{2n}E_{7,8}), \tag{5.34}$$

$$x^-_{1,n} = (q^{\alpha+2}z)^n(E_{4,3}+E_{6,5}), \tag{5.35}$$

$$x^-_{2,n} = (q^{\alpha+2}z)^n(q^{-n}E_{3,2}+q^nE_{7,6}), \tag{5.36}$$

$$x^-_{3,n} = (q^{\alpha+2}z)^n(\sqrt{[\alpha]}q^{-2n}E_{2,1} - \sqrt{[\alpha+1]}(E_{5,3}+E_{6,4}) + \sqrt{[\alpha+2]}q^{2n}E_{8,7}). \tag{5.37}$$

On $V^{*S}_{\alpha,z}$, the Drinfeld generators of $U_q(\widehat{sl}(3|1))$ are realized by

$$h_{1,m} = \frac{[m]_q}{m}(q^{-\alpha-2}z)^m(-q^mE_{3,3} + q^{-m}E_{4,4} - q^mE_{5,5} + q^{-m}E_{6,6}), \tag{5.38}$$

$$h_{2,m} = \frac{[m]_q}{m}(q^{-\alpha-2}z)^m(-q^{2m}E_{2,2} + E_{3,3} - E_{6,6} + q^{-2m}E_{7,7}), \tag{5.39}$$

$$h_{3,m} = \frac{-1}{m}z^m([\alpha m](E_{1,1}+E_{2,2}) + [(\alpha+1)m]q^{-m}(E_{3,3}+E_{4,4}+E_{5,5}+E_{6,6}) + [(\alpha+2)m]q^{-2m}(E_{7,7}+E_{8,8})), \tag{5.40}$$

$$x_{1,n}^{+} = -q^{-1}(q^{-\alpha-2}z)^n(E_{4,3} + E_{6,5}), \tag{5.41}$$

$$x_{2,n}^{+} = -q^{-1}(q^{-\alpha-2}z)^n(q^n E_{3,2} + q^{-n} E_{7,6}), \tag{5.42}$$

$$x_{3,n}^{+} = -(q^{-\alpha-2}z)^n(\sqrt{[\alpha]}q^{-\alpha+2n}E_{2,1} + \sqrt{[\alpha+1]}q^{-\alpha-1}(E_{5,3} + E_{6,4}) \\ +\sqrt{[\alpha+2]}q^{-\alpha-2-2n}E_{8,7}), \tag{5.43}$$

$$x_{1,n}^{-} = -q(q^{-\alpha}z)^n(E_{3,4} + E_{5,6}), \tag{5.44}$$

$$x_{2,n}^{-} = -q(q^{-\alpha-2}z)^n(q^n E_{2,3} + q^{-n} E_{6,7}), \tag{5.45}$$

$$x_{3,n}^{-} = (q^{-\alpha-2}z)^n(\sqrt{[\alpha]}q^{\alpha+2n}E_{1,2} + \sqrt{[\alpha+1]}q^{\alpha+1}(E_{3,5} + E_{4,6}) \\ +\sqrt{[\alpha+2]}q^{\alpha+2-2n}E_{7,8}). \tag{5.46}$$

In Appendix B, we summarize the case of $U_q(\widehat{sl}(4|1))$. The case of $U_q(\widehat{sl}(2|1))$ is summarized in Ref. 11.

B. Vertex operator

Let $\mathcal{F}$ and $\mathcal{F}'$ be level k highest weight $U_q(\widehat{sl}(N|1))$-modules. Let V_α and V_α^{*S} be 2^N-dimensional typical representation with a parameter α.[38] The representations V_α and V_α^{*S} are irreducible if and only if $\alpha \neq 0, -1, -2, \ldots, -N+1$. Let $V_{\alpha,z}$ and $V_{\alpha,z}^{*S}$ be the evaluation module and its dual of the typical representation. Consider the following intertwiners of $U_q(\widehat{sl}(N|1))$-module.[31]

$$\Phi(z) : \mathcal{F} \longrightarrow \mathcal{F}' \otimes V_{\alpha,z},\ \Phi^*(z) : \mathcal{F} \longrightarrow \mathcal{F}' \otimes V_{\alpha,z}^{*S}. \tag{5.47}$$

They are intertwiners in the sense that for any $x \in U_q(\widehat{sl}(N|1))$,

$$\Phi(z)\cdot x = \Delta(x)\cdot\Phi(z),\ \Phi^*(z)\cdot x = \Delta(x)\cdot\Phi^*(z). \tag{5.48}$$

We expand the intertwining operators.

$$\Phi(z) = \sum_{j=1}^{2^N}\Phi_j(z)\otimes v_j,\ \Phi^*(z) = \sum_{j=1}^{2^N}\Phi_j^*(z)\otimes v_j^*. \tag{5.49}$$

We set the $\mathbf{Z}_2$-grading of the intertwiner be $|\Phi(z)| = |\Phi^*(z)| = 0$. In what follows we focus our attention on rank $N \le 4$ case.

Proposition 5.2: For $\alpha \neq 0, -1, -2$, the operator $\Phi(z)$ for $U_q(\widehat{sl}(3|1))$ is determined by the component $\Phi_8(z)$. More explicitly, we have

$$\Phi_3(z) = [\Phi_4(z), f_1]_q,\ \Phi_5(z) = [\Phi_6(z), f_1]_q, \tag{5.50}$$

$$\Phi_2(z) = [\Phi_3(z), f_2]_q,\ \Phi_6(z) = [\Phi_7(z), f_2]_q, \tag{5.51}$$

$$\Phi_1(z) = \frac{1}{\sqrt{[\alpha]}}[\Phi_2(z), f_3]_{q^{-\alpha}},\ \Phi_3(z) = \frac{-1}{\sqrt{[\alpha+1]}}[\Phi_5(z), f_3]_{q^{-\alpha-1}}, \tag{5.52}$$

$$\Phi_4(z) = \frac{-1}{\sqrt{[\alpha+1]}}[\Phi_6(z), f_3]_{q^{-\alpha-1}},\ \Phi_7(z) = \frac{1}{\sqrt{[\alpha+2]}}[\Phi_8(z), f_3]_{q^{-\alpha-2}}. \tag{5.53}$$

For $\alpha \neq 0, -1, -2$, the operator $\Phi^*(z)$ for $U_q(\widehat{sl}(3|1))$ is determined by the component $\Phi_1^*(z)$. More explicitly, we have

$$\Phi_4^*(z) = [f_1, \Phi_3^*(z)]_{q^{-1}}, \quad \Phi_6^*(z) = [f_1, \Phi_5^*(z)]_{q^{-1}}, \tag{5.54}$$

$$\Phi_3^*(z) = [f_2, \Phi_2^*(z)]_{q^{-1}}, \quad \Phi_7^*(z) = [f_2, \Phi_6^*(z)]_{q^{-1}}, \tag{5.55}$$

$$\Phi_2^*(z) = \frac{1}{\sqrt{[\alpha]}}[f_3, \Phi_1^*(z)]_{q^{-\alpha}}, \quad \Phi_5^*(z) = \frac{-1}{\sqrt{[\alpha+1]}}[f_3, \Phi_3^*(z)]_{q^{-\alpha-1}}, \tag{5.56}$$

$$\Phi_6^*(z) = \frac{-1}{\sqrt{[\alpha+1]}}[f_3, \Phi_4^*(z)]_{q^{-\alpha-1}}, \quad \Phi_8^*(z) = \frac{1}{\sqrt{[\alpha+2]}}[f_3, \Phi_7^*(z)]_{q^{-\alpha-2}}. \tag{5.57}$$

Proposition 5.3: For $\alpha \neq 0, -1, -2, -3$, the operator $\Phi(z)$ for $U_q(\widehat{sl}(4|1))$ is determined by the component $\Phi_{16}(z)$. More explicitly, we have

$$\Phi_4(z) = [\Phi_6(z), f_1]_q, \quad \Phi_7(z) = [\Phi_8(z), f_1]_q, \tag{5.58}$$

$$\Phi_9(z) = [\Phi_{10}(z), f_1]_q, \quad \Phi_{11}(z) = [\Phi_{13}(z), f_1]_q, \tag{5.59}$$

$$\Phi_3(z) = [\Phi_4(z), f_2]_q, \quad \Phi_5(z) = [\Phi_7(z), f_2]_q, \tag{5.60}$$

$$\Phi_9(z) = [\Phi_{10}(z), f_2]_q, \quad \Phi_{11}(z) = [\Phi_{13}(z), f_2]_q, \tag{5.61}$$

$$\Phi_2(z) = [\Phi_3(z), f_3]_q, \quad \Phi_7(z) = [\Phi_9(z), f_3]_q, \tag{5.62}$$

$$\Phi_8(z) = [\Phi_{10}(z), f_3]_q, \quad \Phi_{14}(z) = [\Phi_{15}(z), f_2]_q, \tag{5.63}$$

$$\Phi_1(z) = \frac{-1}{\sqrt{[\alpha]}}[\Phi_2(z), f_4]_{q^{-\alpha}}, \quad \Phi_3(z) = \frac{-1}{\sqrt{[\alpha+1]}}[\Phi_5(z), f_4]_{q^{-\alpha-1}}, \tag{5.64}$$

$$\Phi_4(z) = \frac{-1}{\sqrt{[\alpha+1]}}[\Phi_7(z), f_4]_{q^{-\alpha-1}}, \quad \Phi_6(z) = \frac{-1}{\sqrt{[\alpha+1]}}[\Phi_8(z), f_4]_{q^{-\alpha-1}}, \tag{5.65}$$

$$\Phi_9(z) = \frac{-1}{\sqrt{[\alpha+2]}}[\Phi_{11}(z), f_4]_{q^{-\alpha-2}}, \quad \Phi_{10}(z) = \frac{-1}{\sqrt{[\alpha+2]}}[\Phi_{13}(z), f_4]_{q^{-\alpha-2}}, \tag{5.66}$$

$$\Phi_{12}(z) = \frac{-1}{\sqrt{[\alpha+2]}}[\Phi_{14}(z), f_4]_{q^{-\alpha-2}}, \quad \Phi_{15}(z) = \frac{-1}{\sqrt{[\alpha+3]}}[\Phi_{16}(z), f_4]_{q^{-\alpha-3}}. \tag{5.67}$$

For $\alpha \neq 0, -1, -2, -3$, the operator $\Phi^*(z)$ for $U_q(\widehat{sl}(4|1))$ is determined by the component $\Phi_1^*(z)$. More explicitly, we have

$$\Phi_6^*(z) = [f_1, \Phi_4^*(z)]_{q^{-1}}, \quad \Phi_8^*(z) = [f_1, \Phi_7^*(z)]_{q^{-1}}, \tag{5.68}$$

$$\Phi_{10}^*(z) = [f_1, \Phi_9^*(z)]_{q^{-1}}, \quad \Phi_{13}^*(z) = [f_1, \Phi_{11}^*(z)]_{q^{-1}}, \tag{5.69}$$

$$\Phi_4^*(z) = [f_2, \Phi_3^*(z)]_{q^{-1}}, \quad \Phi_7^*(z) = [f_2, \Phi_5^*(z)]_{q^{-1}}, \tag{5.70}$$

$$\Phi_{10}^*(z) = [f_2, \Phi_9^*(z)]_{q^{-1}}, \quad \Phi_{13}^*(z) = [f_2, \Phi_{11}^*(z)]_{q^{-1}}, \tag{5.71}$$

$$\Phi_3^*(z) = [f_3, \Phi_2^*(z)]_{q^{-1}}, \quad \Phi_9^*(z) = [f_3, \Phi_7^*(z)]_{q^{-1}}, \tag{5.72}$$

$$\Phi_{10}^*(z) = [f_3, \Phi_8^*(z)]_{q^{-1}}, \quad \Phi_{15}^*(z) = [f_3, \Phi_{14}^*(z)]_{q^{-1}}, \tag{5.73}$$

$$\Phi_2^*(z) = \frac{-1}{\sqrt{[\alpha]}}[f_4, \Phi_1^*(z)]_{q^{-\alpha}}, \quad \Phi_5^*(z) = \frac{1}{\sqrt{[\alpha+1]}}[f_4, \Phi_3^*(z)]_{q^{-\alpha-1}}, \tag{5.74}$$

$$\Phi_7^*(z) = \frac{1}{\sqrt{[\alpha+1]}}[f_4, \Phi_4^*(z)]_{q^{-\alpha-1}}, \quad \Phi_8^*(z) = \frac{1}{\sqrt{[\alpha+1]}}[f_4, \Phi_6^*(z)]_{q^{-\alpha-1}}, \tag{5.75}$$

$$\Phi_{11}^*(z) = \frac{-1}{\sqrt{[\alpha+2]}}[f_4, \Phi_9^*(z)]_{q^{-\alpha-2}}, \quad \Phi_{13}^*(z) = \frac{-1}{\sqrt{[\alpha+2]}}[f_4, \Phi_{10}^*(z)]_{q^{-\alpha-2}}, \tag{5.76}$$

$$\Phi_{14}^*(z) = \frac{-1}{\sqrt{[\alpha+2]}}[f_4, \Phi_{12}^*(z)]_{q^{-\alpha-2}}, \quad \Phi_{16}^*(z) = \frac{-1}{\sqrt{[\alpha+3]}}[f_4, \Phi_{15}^*(z)]_{q^{-\alpha-3}}. \tag{5.77}$$

The case of $U_q(\widehat{sl}(2|1))$ is summarized in Ref. 11. Next we determine the relations between the components $\Phi_{2^N}(z)$, $\Phi_1^*(z)$ and the Drinfeld generators. We use the coproduct

$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \tag{5.78}$$

$$\Delta(h_{i,m}) = h_{i,m} \otimes q^{\frac{cm}{2}} + q^{\frac{3cm}{2}} \otimes h_{i,m} \quad (m > 0), \tag{5.79}$$

$$\Delta(h_{i,-m}) = h_{i,-m} \otimes q^{-\frac{3cm}{2}} + q^{-\frac{cm}{2}} \otimes h_{i,-m} \quad (m > 0). \tag{5.80}$$

Proposition 5.4: The component $\Phi_{2^N}(z)$ associated with $U_q(\widehat{sl}(N|1))$ satisfy

$$[h_i, \Phi_{2^N}(z)] = -\delta_{i,N}(\alpha + N - 1)\Phi_{2^N}(z)(1 \le i \le N), \tag{5.81}$$

$$[h_{i,m}, \Phi_{2^N}(z)] = -\delta_{i,N} q^{(N-1+\frac{3k}{2})m} \frac{[(\alpha+N-1)m]}{m} z^m \Phi_{2^N}(z) \quad (m > 0, 1 \le i \le N), \tag{5.82}$$

$$[h_{i,-m}, \Phi_{2^N}(z)] = -\delta_{i,N} q^{(-N+1-\frac{k}{2})m} \frac{[(\alpha+N-1)m]}{m} z^{-m} \Phi_{2^N}(z) \quad (m > 0, 1 \le i \le N), \tag{5.83}$$

$$[X_i^+(z_1), \Phi_{2^N}(z_2)] = 0 \quad (1 \le i \le N). \tag{5.84}$$

The component $\Phi_1^*(z)$ associated with $U_q(\widehat{sl}(N|1))$ satisfy

$$[h_i, \Phi_1^*(z)] = \delta_{i,N}\alpha\Phi_1^*(z) \quad (1 \le i \le N), \tag{5.85}$$

$$[h_{i,m}, \Phi_1^*(z)] = \delta_{i,N} q^{\frac{3k}{2}m} \frac{[\alpha m]}{m} z^m \Phi_1^*(z) \quad (m > 0, 1 \le i \le N), \tag{5.86}$$

$$[h_{i,-m}, \Phi_1^*(z)] = \delta_{i,N} q^{-\frac{k}{2}m} \frac{[\alpha m]}{m} z^{-m} \Phi_1^*(z) \quad (m > 0, 1 \le i \le N), \tag{5.87}$$

$$[X_i^+(z_1), \Phi_1^*(z_2)] = 0 \quad (1 \le i \le N). \tag{5.88}$$

We have checked this proposition for rank $N = 2, 3, 4$.

In order to construct bosonizations of $\Phi_{2^N}(z)$ and $\Phi_1^*(z)$, we introduce a bosonic operator $\phi^{l_a}(z|\beta)$.

Definition 5.5: For $l_a = (l_a^1, l_a^2, \cdots, l_a^N) \in \mathbf{C}^N$ and $\beta \in \mathbf{C}$, we set the bosonic operator $\phi^{l_a}(z|\beta)$ by

$$\phi^{l_a}(z|\beta) =: \exp\left(\sum_{i,j=1}^N \left(\frac{l_a^i}{k+N-1}\frac{\mathrm{Min}(i,j)}{N-1}\frac{N-1-\mathrm{Max}(i,j)}{1}a^j\right)(z|\beta)\right) : . \tag{5.89}$$

We call the operator $\phi^{l_a}(z|\beta)$ the "elementary vertex operator."

Proposition 5.6: The highest vector $|\lambda\rangle = |l_a, 0, 0\rangle$ *of* $U_q(\widehat{sl}(N|1))$ *is created from the Fock vacuum* $|0\rangle$ *and* $\phi^{l_a}(z|\beta)$.

$$|\lambda\rangle = \lim_{z\to 0} \phi^{l_a}(z|\beta)|0\rangle. \tag{5.90}$$

Here $|\lambda\rangle$ *is the highest weight vector of the highest weight whose classical part* $\bar{\lambda} = \sum_{i=1}^N l_a^i \bar{\Lambda}_i$.

The elementary vertex operators $\phi^{l_a}(z|\beta)$ give rise to the following map.

$$\phi^{l_a}(z|\beta) : F(p_a) \longrightarrow F(p_a + l_a). \tag{5.91}$$

Using the inversion relation,

$$\sum_{r=1}^N \frac{[A_{i,r}m]}{[m]} \frac{[\mathrm{Min}(r, j)m][(N-1-\mathrm{Max}(r, j))m]}{[(N-1)m][m]} = \delta_{i,j}, \tag{5.92}$$

we have the following proposition.

Proposition 5.7: The elementary vertex operators $\phi^{l_a}(z|\beta)$ *satisfy the following relations.*

$$[h_{i,m}, \phi^{l_a}(z|\beta)] = \frac{1}{m}[l_a^i m] q^{-(\beta+\frac{N-1}{2})|m|} z^m \phi^{l_a}(z|\beta) \quad (1 \le i \le N), \tag{5.93}$$

$$[X_i^+(z_1), \phi^{l_a}(z_2|\beta)] = 0 \quad (1 \le i \le N), \tag{5.94}$$

$$(z_1 - q^{l_a^i} z_2) X_i^-(z_1) \phi^{l_a}\left(z_2 \left| -\frac{k+N-1}{2}\right.\right)$$
$$= (q^{l_a^i} z_1 - z_2) \phi^{l_a}\left(z_2 \left| -\frac{k+N-1}{2}\right.\right) X_i^-(z_1) (1 \le i \le N). \tag{5.95}$$

Proposition 5.8: For $k = \alpha \ne 0, -1, -2, \ldots, -N+1$, *bosonizations of the components* $\Phi_{2^N}(z)$ *and* $\Phi_1^*(z)$ *associated with* $U_q(\widehat{sl}(N|1))$ *are given by*

$$\Phi_{2^N}(z) = \phi^{\hat{l}}\left(q^{k+N-1} z \left| -\frac{k+N-1}{2}\right.\right), \quad \Phi_1^*(z) = \phi^{\hat{l}^*}\left(q^k z \left| -\frac{k+N-1}{2}\right.\right), \tag{5.96}$$

where we have set $\hat{l} = -(0, \cdots, 0, \alpha + N - 1)$ *and* $\hat{l}^* = (0, \cdots, 0, \alpha)$. *The other components* $\Phi_j(z)$ *and* $\Phi_j^*(z) (1 \le j \le 2^N)$ *are represented by multiple contour integrals of Drinfeld currents (cf. Propositions 5.2 and 5.3). We have checked this proposition for* $N = 2, 3, 4$.

These bosonizations of the vertex operators are determined from the commutation relations with the superalgebra $U_q(\widehat{sl}(N|1))$. The construction is completely independent of which infinite dimensional modules the vertex operators intertwine. In what follows we shall clarify on which space these vertex operators act. We balance the "background charge" of the vertex operators by using the screening currents. For $x = (x_1, x_2, \ldots, x_N) \in \mathbf{N}^N$, we set the screening operator

$$\mathcal{Q}^{(x)} =: Q_1^{x_1} Q_2^{x_2} \cdots Q_N^{x_N} : . \tag{5.97}$$

The screening operator $\mathcal{Q}^{(x)}$ gives rise to the map,

$$\mathcal{Q}^{(x)} : F(p_a) \longrightarrow F(p_a + \hat{x}). \tag{5.98}$$

Here $\hat{x} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_N)$, where $\hat{x}_i = \sum_{j=1}^N A_{i,j} x_j$. The $\mathcal{Q}^{(x)}$ commute with the projection operator $\eta_0 \xi_0$. Hence we have the map on the Fock-Wakimoto module.

$$\mathcal{Q}^{(x)} : \mathcal{F}(p_a) \longrightarrow \mathcal{F}(p_a + \hat{x}). \tag{5.99}$$

Definition 5.9: For $k=\alpha\neq 0,\ -1,\ -2,\ldots,\ -N$, we set the bosonic operators

$$\widetilde{\Phi}^{(x)}(z)=\sum_{j=1}^{2^N}\widetilde{\Phi}_j^{(x)}(z)\otimes v_j,\quad \widetilde{\Phi}^{(y)*}(z)=\sum_{j=1}^{2^N}\widetilde{\Phi}_j^{(y)*}(z)\otimes v_j^*. \tag{5.100}$$

Here we have set

$$\widetilde{\Phi}_j^{(x)}(z)=\eta_0\xi_0\cdot\mathcal{Q}^{(x)}\cdot\Phi_j(z)\cdot\eta_0\xi_0, \tag{5.101}$$

$$\widetilde{\Phi}_j^{(y)*}(z)=\eta_0\xi_0\cdot\mathcal{Q}^{(y)}\cdot\Phi_j^*(z)\cdot\eta_0\xi_0, \tag{5.102}$$

where $x=(x_1,x_2,\ldots,x_N)\in\mathbf{N}^N$ and $y=(y_1,y_2,\ldots,y_N)\in\mathbf{N}^N$. We call the operators $\widetilde{\Phi}^{(x)}(z)$, $\widetilde{\Phi}^{(y)*}(z)$ the "projected vertex operators" for $U_q(\widehat{sl}(N|1))$.

Proposition 5.10: For $k=\alpha\neq 0,\ -1,\ -2,\ldots,\ -N+1$, the projected vertex operators $\widetilde{\Phi}^{(x)}(z)$ and $\widetilde{\Phi}^{(y)*}(z)$ are the intertwiners among the Fock-Wakimoto module and the typical representation.

$$\widetilde{\Phi}^{(x)}(z):\mathcal{F}(p_a)\longrightarrow\mathcal{F}(p_a+\hat{l}+\hat{x})\otimes V_{\alpha,z}, \tag{5.103}$$

$$\widetilde{\Phi}^{(y)*}(z):\mathcal{F}(p_a)\longrightarrow\mathcal{F}(p_a+\hat{l}^*+\hat{y})\otimes V_{\alpha,z}^{*S}. \tag{5.104}$$

Here we have set $\hat{l}=-(0,\cdots,0,\alpha+N-1)$ and $\hat{l}^*=(0,\cdots,0,\alpha)$. Here we have set $\hat{x}=(\hat{x}_1,\hat{x}_2,\cdots,\hat{x}_N)$ and $\hat{y}=(\hat{y}_1,\hat{y}_2,\cdots,\hat{y}_N)$ where $\hat{x}_i=\sum_{j=1}^N A_{i,j}x_j$ and $\hat{y}_i=\sum_{j=1}^N A_{i,j}y_j$. We have checked this proposition for rank $N=2,3,4$.

C. Correlation function

In this section we discuss an application of the projected vertex operators $\widetilde{\Phi}^{(x)}(z)$ and $\widetilde{\Phi}^{*(y)}(z)$. We study non-vanishing property of the correlation function which is defined to be the trace of the vertex operators over the Fock-Wakimoto module of $U_q(\widehat{sl}(N|1))$, that is

$$\mathrm{Tr}_{\mathcal{F}(l_a)}\left(q^{L_0}\widetilde{\Phi}_{j_1}^{(x_{(1)})}(z_1)\widetilde{\Phi}_{j_2}^{(x_{(2)})}(z_2)\cdots\widetilde{\Phi}_{j_n}^{(x_{(n)})}(z_n)\right). \tag{5.105}$$

Here we propose the q-Virasoro operator L_0 for $k=\alpha\neq -N+1$ as follows.

$$\begin{aligned}L_0=&\frac{1}{2}\sum_{i,j=1}^N\sum_{m\in\mathbf{Z}}:a_{-m}^i\frac{m^2}{[m][(k+N-1)m]}\frac{[\mathrm{Min}(i,\,j)m][(N-1-\mathrm{Max}(i,\,j))m]}{[(N-1)m][m]}a_m^j:\\&+\sum_{i,j=1}^N\frac{\mathrm{Min}(i,\,j)(N-1-\mathrm{Max}(i,\,j))}{(k+N-1)(N-1)}a_0^j\\&-\frac{1}{2}\sum_{1\le i<j\le N}\sum_{m\in\mathbf{Z}}:b_{-m}^{i,j}\frac{m^2}{[m]^2}b_m^{i,j}:+\frac{1}{2}\sum_{1\le i<j\le N}\sum_{m\in\mathbf{Z}}:c_{-m}^{i,j}\frac{m^2}{[m]^2}c_m^{i,j}:\\&+\frac{1}{2}\sum_{1\le i\le N}\sum_{m\in\mathbf{Z}}:b_{-m}^{i,N+1}\frac{m^2}{[m]^2}b_m^{i,N+1}:+\frac{1}{2}\sum_{1\le i\le N}b_0^{i,N+1}.\end{aligned} \tag{5.106}$$

The L_0 eigenvalue of $|l_a,0,0\rangle$ is $\frac{1}{2(k+N-1)}(\bar{\lambda}|\bar{\lambda}+2\bar{\rho})$, where $\bar{\rho}=\sum_{i=1}^N\bar{\Lambda}_i$ and $\bar{\lambda}=\sum_{i=1}^N l_a^i\bar{\Lambda}_i$.

Proposition 5.11: The correlation function of the vertex operators,

$$\mathrm{Tr}_{\mathcal{F}(l_a)}\left(q^{L_0}\widetilde{\Phi}_{j_1}^{(x_{(1)})}(z_1)\widetilde{\Phi}_{j_2}^{(x_{(2)})}(z_2)\cdots\widetilde{\Phi}_{j_n}^{(x_{(n)})}(z_n)\right)\neq 0, \tag{5.107}$$

if and only if $k=\alpha\neq 0,\ -1,\ -2,\ldots,\ -N+1$ *and* $x_{(s)}=(x_{(s),1},x_{(s),2}\ldots,x_{(s),N})\in\mathbf{N}^N$ $(1\leq s\leq n)$ *satisfy the condition*,

$$\sum_{s=1}^{n}x_{(s),i}=\frac{n\cdot i}{N-1}\alpha+n\cdot i(1\leq i\leq N). \tag{5.108}$$

We note that there does not exist non-rational solution $k=\alpha\notin\mathbf{Q}$ of the relation (5.108). Next, we consider the correlation function involving also dual vertex operators.

Proposition 5.12: The correlation function of the vertex operators and the dual vertex operators,

$$\mathrm{Tr}_{\mathcal{F}(l_a)}\left(q^{L_0}\widetilde{\Phi}_{i_1}^{*(y_{(1)})}(w_1)\widetilde{\Phi}_{i_2}^{*(y_{(2)})}(w_2)\cdots\widetilde{\Phi}_{i_m}^{*(y_{(m)})}(w_m)\widetilde{\Phi}_{j_1}^{(x_{(1)})}(z_1)\widetilde{\Phi}_{j_2}^{(x_{(2)})}(z_2)\cdots\widetilde{\Phi}_{j_n}^{(x_{(n)})}(z_n)\right)\neq 0, \tag{5.109}$$

if and only if $k=\alpha\neq 0,\ -1,\ -2,\ldots,\ -N+1$, $x_{(s)}=(x_{(s),1},x_{(s),2},\ldots,x_{(s),N})\in\mathbf{N}^N$ $(1\leq s\leq n)$ *and* $y_{(t)}=(y_{(t),1},y_{(t),2},\ldots,y_{(t),N})\in\mathbf{N}^N(1\leq t\leq m)$ *satisfy the condition*

$$\sum_{s=1}^{n}x_{(s),i}+\sum_{t=1}^{m}y_{(t),i}=\frac{(n-m)i}{N-1}\alpha+n\cdot i(1\leq i\leq N). \tag{5.110}$$

We note that there exist non-rational solutions $k=\alpha\notin\mathbf{Q}$ of the relation (5.110). Upon $k=\alpha\notin\mathbf{Q}$, the relation (5.110) is equivalent to

$$m=n\quad\text{and}\quad\sum_{s=1}^{n}(x_{(s),i}+y_{(s),i})=n\cdot i\quad(1\leq i\leq N). \tag{5.111}$$

We conclude that the screening operators Q_i are needed to ensure non-vanishing property of correlation functions. In other words, we have to balance the "background charge" of the vertex operators to construct non-zero correlation functions. We can write down integral representations of the correlation functions by using bosonizations of the vertex operators.[22] It is open and nontrivial problem to deform these integral representations to convenient formulae for physical applications.

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APPENDIX A: NORMAL ORDERINGS

In this appendix we summarize formulae of normal orderings. In order to get the following delta-function formulae, the following normal orderings are useful.

$$\exp\left(a_+^i(q^{\frac{k+N-1}{2}}z_1)\right):\exp\left(-\left(\frac{1}{k+N-1}a^j\right)\left(z_2\left|\frac{k+N-1}{2}\right.\right)\right):$$

$$=::q^{-A_{i,j}}\frac{(1-q^{A_{i,j}-k-N+1}z_2/z_1)}{(1-q^{-A_{i,j}-k-N+1}z_2/z_1)},$$

$$:\exp\left(-\left(\frac{1}{k+N-1}a^j\right)\left(z_2\left|\frac{k+N-1}{2}\right.\right)\right):\exp\left(a_+^i(q^{\frac{k+N-1}{2}}z_1)\right)=::1,$$

$$:\exp\left(-\left(\frac{1}{k+N-1}a^i\right)\left(z_1\left|\frac{k+N-1}{2}\right.\right)\right):\exp\left(a_-^j(q^{-\frac{k+N-1}{2}}z_2)\right)$$

$$=:: \frac{(1-q^{A_{i,j}-k-N+1}z_2/z_1)}{(1-q^{-A_{i,j}-k-N+1}z_2/z_1)},$$

$$\exp\left(a_-^i(q^{-\frac{k+N-1}{2}}z_2)\right):\exp\left(-\left(\frac{1}{k+N-1}a^j\right)\left(z_1\left|\frac{k+N-1}{2}\right.\right)\right):=:: q^{A_{i,j}}.$$

In order to get the following specialization relations, the following formula is useful.

$$b^{i,j}(qz)-b^{i,j}(q^{-1}z)=b_+^{i,j}(z)-b_-^{i,j}(z).$$

1. Delta-function

$$[X_i^{-(N,0)}(z_1), S_i^{(N+1,0)}(z_2)] = \frac{-1}{(q-q^{-1})q^{k+N}z_1z_2}\left(\delta\left(\frac{q^{-N-k+1}z_2}{z_1}\right)-\delta\left(\frac{q^{-N-k-1}z_2}{z_1}\right)\right)::(1\le i\le N-1), \quad \text{(A1)}$$

$$[X_i^{-(i,1)}(z_1), S_i^{(i+1,1)}(z_2)] = (q-q^{-1})\delta\left(\frac{q^{N+k-1}z_2}{z_1}\right)::(1\le i\le N-1), \quad \text{(A2)}$$

$$[X_i^{-(N-1,1)}(z_1), S_i^{(N,2)}(z_2)] = (q^{-1}-q)\delta\left(\frac{q^{-N-k-1}z_2}{z_1}\right)::(1\le i\le N-1), \quad \text{(A3)}$$

$$[X_i^{-(l+1,2)}(z_1), S_i^{(l+2,1)}(z_2)] = (q-q^{-1})\delta\left(\frac{q^{N-3-k-2l}z_2}{z_1}\right)::(1\le i\le l\le N-2), \quad \text{(A4)}$$

$$[X_i^{-(l+1,1)}(z_1), S_i^{(l+2,2)}(z_2)] = (q^{-1}-q)\delta\left(\frac{q^{N-3-k-2l}z_2}{z_1}\right)::(1\le i\le l\le N-3), \quad \text{(A5)}$$

$$[X_i^{-(i,2)}(z_1), S_i^{(i+1,2)}(z_2)] = (q^{-1}-q)\delta\left(\frac{q^{N-3-k-2i}z_2}{z_1}\right)::(1\le i\le N-2), \quad \text{(A6)}$$

$$[X_N^{-(N,1)}(z_1), S_N^{(N+1,0)}(z_2)] = \frac{1}{q^{-N-k+1}z_1}\delta\left(\frac{q^{N+k-1}z_2}{z_1}\right)::, \quad \text{(A7)}$$

$$[X_N^{-(N,2)}(z_1), S_N^{(N+1,0)}(z_2)] = \frac{1}{q^{N+k-1}z_1}\delta\left(\frac{q^{-N-k+1}z_2}{z_1}\right)::, \quad \text{(A8)}$$

$$[X_i^{-(l,1)}(z_1), S_{i+1}^{(l+1,2)}(z_2)] = (q-q^{-1})\delta\left(\frac{q^{N-k-2l-2}z_2}{z_1}\right)::(1\le i\le N-2, i+1\le l\le N-1), \quad \text{(A9)}$$

$$[X_i^{-(l,2)}(z_1), S_{i+1}^{(l+1,1)}(z_2)] = (q^{-1}-q)\delta\left(\frac{q^{N-k-2l-2}z_2}{z_1}\right)::(1\le i\le N-2, i+1\le l\le N-1), \quad \text{(A10)}$$

$$[X_i^{-(j,1)}(z_1), S_j^{(i+1,1)}(z_2)] = (q-q^{-1})\delta\left(\frac{q^{N+k-i+j-1}z_2}{z_1}\right)::(1\le j<i\le N-1), \quad \text{(A11)}$$

$$[X_i^{-(j+1,1)}(z_1), S_j^{(i,1)}(z_2)] = (q^{-1}-q)\delta\left(\frac{q^{N+k-i+j-1}z_2}{z_1}\right)::(1\le j<i\le N), \quad \text{(A12)}$$

$$[X_i^{-(j,2)}(z_1), S_j^{(i+1,1)}(z_2)] = (q-q^{-1})\delta\left(\frac{q^{N+k-i+j-1}z_2}{z_1}\right)::(1\le j<i\le N-1), \quad \text{(A13)}$$

$$[X_i^{-(j+1,1)}(z_1), S_j^{(i,2)}(z_2)] = (q^{-1}-q)\delta\left(\frac{q^{N+k-i+j-1}z_2}{z_1}\right)::(1\le j<i\le N), \quad \text{(A14)}$$

$$[X_i^{-(j,1)}(z_1), S_j^{(i,2)}(z_2)] = (1-q^{-2})\delta\left(\frac{q^{N+k-i+j-1}z_2}{z_1}\right) :: (1 \le j < i \le N), \tag{A15}$$

$$[X_i^{-(j,2)}(z_1), S_j^{(i,1)}(z_2)] = (q^{-2}-1)\delta\left(\frac{q^{N+k-i+j-1}z_2}{z_1}\right) :: (1 \le j < i \le N), \tag{A16}$$

$$[X_N^{-(j,1)}(z_1), S_j^{(N+1,0)}(z_2)] = \frac{1}{q^{-k-j+1}z_1}\delta\left(\frac{q^{k+j-1}z_2}{z_1}\right) :: (1 \le j \le N-1), \tag{A17}$$

$$[X_N^{-(j,2)}(z_1), S_j^{(N+1,0)}(z_2)] = \frac{1}{q^{-k-j+1}z_1}\delta\left(\frac{q^{k+j-1}z_2}{z_1}\right) :: (1 \le j \le N-1), \tag{A18}$$

$$[X_i^{+(i,1)}(z_1), S_i^{(i+1,2)}(z_2)] = (q-q^{-1})\delta\left(\frac{q^{N-2i-1}z_2}{z_1}\right) :: (1 \le i \le N-1), \tag{A19}$$

$$[X_i^{+(i,2)}(z_1), S_i^{(i+1,1)}(z_2)] = (q^{-1}-q)\delta\left(\frac{q^{N-2i-1}z_2}{z_1}\right) :: (1 \le i \le N-1), \tag{A20}$$

$$[X_i^{+(j,1)}(z_1), S_j^{(i,2)}(z_2)] = (1-q^{2})\delta\left(\frac{q^{N-i-j-1}z_2}{z_1}\right) :: (1 \le i \le N, 1 \le j \le i-1), \tag{A21}$$

$$[X_i^{+(j+1,2)}(z_1), S_j^{(i+1,1)}(z_2)] = (q^{2}-1)\delta\left(\frac{q^{N-i-j-1}z_2}{z_1}\right) :: (1 \le i \le N, 1 \le j \le i-1), \tag{A22}$$

$$[X_N^{+(j,0)}(z_1), S_j^{(N,2)}(z_2)] = (1-q^{2})\delta\left(\frac{q^{-j-1}z_2}{z_1}\right) :: (1 \le j \le N-1), \tag{A23}$$

$$[X_N^{+(j+1,0)}(z_1), S_j^{(N+1,0)}(z_2)] = \frac{-1}{q^{j+1}z_1}\delta\left(\frac{q^{-j-1}z_2}{z_1}\right) :: (1 \le j \le N-1), \tag{A24}$$

$$[X_i^{+(j,1)}(z_1), S_j^{(i+1,2)}(z_2)] = (q-q^{-1})\delta\left(\frac{q^{N-i-j-1}z_2}{z_1}\right) :: (1 \le i \le N-1, 1 \le j \le i-1), \tag{A25}$$

$$[X_i^{+(j,2)}(z_1), S_j^{(i+1,1)}(z_2)] = (q^{-1}-q)\delta\left(\frac{q^{N-i-j-1}z_2}{z_1}\right) :: (1 \le i \le N-1, 1 \le j \le i-1), \tag{A26}$$

$$[X_i^{+(j,2)}(z_1), S_j^{(i,2)}(z_2)] = (1-q^{2})\delta\left(\frac{q^{N-i-j-1}z_2}{z_1}\right) :: (1 \le i \le N-1, 1 \le j \le i-1), \tag{A27}$$

$$[X_i^{+(j+1,2)}(z_1), S_j^{(i+1,2)}(z_2)] = (q^{2}-1)\delta\left(\frac{q^{N-i-j-1}z_2}{z_1}\right) :: (1 \le i \le N-1, 1 \le j \le i-1), \tag{A28}$$

2. Specialization

$$: X_i^{-(N,0)}(z)S_i^{(N+1,0)}(q^{N+k-1}z) :=: X_i^{-(i,1)}(z)S_i^{(i+1,1)}(q^{-N-k+1}z) :$$
$$=: \exp\left(-\left(\frac{1}{k+N-1}a^i\right)\left(z\left|-\frac{k+N-1}{2}\right.\right)\right) : (1 \le i \le N-1), \tag{A29}$$

$$: X_N^{-(N,1)}(z)S_N^{(N+1,0)}(q^{-N-k+1}z) :=: X_N^{-(N,2)}(z)S_N^{(N+1,0)}(q^{N+k-1}z) :$$
$$=: \exp\left(-\left(\frac{1}{k+N-1}a^i\right)\left(z\left|-\frac{k+N-1}{2}\right.\right)\right) :, \tag{A30}$$

$$: X_i^{-(N,0)}(z)S_i^{(N+1,0)}(q^{N+k+1}z) :=: X_i^{-(N-1,1)}(z)S_i^{(N,2)}(q^{N+k+1}z) : \quad (1 \le i \le N-1), \tag{A31}$$

$$: X_i^{-(i+1,2)}(z)S_i^{(i+2,1)}(q^{-N+3+k+2i}z) :=: X_i^{-(i,2)}(z)S_i^{(i+1,2)}(q^{-N+3+k+2i}z) : \quad (1 \le i \le N-1), \tag{A32}$$

$$: X_i^{-(l+1,1)}(z)S_i^{(l+2,2)}(q^{-N+3+k+2l}z) :=: X_i^{-(l+1,2)}(z)S_i^{(l+2,1)}(q^{-N+3+k+2l}z) : \quad (1 \le i \le N-1, i+1 \le l \le N-2). \tag{A33}$$

$$: X_i^{-(l,1)}(z)S_{i+1}^{(l+1,2)}(q^{-N+k+2l+2}z) :=: X_i^{-(l,2)}(z)S_{i+1}^{(l+1,1)}(q^{-N+k+2l+2}z) :, \quad (1 \le i \le N-2, i+1 \le l \le N-1), \tag{A34}$$

$$: X_i^{-(j,1)}(z)S_j^{(i+1,1)}(q^{-N-k+i-j+1}z) :=: X_i^{-(j+1,1)}(z)S_j^{(i,1)}(q^{-N-k+i-j+1}z) : \quad (1 \le j < i \le N-1), \tag{A35}$$

$$: X_i^{-(j,2)}(z)S_j^{(i+1,1)}(q^{-N-k+i-j+1}z) :=: X_i^{-(j+1,1)}(z)S_j^{(i,2)}(q^{-N-k+i-j+1}z) : \quad (1 \le j < i \le N-1), \tag{A36}$$

$$: X_i^{-(j,2)}(z)S_j^{(i,1)}(q^{-N-k+i-j+1}z) :=: X_i^{-(j,1)}(z)S_j^{(i,2)}(q^{-N-k+i-j+1}z) : \quad (1 \le j < i \le N), \tag{A37}$$

$$: X_N^{-(j,1)}(z)S_j^{(N+1,0)}(q^{-k-j+1}z) :=: X_N^{-(j+1,1)}(z)S_j^{(N,1)}(q^{-k-j+1}z) : \quad (1 \le j < i \le N-1), \tag{A38}$$

$$: X_N^{-(j,2)}(z)S_j^{(N+1,0)}(q^{-k-j+1}z) :=: X_N^{-(j+1,1)}(z)S_j^{(N,2)}(q^{-k-j+1}z) : \quad (1 \le j < i \le N-1), \tag{A39}$$

$$: X_i^{+(i,1)}(z)S_i^{(i+1,2)}(q^{-N+2i+1}z) :=: X_i^{+(i,2)}(z)S_i^{(i+1,1)}(q^{-N+2i+1}z) : \quad (1 \le i \le N-1), \tag{A40}$$

$$: X_i^{+(j,1)}(z)S_j^{(i,2)}(q^{-N+i+j+1}z) :=: X_i^{+(j+1,2)}(z)S_j^{(i+1,1)}(q^{-N+i+j+1}z) : \quad (1 \le i \le N-1, 1 \le j \le i-1), \tag{A41}$$

$$: X_N^{+(j,0)}(z)S_j^{(N,2)}(q^{j+1}z) :=: X_N^{+(j+1,0)}(z)S_j^{(N+1,0)}(q^{j+1}z) : \quad (1 \le j \le N-1), \tag{A42}$$

$$: X_i^{+(j,1)}(z)S_j^{(i+1,2)}(q^{-N+i+j+1}z) :=: X_i^{+(j,2)}(z)S_j^{(i+1,1)}(q^{-N+i+j+1}z) : \quad (1 \le i \le N-1, 1 \le j \le i-1), \tag{A43}$$

$$: X_i^{+(j,2)}(z)S_j^{(i,2)}(q^{-N+i+j+1}z) :=: X_i^{+(j+1,2)}(z)S_j^{(i+1,2)}(q^{-N+i+j+1}z) : \quad (1 \le i \le N-1, 1 \le j \le i-1), \tag{A44}$$

APPENDIX B: LEVEL-ZERO REPRESENTATION OF $U_q(\widehat{sl}(4|1))$

In this appendix we summarize the level-zero representation of $U_q(\widehat{sl}(4|1))$. Let V_α be the one parameter family of the 16(= 2^4)-dimensional typical representation of $U_q(sl(4|1))$.[37,38] In the case of $U_q(sl(4|1))$, we choose the basis $\{v_j\}_{1\le j\le 16}$ of V_α and assign them the $\mathbf{Z}_2$-gradings as following.

$$|v_1| = |v_5| = |v_7| = |v_8| = |v_9| = |v_{10}| = |v_{12}| = |v_{16}| = 0,$$
$$|v_2| = |v_3| = |v_4| = |v_6| = |v_{11}| = |v_{13}| = |v_{14}| = |v_{15}| = 1. \tag{B1}$$

In the homogeneous gradation, the evaluation representation $V_{\alpha,z}$ of $U_q(\widehat{sl}(4|1))$ is given by

$$h_1 = E_{4,4} - E_{6,6} + E_{7,7} - E_{8,8} + E_{9,9} - E_{10,10} + E_{11,11} - E_{13,13}, \tag{B2}$$

$$h_2 = E_{3,3} - E_{4,4} + E_{5,5} - E_{7,7} + E_{10,10} - E_{12,12} + E_{13,13} - E_{14,14}, \tag{B3}$$

$$h_3 = E_{2,2} - E_{3,3} + E_{7,7} - E_{9,9} + E_{8,8} - E_{10,10} + E_{14,14} - E_{15,15}, \tag{B4}$$

$$h_4 = \alpha\sum_{j=1}^{2} E_{j,j} + (\alpha+1)\sum_{j=3}^{8} E_{j,j} + (\alpha+2)\sum_{j=9}^{14} E_{j,j} + (\alpha+3)\sum_{j=15}^{16} E_{j,j}, \tag{B5}$$

$$e_1 = E_{4,6} + E_{7,8} + E_{9,10} + E_{11,13}, \tag{B6}$$

$$e_2 = E_{3,4} + E_{5,7} + E_{10,12} + E_{13,14}, \tag{B7}$$

$$e_3 = E_{2,3} + E_{7,9} + E_{8,10} + E_{14,15}, \tag{B8}$$

$$e_4 = -\sqrt{[\alpha]}E_{1,2} + \sqrt{[\alpha+1]}(E_{3,5} + E_{4,7} + E_{6,8}) \\ -\sqrt{[\alpha+2]}(E_{9,11} + E_{10,13} + E_{12,14}) + \sqrt{[\alpha+3]}E_{15,16}, \tag{B9}$$

$$f_1 = E_{6,4} + E_{8,7} + E_{10,9} + E_{13,11}, \tag{B10}$$

$$f_2 = E_{4,3} + E_{7,5} + E_{12,10} + E_{14,13}, \tag{B11}$$

$$f_3 = E_{3,2} + E_{9,7} + E_{10,8} + E_{15,14}, \tag{B12}$$

$$f_4 = -\sqrt{[\alpha]}E_{2,1} + \sqrt{[\alpha+1]}(E_{5,3} + E_{7,4} + E_{8,6}) \\ -\sqrt{[\alpha+2]}(E_{11,9} + E_{13,10} + E_{14,12}) + \sqrt{[\alpha+3]}E_{16,15}, \tag{B13}$$

$$h_0 = -\alpha(E_{1,1} + E_{6,6}) - (\alpha+1)(E_{2,2} + E_{3,3} + E_{4,4} + E_{8,8} + E_{10,10} + E_{12,12}) \\ -(\alpha+2)(E_{5,5} + E_{7,7} + E_{9,9} + E_{13,13} + E_{14,14} + E_{15,15}) - (\alpha+3)(E_{11,11} + E_{16,16}), \tag{B14}$$

$$e_0 = z(\sqrt{[\alpha]}E_{6,1} - \sqrt{[\alpha+1]}(E_{8,2} + E_{10,3} + E_{12,4}) \\ +\sqrt{[\alpha+2]}(E_{13,5} + E_{14,7} + E_{15,9}) - \sqrt{[\alpha+3]}E_{16,11}), \tag{B15}$$

$$f_0 = -z^{-1}(\sqrt{[\alpha]}E_{1,6} - \sqrt{[\alpha+1]}(E_{2,8} + E_{3,10} + E_{4,12}) \\ +\sqrt{[\alpha+2]}(E_{5,13} + E_{7,14} + E_{9,15}) - \sqrt{[\alpha+3]}E_{11,16}). \tag{B16}$$

We choose the dual basis $\{v_j^*\}_{1\le j\le 16}$ of $V_\alpha^{*,S}$ and assign them the $\mathbf{Z}_2$-gradings as following.

$$|v_1^*|=|v_5^*|=|v_7^*|=|v_8^*|=|v_9^*|=|v_{10}^*|=|v_{12}^*|=|v_{16}^*|=0,$$
$$|v_2^*|=|v_3^*|=|v_4^*|=|v_6^*|=|v_{11}^*|=|v_{13}^*|=|v_{14}^*|=|v_{15}^*|=1. \tag{B17}$$

In the homogeneous gradation, the evaluation representation $V_{\alpha,z}^{*S}$ of $U_q(\widehat{sl}(4|1))$ is given by

$$h_1=-E_{4,4}+E_{6,6}-E_{7,7}+E_{8,8}-E_{9,9}+E_{10,10}-E_{11,11}+E_{13,13}, \tag{B18}$$

$$h_2=-E_{3,3}+E_{4,4}-E_{5,5}+E_{7,7}-E_{10,10}+E_{12,12}-E_{13,13}+E_{14,14}, \tag{B19}$$

$$h_3=-E_{2,2}+E_{3,3}-E_{7,7}+E_{9,9}-E_{8,8}+E_{10,10}-E_{14,14}+E_{15,15}, \tag{B20}$$

$$h_4=-\alpha\sum_{j=1}^{2}E_{j,j}-(\alpha+1)\sum_{j=3}^{8}E_{j,j}-(\alpha+2)\sum_{j=9}^{14}E_{j,j}-(\alpha+3)\sum_{j=15}^{16}E_{j,j}, \tag{B21}$$

$$e_1=-q^{-1}(E_{6,4}+E_{8,7}+E_{10,9}+E_{13,11}), \tag{B22}$$

$$e_2=-q^{-1}(E_{4,3}+E_{7,5}+E_{12,10}+E_{14,13}), \tag{B23}$$

$$e_3=-q^{-1}(E_{3,2}+E_{9,7}+E_{10,8}+E_{15,14}), \tag{B24}$$

$$e_4=\sqrt{[\alpha]}q^{-\alpha}E_{2,1}+\sqrt{[\alpha+1]}q^{-\alpha-1}(E_{5,3}+E_{7,4}+E_{8,6})$$
$$+\sqrt{[\alpha+2]}q^{-\alpha-2}(E_{11,9}+E_{13,10}+E_{14,12})+\sqrt{[\alpha+3]}q^{-\alpha-3}E_{16,15}, \tag{B25}$$

$$f_1=-q(E_{4,6}+E_{7,8}+E_{9,10}+E_{11,13}), \tag{B26}$$

$$f_2=-q(E_{3,4}+E_{5,7}+E_{10,12}+E_{13,14}), \tag{B27}$$

$$f_3=-q(E_{2,3}+E_{7,9}+E_{8,10}+E_{14,15}), \tag{B28}$$

$$f_4=-\sqrt{[\alpha]}q^{\alpha}E_{1,2}-\sqrt{[\alpha+1]}q^{\alpha+1}(E_{3,5}+E_{4,7}+E_{6,8})$$
$$-\sqrt{[\alpha+2]}q^{\alpha+2}(E_{9,11}+E_{10,13}+E_{12,14})-\sqrt{[\alpha+3]}q^{\alpha+3}E_{15,16}, \tag{B29}$$

$$h_0=\alpha(E_{1,1}+E_{6,6})+(\alpha+1)(E_{2,2}+E_{3,3}+E_{4,4}+E_{8,8}+E_{10,10}+E_{12,12})$$
$$+(\alpha+2)(E_{5,5}+E_{7,7}+E_{9,9}+E_{13,13}+E_{14,14}+E_{15,15})+(\alpha+3)(E_{11,11}+E_{16,16}), \tag{B30}$$

$$e_0=z(\sqrt{[\alpha]}q^{\alpha}E_{1,6}+\sqrt{[\alpha+1]}q^{\alpha+1}(E_{2,8}+E_{3,10}+E_{4,12})$$
$$+\sqrt{[\alpha+2]}q^{\alpha+2}(E_{5,13}+E_{7,14}+E_{9,15})+\sqrt{[\alpha+3]}q^{\alpha+3}E_{11,16}), \tag{B31}$$

$$f_0=z^{-1}(\sqrt{[\alpha]}q^{-\alpha}E_{6,1}+\sqrt{[\alpha+1]}q^{-\alpha-1}(E_{8,2}+E_{10,3}+E_{12,4})$$
$$+\sqrt{[\alpha+2]}q^{-\alpha-2}(E_{13,5}+E_{14,7}+E_{15,9})+\sqrt{[\alpha+3]}q^{-\alpha-3}E_{16,11}). \tag{B32}$$

We give the level-zero realization of the Drinfeld generators.

Proposition B.1: On $V_{\alpha,z}$, the Drinfeld generators of $U_q(\widehat{sl}(4|1))$ are given by

$$h_{1,m} = \frac{[m]}{m}(q^{\alpha+3}z)^m(q^{-m}E_{4,4} - q^m E_{6,6} + q^{-m}E_{7,7} - q^m E_{8,8} + q^{-m}E_{9,9} - q^m E_{10,10} + q^{-m}E_{11,11} - q^m E_{13,13}), \tag{B33}$$

$$h_{2,m} = \frac{[m]}{m}(q^{\alpha+3}z)^m(q^{-2m}E_{3,3} - E_{4,4} + q^{-2m}E_{5,5} - E_{7,7} + E_{10,10} - q^{2m}E_{12,12} + E_{13,13} - q^{2m}E_{14,14}), \tag{B34}$$

$$h_{3,m} = \frac{[m]}{m}(q^{\alpha+3}z)^m(q^{-3m}E_{2,2} - q^{-m}E_{3,3} + q^{-m}E_{7,7} - q^m E_{9,9} + q^{-m}E_{8,8} - q^m E_{10,10} + q^m E_{14,14} - q^{3m}E_{15,15}), \tag{B35}$$

$$h_{4,m} = \frac{1}{m}z^m\left([\alpha m]\sum_{j=1}^{2}E_{j,j} + [(\alpha+1)m]q^m\sum_{j=3}^{8}E_{j,j} + [(\alpha+2)m]q^{2m}\sum_{j=9}^{14}E_{j,j} + [(\alpha+3)m]q^{3m}\sum_{j=15}^{16}E_{j,j}\right), \tag{B36}$$

$$x^+_{1,n} = (q^{\alpha+3}z)^m(E_{4,6} + E_{7,8} + E_{9,10} + E_{11,13}), \tag{B37}$$

$$x^+_{2,n} = (q^{\alpha+3}z)^n(q^{-3n}E_{3,4} + q^{-n}E_{5,7} + q^n E_{10,12} + q^n E_{13,14}), \tag{B38}$$

$$x^+_{3,n} = (q^{\alpha+3}z)^n(q^{-2n}E_{2,3} + E_{7,9} + E_{8,10} + q^{2n}E_{14,15}), \tag{B39}$$

$$x^+_{4,n} = (q^{\alpha+3}z)^n(-\sqrt{[\alpha]}q^{-3n}E_{1,2} + \sqrt{[\alpha+1]}q^{-n}(E_{3,5} + E_{4,7} + E_{6,8}) - \sqrt{[\alpha+2]}q^n(E_{9,11} + E_{10,13} + E_{12,14}) + \sqrt{[\alpha+3]}q^{3n}E_{15,16}), \tag{B40}$$

$$x^-_{1,n} = (q^{\alpha+3}z)^m(E_{6,4} + E_{8,7} + E_{10,9} + E_{13,11}), \tag{B41}$$

$$x^-_{2,n} = (q^{\alpha+3}z)^n(q^{-n}E_{4,3} + q^{-n}E_{7,5} + q^n E_{12,10} + q^n E_{14,13}), \tag{B42}$$

$$x^-_{3,n} = (q^{\alpha+3}z)^n(q^{-2n}E_{3,2} + E_{9,7} + E_{10,8} + q^{2n}E_{15,14}), \tag{B43}$$

$$x^-_{4,n} = (q^{\alpha+3}z)^n(-\sqrt{[\alpha]}q^{-3n}E_{2,1} + \sqrt{[\alpha+1]}q^{-n}(E_{5,3} + E_{7,4} + E_{8,6}) - \sqrt{[\alpha+2]}q^n(E_{11,9} + E_{13,10} + E_{14,12}) + \sqrt{[\alpha+3]}q^{3n}E_{16,15}). \tag{B44}$$

On $V^{*S}_{\alpha,z}$, the Drinfeld generators of $U_q(\widehat{sl}(4|1))$ are given by

$$h_{1,m} = \frac{[m]}{m}(q^{-\alpha-3}z)^m(-q^m E_{4,4} + q^{-m}E_{6,6} - q^m E_{7,7} + q^{-m}E_{8,8} - q^m E_{9,9} + q^{-m}E_{10,10} - q^m E_{11,11} + q^{-m}E_{13,13}), \tag{B45}$$

$$h_{2,m} = \frac{[m]}{m}(q^{-\alpha-3}z)^m(-q^{2m}E_{3,3} + E_{4,4} - q^{2m}E_{5,5} + E_{7,7} - E_{10,10} + q^{-2m}E_{12,12} - E_{13,13} + q^{-2m}E_{14,14}), \tag{B46}$$

$$h_{3,m} = \frac{[m]}{m}(q^{-\alpha-3}z)^m(-q^{3m}E_{2,2} + q^m E_{3,3} - q^m E_{7,7} + q^{-m}E_{9,9} -q^m E_{8,8} + q^{-m}E_{10,10} - q^{-m}E_{14,14} + q^{-3m}E_{15,15}), \tag{B47}$$

$$h_{4,m} = \frac{-1}{m}z^m([\alpha m]\sum_{j=1}^{2}E_{j,j} + [(\alpha+1)m]q^{-m}\sum_{j=3}^{8}E_{j,j} +[(\alpha+2)m]q^{-2m}\sum_{j=9}^{14}E_{j,j} + [(\alpha+3)m]q^{-3m}\sum_{j=15}^{16}E_{j,j}), \tag{B48}$$

$$x^+_{1,n} = -q^{-1}(q^{-\alpha-3}z)^n(E_{6,4} + E_{8,7} + E_{10,9} + E_{13,11}), \tag{B49}$$

$$x^+_{2,n} = -q^{-1}(q^{-\alpha-3}z)^n(q^n E_{4,3} + q^n E_{7,5} + q^{-n}E_{12,10} + q^{-n}E_{14,13}), \tag{B50}$$

$$x^+_{3,n} = -q^{-1}(q^{-\alpha-3}z)^n(q^{2n}E_{3,2} + E_{9,7} + E_{10,8} + q^{-2n}E_{15,14}), \tag{B51}$$

$$x^+_{4,n} = (q^{-\alpha-3}z)^n(\sqrt{[\alpha]}q^{-\alpha+3n}E_{2,1} + \sqrt{[\alpha+1]}q^{-\alpha-1+n}(E_{5,3} + E_{7,4} + E_{8,6}) +\sqrt{[\alpha+2]}q^{-\alpha-2-n}(E_{11,9} + E_{13,10} + E_{14,12}) + \sqrt{[\alpha+3]}q^{-\alpha-3-3n}E_{16,15}), \tag{B52}$$

$$x^-_{1,n} = -q(q^{-\alpha-3}z)^n(E_{4,6} + E_{7,8} + E_{9,10} + E_{11,13}), \tag{B53}$$

$$x^-_{2,n} = -q(q^{-\alpha-3}z)^n(q^n E_{3,4} + q^n E_{5,7} + q^{-n}E_{10,12} + q^{-n}E_{13,14}), \tag{B54}$$

$$x^-_{3,n} = -q(q^{-\alpha-3}z)^n(q^{2n}E_{2,3} + E_{7,9} + E_{8,10} + q^{-2n}E_{14,15}), \tag{B55}$$

$$x^-_{4,n} = -(q^{-\alpha-3}z)^n(\sqrt{[\alpha]}q^{\alpha+3n}E_{1,2} + \sqrt{[\alpha+1]}q^{\alpha+1+n}(E_{3,5} + E_{4,7} + E_{6,8}) +\sqrt{[\alpha+2]}q^{\alpha+2-n}(E_{9,11} + E_{10,13} + E_{12,14}) + \sqrt{[\alpha+3]}q^{\alpha+3-3n}E_{15,16}). \tag{B56}$$

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