

ELLIPTIC DEFORMED SUPERALGEBRA $U_{q,p}(\widehat{sl}(M|N))$

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Abstract

We introduce the elliptic superalgebra $U_{q,p}(\widehat{sl}(M|N))$ as one parameter deformation of the quantum superalgebra $U_q(\widehat{sl}(M|N))$. For an arbitrary level $k \neq 1$ we give the bosonization of the elliptic superalgebra $U_{q,p}(\widehat{sl}(1|2))$ and the screening currents that commute with $U_{q,p}(\widehat{sl}(1|2))$ modulo total difference.

1 Introduction

Infinite dimensional symmetry has been an impressive success in conformal field theory (CFT) [1]. Solvable lattice model is an off-critical extension of CFT and infinite dimensional symmetry plays an important role in algebraic analysis of solvable lattice model [2]. The lattice counterpart of minimal unitary CFT is Andrews-Baxter-Forrester (ABF) model [3], whose Boltzmann weights are elliptic solutions of the Yang-Baxter equation (YBE) of the face-type. Among the solvable models based on YBE, those related to elliptic solutions occupy a fundamental place.

Elliptic algebras are certain algebraic structures introduced to investigate these elliptic models. In study of k -fusion hierarchy of ABF model, Konno [4] introduced the elliptic algebra $U_{q,p}(\widehat{sl}(2))$ and constructed bosonization of the vertex operator by using this algebra. Jimbo-Konno-Odake-Shiraishi [5] constructed the elliptic algebra $U_{q,p}(g)$ by dressing the usual Drinfeld currents [20] of the quantum group $U_q(g)$ for non-twisted affine Lie algebra g . In this paper we introduce the elliptic deformed superalgebra $U_{q,p}(\widehat{sl}(M|N))$ as one parameter deformation of the quantum superalgebra $U_q(\widehat{sl}(M|N))$. We give the bosonization of the elliptic superalgebra $U_{q,p}(\widehat{sl}(1|2))$ and $U_{q,p}(\widehat{sl}(2|1))$ for generic level k , and give the screening currents that commute with $U_{q,p}(\widehat{sl}(1|2))$ and $U_{q,p}(\widehat{sl}(2|1))$ modulo total difference.

In this paper we aim to contribute mathematical tools for the study of super $\widehat{sl}(M|N)$ -family of the ABF model [19]. Mathematical tools are the elliptic algebra $U_{q,p}(\widehat{sl}(M|N))$ and bosonizations. We give comments on $\widehat{sl}(N)$ -family of the ABF model, where such mathematical tools have been used previously in analogous, but simpler case. Andrews-Baxter-Forrester [3] introduced the ABF model, that gives an extension of the hard hexagon model, and derived local height probabilities by Baxter's corner transfer matrix method (CTM) [6]. The k -fusion and higher-rank generalization, that we call $\widehat{sl}(N)$ -family of the ABF model, have been studied in [7, 8, 10, 11]. Inspired by the vertex operator approach to the 6-vertex model [2, 12, 13, 14], that originated from CTM, Lukyanov-Pugai [15] studied the vertex operator approach to the ABF model, and derived integral representations of multi-point local height probabilities. In study of k -fusion hierarchy of ABF model, Konno [4] introduced the elliptic algebra $U_{q,p}(\widehat{sl}(2))$ and constructed bosonization of the vertex operator by using this algebra. The vertex operator approach to the higher-rank generalization of the ABF model have been studied in [16, 17, 18]. In the vertex operator approach to $\widehat{sl}(N)$ -family of the ABF model, bosonization of the vertex operator played important role. In construction of the vertex operator, the current of the elliptic algebra $U_{q,p}(\widehat{sl}(N))$ and its bosonization played important roles. In order to derive integral representation of multi-point local height probabilities of super $\widehat{sl}(M|N)$ -family of the ABF model, we have to construct bosonizations of the vertex operators by using the current of the elliptic algebra $U_{q,p}(\widehat{sl}(M|N))$, and understand the structure of the space of state of the model by CTM [6], that has been open problem for superalgebra $\widehat{sl}(M|N)$.

Next we give comments on pure mathematical aspects. Through an attempt to understand solvable models based on elliptic solutions of YBE, various versions of elliptic algebras have been introduced [4, 5, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. It is important to understand not only themselves but also relations between them. Here we summarize some basic facts on the elliptic quantum group $\mathcal{B}_{q,\lambda}(g)$ and the elliptic algebra $U_{q,p}(g)$. The elliptic quantum group

$\mathcal{B}_{q,\lambda}(g)$, was introduced by twisting the standard quantum group $U_q(g)$ [25, 26, 27, 28, 29], where g is the symmetrizable Kac-Moody algebra. The elliptic quantum group $\mathcal{B}_{q,\lambda}(g)$ has quasi-Hopf structure and the elliptic algebra $U_{q,p}(\widehat{sl}(2))$ has H -Hopf algebroid structure [31, 32]. The realizations of the L -operators of the elliptic quantum group $\mathcal{B}_{q,\lambda}(g)$ were constructed in [5, 21, 22] by using the currents of the elliptic algebra $U_{q,p}(g)$ for $g = \widehat{sl}(N), A_2^{(2)}$. This suggests that the currents of $U_{q,p}(g)$ give the Drinfeld currents [20] of the elliptic quantum group $\mathcal{B}_{q,\lambda}(g)$. The construction of the elliptic quantum group $\mathcal{B}_{q,\lambda}(g)$ has been extended to the superalgebra $g = \widehat{sl}(M|N)$ [30]. In this paper we introduce the elliptic algebra $U_{q,p}(\widehat{sl}(M|N))$. We conjecture that the L -operator of $\mathcal{B}_{q,\lambda}(\widehat{sl}(M|N))$ is constructed by using the currents of $U_{q,p}(\widehat{sl}(M|N))$ and that there exists H -Hopf algebroid structure for $U_{q,p}(\widehat{sl}(M|N))$. The bosonizations of the vertex operators give useful information for construction of the L -operator, thorough so-called Miki's construction [33] of the L -operator. The above is background mathematical theory of the vertex operator. Next we give a comment on mathematical phenomenon of the space of state. Date-Jimbo-Kuniba-Miwa-Okado [8, 9, 10, 11] found that local height probabilities of $\widehat{sl}(N)$ -family of the ABF model were expressed in terms of the branching coefficients appearing in the irreducible decomposition of the character of $\widehat{sl}(N)$ [34, 35]. In order to extend this to super $\widehat{sl}(M|N)$ -family of the ABF model, we have to know the character formulae of the superalgebra $\widehat{sl}(M|N)$, that gives the affine generalization of formulae [36, 37].

The text is organized as follows. In section 2, after preparing the notations and giving the definition of the quantum group $U_q(\widehat{sl}(M|N))$, we introduce the elliptic deformed superalgebra $U_{q,p}(\widehat{sl}(M|N))$. Our approach is based on the dressing procedure of the Drinfeld current of the quantum group. In section 3 we give bosonizations of the superalgebra $U_q(g), U_{q,p}(g)$ ($g = \widehat{sl}(1|2), \widehat{sl}(2|1)$) for an arbitrary level k . We give the screening currents that commute with $U_q(g), U_{q,p}(g)$ ($g = \widehat{sl}(1|2), \widehat{sl}(2|1)$) modulo total difference. In appendix we summarize some useful formulae of bosonizations and screening currents.

2 Elliptic deformed superalgebra $U_{q,p}(\widehat{sl}(M|N))$

In this section we introduce the elliptic superalgebra $U_{q,p}(\widehat{sl}(M|N))$. Kac [38] introduced the superalgebra generalization of contragredient Lie algebra. Van de Leur [39] classified the contragredient superalgebra g of finite growth. Yamane [40] introduced quantum affine superalgebra $U_q(g)$ and constructed the Drinfeld currents. We give elliptic deformation of the quantum affine superalgebra by developing the dressing procedure [5].

2.1 Quantum superalgebra $U_q(\widehat{sl}(M|N))$

In this section we review the Drinfeld realization of the quantum superalgebra $U_q(\widehat{sl}(M|N))$ for $M, N = 1, 2, 3, \dots$ [40]. We restrict our consideration to $M \neq N$. The quantum superalgebra $U_q(\widehat{sl}(M|N))$ in [40] is a q -deformation of the universal enveloping algebra of $\widehat{sl}(M|N)$ [39]. Hereafter we fix a complex number $q \neq 0, |q| < 1$. Let us set

$$[x, y] = xy - yx, \quad \{x, y\} = xy + yx, \quad [a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}. \quad (2.1)$$

The Cartan matrix of the Lie superalgebra $\widehat{sl}(M|N)$ is given by

$$(A_{i,j})_{0 \leq i,j \leq M+N-1} = \begin{pmatrix} 0 & -1 & 0 & \cdots & & & \cdots & 0 & 1 \\ -1 & 2 & -1 & \cdots & & & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & & & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & -1 & \cdots & & & \\ & & \cdots & -1 & 2 & -1 & \cdots & & \\ & & \cdots & -1 & 0 & 1 & \cdots & & \\ & & & \cdots & 1 & -2 & 1 & \cdots & \\ & & & \cdots & 1 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & & & & \cdots & -2 & 1 & 0 \\ 0 & 0 & \cdots & & & & \cdots & 1 & -2 & 1 \\ 1 & 0 & \cdots & & & & \cdots & 0 & 1 & -2 \end{pmatrix}, \quad (2.2)$$

where the diagonal part is $(A_{i,i})_{0 \leq i \leq M+N-1} = (0, \overbrace{2, \dots, 2}^{M-1}, 0, \overbrace{-2, \dots, -2}^{N-1})$.

Definition 2.1 [40] *The generators of the quantum superalgebra $U_q(\widehat{sl}(M|N))$, which we call the Drinfeld generators, are given by*

$$x_{i,m}^\pm, a_{i,n}, h_i, c, \quad (1 \leq i \leq M+N-1, m \in \mathbb{Z}, n \in \mathbb{Z}_{\neq 0}). \quad (2.3)$$

Defining relations are

$$c : \text{central}, \quad [h_i, a_{j,m}] = 0, \quad (2.4)$$

$$[a_{i,m}, a_{j,n}] = \frac{[A_{i,j}m]_q [cm]_q}{m} q^{-c|m|} \delta_{m+n,0}, \quad (2.5)$$

$$[h_i, x_j^\pm(z)] = \pm A_{i,j} x_j^\pm(z), \quad (2.6)$$

$$[a_{i,m}, x_j^+(z)] = \frac{[A_{i,j}m]_q}{m} q^{-c|m|} z^m x_j^+(z), \quad (2.7)$$

$$[a_{i,m}, x_j^-(z)] = -\frac{[A_{i,j}m]_q}{m} z^m x_j^-(z), \quad (2.8)$$

$$(z_1 - q^{\pm A_{i,j}} z_2) x_i^\pm(z_1) x_j^\pm(z_2) = (q^{\pm A_{j,i}} z_1 - z_2) x_j^\pm(z_2) x_i^\pm(z_1), \quad \text{for } |A_{i,j}| \neq 0, \quad (2.9)$$

$$x_i^\pm(z_1) x_j^\pm(z_2) = x_j^\pm(z_2) x_i^\pm(z_1), \quad \text{for } |A_{i,j}| = 0, (i, j) \neq (M, M), \quad (2.10)$$

$$\{x_M^\pm(z_1), x_M^\pm(z_2)\} = 0, \quad (2.11)$$

$$[x_i^+(z_1), x_j^-(z_2)] = \frac{\delta_{i,j}}{(q - q^{-1})z_1 z_2} \left(\delta(q^{-c} z_1/z_2) \psi_i^+(q^{\frac{c}{2}} z_2) - \delta(q^c z_1/z_2) \psi_i^-(q^{-\frac{c}{2}} z_2) \right),$$

for $(i, j) \neq (M, M),$ (2.12)

$$\{x_M^+(z_1), x_M^-(z_2)\} = \frac{1}{(q - q^{-1})z_1 z_2} \left(\delta(q^{-c} z_1/z_2) \psi_M^+(q^{\frac{c}{2}} z_2) - \delta(q^c z_1/z_2) \psi_M^-(q^{-\frac{c}{2}} z_2) \right),$$

(2.13)

$$\begin{aligned} & \left(x_i^\pm(z_1) x_i^\pm(z_2) x_j^\pm(z) - (q + q^{-1}) x_i^\pm(z_1) x_j^\pm(z) x_i^\pm(z_2) + x_j^\pm(z) x_i^\pm(z_1) x_i^\pm(z_2) \right) \\ & + (z_1 \leftrightarrow z_2) = 0, \quad \text{for } |A_{i,j}| = 1, i \neq M, \end{aligned} \quad (2.14)$$

$$\begin{aligned} & (x_M^\pm(z_1) x_{M+1}^\pm(w_1) x_M^\pm(z_2) x_{M-1}^\pm(w_2) - q^{-1} x_M^\pm(z_1) x_{M+1}^\pm(w_1) x_{M-1}^\pm(w_2) x_M^\pm(z_2) \\ & - q x_M^\pm(z_1) x_M^\pm(z_2) x_{M-1}^\pm(w_2) x_{M+1}^\pm(w_1) + x_M^\pm(z_1) x_{M-1}^\pm(w_2) x_M^\pm(z_2) x_{M+1}^\pm(w_1) \\ & + x_{M+1}^\pm(w_1) x_M^\pm(z_2) x_{M-1}^\pm(w_2) x_M^\pm(z_1) - q^{-1} x_{M+1}^\pm(w_1) x_{M-1}^\pm(w_2) x_M^\pm(z_2) x_M^\pm(z_1) \\ & - q x_M^\pm(z_2) x_{M-1}^\pm(w_2) x_{M+1}^\pm(w_1) x_M^\pm(z_1) + x_{M-1}^\pm(w_2) x_M^\pm(z_2) x_{M+1}^\pm(w_1) x_M^\pm(z_1)) \\ & + (z_1 \leftrightarrow z_2) = 0, \end{aligned} \quad (2.15)$$

where we have used $\delta(z) = \sum_{m \in \mathbb{Z}} z^m$. Here we have set the generating functions

$$x_j^\pm(z) = \sum_{m \in \mathbb{Z}} x_{j,m}^\pm z^{-m-1}, \quad (2.16)$$

$$\psi_i^+(q^{\frac{c}{2}} z) = q^{h_i} \exp \left((q - q^{-1}) \sum_{m > 0} a_{i,m} z^{-m} \right), \quad (2.17)$$

$$\psi_i^-(q^{-\frac{c}{2}} z) = q^{-h_i} \exp \left(-(q - q^{-1}) \sum_{m > 0} a_{i,-m} z^m \right). \quad (2.18)$$

We changed the gauge of boson $a_{i,m}$ from those of [40]. In what follows we assume $c \in \mathbb{C}$.

2.2 Elliptic deformed superalgebra $U_{q,p}(\widehat{sl}(M|N))$

In this section we introduce the elliptic superalgebra $U_{q,p}(\widehat{sl}(M|N))$ for $M, N = 1, 2, 3, \dots, (M \neq N)$. Let us introduce a deformation parameter r such that

$$r, r^* = r - c > 0. \quad (2.19)$$

We often use the parameterization.

$$p = q^{2r} = e^{-\frac{2\pi i}{\tau}}, \quad p^* = q^{2r^*} = e^{-\frac{2\pi i}{\tau^*}}, \quad z = q^{2u}, \quad w = q^{2v}. \quad (2.20)$$

We have $r\tau = r^*\tau^*$. Let us set the Jacobi theta functions $[u]$, $[u]^*$ by

$$[u] = q^{\frac{u^2}{r}-u} \frac{\Theta_p(q^{2u})}{(p;p)_\infty^3}, \quad [u]^* = q^{\frac{u^2}{r^*}-u} \frac{\Theta_{p^*}(q^{2u})}{(p^*;p^*)_\infty^3}. \quad (2.21)$$

Here we have used the standard symbols.

$$\Theta_p(z) = (p;p)_\infty(z;p)_\infty(pz^{-1};p)_\infty, \quad (2.22)$$

$$(z; t_1, \dots, t_k)_\infty = \prod_{n_1, \dots, n_k \geq 0} (1 - zt_1^{n_1} \dots t_k^{n_k}). \quad (2.23)$$

Definition 2.2 *The elliptic superalgebra $U_{q,p}(\widehat{sl}(M|N))$ is generated by the currents (operator valued function) and elements*

$$E_j(z), F_j(z), B_{j,n}, h_j, c \quad (1 \leq j \leq M+N-1, n \in \mathbb{Z}_{\neq 0}). \quad (2.24)$$

The defining relations are given as follows.

For $1 \leq i, j \leq M+N-1$, the relations are

$$c : \text{central}, [h_i, B_{j,m}] = 0, \quad (2.25)$$

$$[B_{i,m}, B_{j,n}] = \frac{[A_{i,j}m]_q [cm]_q}{m} \frac{[r^*m]_q}{[rm]_q} \delta_{m+n,0}, \quad (2.26)$$

$$[h_i, E_j(z)] = A_{i,j} E_j(z), \quad [h_i, F_j(z)] = -A_{i,j} F_j(z), \quad (2.27)$$

$$[B_{i,m}, E_j(z)] = \frac{[A_{i,j}m]_q}{m} z^m E_j(z), \quad [B_{i,m}, F_j(z)] = -\frac{[A_{i,j}m]_q}{m} \frac{[r^*m]_q}{[rm]_q} z^m F_j(z). \quad (2.28)$$

For $1 \leq i, j \leq M+N-1$ such that $(i, j) \neq (M, M)$, the relations are

$$\left[u_1 - u_2 - \frac{A_{i,j}}{2} \right]^* E_i(z_1) E_j(z_2) = \left[u_1 - u_2 + \frac{A_{i,j}}{2} \right]^* E_j(z_2) E_i(z_1), \quad (2.29)$$

$$\left[u_1 - u_2 + \frac{A_{i,j}}{2} \right] F_i(z_1) F_j(z_2) = \left[u_1 - u_2 - \frac{A_{i,j}}{2} \right] F_j(z_2) F_i(z_1), \quad (2.30)$$

$$[E_i(z_1), F_j(z_2)] = \frac{\delta_{i,j}}{(q - q^{-1})z_1 z_2} (\delta(q^{-c} z_1 / z_2) H_i(q^r z_2) - \delta(q^c z_1 / z_2) H_i(q^{-r} z_2)), \quad (2.31)$$

$$\{E_M(z_1), E_M(z_2)\} = 0, \quad \{F_M(z_1), F_M(z_2)\} = 0, \quad (2.32)$$

$$\{E_M(z_1), F_M(z_2)\} = \frac{1}{(q - q^{-1})z_1 z_2} (\delta(q^{-c} z_1 / z_2) H_M(q^r z_2) - \delta(q^c z_1 / z_2) H_M(q^{-r} z_2)). \quad (2.33)$$

For $1 \leq i, j \leq M+N-1$, the relations are

$$H_i(z_1) H_j(z_2) = \frac{[u_2 - u_1 - \frac{A_{i,j}}{2}]^* [u_2 - u_1 + \frac{A_{i,j}}{2}]}{[u_2 - u_1 + \frac{A_{i,j}}{2}]^* [u_2 - u_1 - \frac{A_{i,j}}{2}]} H_j(z_2) H_i(z_1), \quad (2.34)$$

$$H_i(z_1)E_j(z_2) = \frac{[u_1 - u_2 + \frac{r^*}{2} + \frac{A_{i,j}}{2}]^*}{[u_1 - u_2 + \frac{r^*}{2} - \frac{A_{i,j}}{2}]^*} E_j(z_2)H_i(z_1), \quad (2.35)$$

$$H_i(z_1)F_j(z_2) = \frac{[u_1 - u_2 + \frac{r}{2} + \frac{A_{i,j}}{2}]}{[u_1 - u_2 + \frac{r}{2} - \frac{A_{i,j}}{2}]} F_j(z_2)H_i(z_1). \quad (2.36)$$

For $1 \leq i, j \leq M + N - 1$ ($i \neq M$) such that $|A_{i,j}| = 1$, they satisfy the Serre relations

$$\begin{aligned} & \left(E_i(z_1)E_i(z_2)E_j(z) \frac{\{q^{A_{i,j}} \frac{z}{z_1}\}^* \{q^{A_{i,j}} \frac{z}{z_2}\}^*}{\{q^{-A_{i,j}} \frac{z}{z_1}\}^* \{q^{-A_{i,j}} \frac{z}{z_2}\}^*} \left(\frac{z}{z_2} \right)^{\frac{1}{r^*} A_{i,j}} \right. \\ & - (q + q^{-1})E_i(z_1)E_j(z)E_i(z_2) \frac{\{q^{A_{i,j}} \frac{z}{z_1}\}^* \{q^{A_{i,j}} \frac{z_2}{z}\}^*}{\{q^{-A_{i,j}} \frac{z}{z_1}\}^* \{q^{-A_{i,j}} \frac{z_2}{z}\}^*} \\ & + E_j(z)E_i(z_1)E_i(z_2) \frac{\{q^{A_{i,j}} \frac{z_1}{z}\}^* \{q^{A_{i,j}} \frac{z_2}{z}\}^*}{\{q^{-A_{i,j}} \frac{z_1}{z}\}^* \{q^{-A_{i,j}} \frac{z_2}{z}\}^*} \left(\frac{z_1}{z} \right)^{\frac{1}{r^*} A_{i,j}} \left. \right) \frac{\{q^{A_{i,i}} \frac{z_2}{z_1}\}^*}{\{q^{-A_{i,i}} \frac{z_2}{z_1}\}^*} z_1^{-\frac{1}{r^*} (A_{i,i} + A_{i,j})} \\ & + (z_1 \leftrightarrow z_2) = 0, \end{aligned} \quad (2.37)$$

$$\begin{aligned} & \left(F_i(z_1)F_i(z_2)F_j(z) \frac{\{q^{-A_{i,j}} \frac{z}{z_1}\} \{q^{-A_{i,j}} \frac{z}{z_2}\}}{\{q^{A_{i,j}} \frac{z}{z_1}\} \{q^{A_{i,j}} \frac{z}{z_2}\}} \left(\frac{z_2}{z} \right)^{\frac{1}{r}} \right. \\ & - (q + q^{-1})F_i(z_1)F_j(z)F_i(z_2) \frac{\{q^{-A_{i,j}} \frac{z}{z_1}\} \{q^{-A_{i,j}} \frac{z_2}{z}\}}{\{q^{A_{i,j}} \frac{z}{z_1}\} \{q^{A_{i,j}} \frac{z_2}{z}\}} \\ & + F_j(z)F_i(z_1)F_i(z_2) \frac{\{q^{-A_{i,j}} \frac{z_1}{z}\} \{q^{-A_{i,j}} \frac{z_2}{z}\}}{\{q^{A_{i,j}} \frac{z_1}{z}\} \{q^{A_{i,j}} \frac{z_2}{z}\}} \left(\frac{z}{z_1} \right)^{\frac{1}{r}} \left. \right) \frac{\{q^{-A_{i,i}} \frac{z_2}{z_1}\}}{\{q^{A_{i,i}} \frac{z_2}{z_1}\}} z_1^{\frac{1}{r} (A_{i,i} + A_{i,j})} \\ & + (z_1 \leftrightarrow z_2) = 0, \end{aligned} \quad (2.38)$$

and

$$\begin{aligned} & \left(E_M(z_1)E_{M+1}(w_1)E_M(z_2)E_{M-1}(w_2) \frac{\{qw_1\}^* \{qz_2\}^* \{w_2\}^* \{w_2\}^*}{\{w_1\}^* \{z_2\}^* \{qw_2\}^* \{qw_2\}^*} \left(\frac{w_2}{z_2} \right)^{\frac{1}{r^*}} \right. \\ & - q^{-1}E_M(z_1)E_{M+1}(w_1)E_{M-1}(w_2)E_M(z_2) \frac{\{qw_1\}^* \{w_2\}^* \{qz_2\}^* \{z_2\}^*}{\{w_1\}^* \{qw_2\}^* \{z_2\}^* \{qw_2\}^*} \\ & - qE_M(z_1)E_M(z_2)E_{M-1}(w_2)E_{M+1}(w_1) \frac{\{w_2\}^* \{w_2\}^* \{qw_1\}^* \{qw_1\}^*}{\{qw_2\}^* \{qw_1\}^* \{w_1\}^* \{z_2\}^*} \left(\frac{w_2}{w_1} \right)^{\frac{1}{r^*}} \\ & + E_M(z_1)E_{M-1}(w_2)E_M(z_2)E_{M+1}(w_1) \frac{\{w_2\}^* \{z_2\}^* \{qw_1\}^* \{qw_1\}^*}{\{qw_2\}^* \{qz_2\}^* \{w_1\}^* \{w_1\}^*} \left(\frac{z_2}{w_1} \right)^{\frac{1}{r^*}} \\ & + E_{M+1}(w_1)E_M(z_2)E_{M-1}(w_2)E_M(z_1) \frac{\{qz_2\}^* \{w_2\}^* \{qz_1\}^* \{z_1\}^*}{\{z_2\}^* \{qw_2\}^* \{z_1\}^* \{qw_2\}^*} \left(\frac{w_1}{z_2} \right)^{\frac{1}{r^*}} \\ & - q^{-1}E_{M+1}(w_1)E_{M-1}(w_2)E_M(z_2)E_M(z_1) \frac{\{qz_2\}^* \{z_2\}^* \{qw_1\}^* \{z_1\}^*}{\{z_2\}^* \{qw_2\}^* \{z_1\}^* \{qw_2\}^*} \left(\frac{w_1}{w_2} \right)^{\frac{1}{r^*}} \\ & - qE_M(z_2)E_{M-1}(w_2)E_{M+1}(w_1)E_M(z_1) \frac{\{w_2\}^* \{w_1\}^* \{qz_1\}^* \{qz_1\}^*}{\{qw_2\}^* \{qw_1\}^* \{z_1\}^* \{z_1\}^*} \\ & + E_{M-1}(w_2)E_M(z_2)E_{M+1}(w_1)E_M(z_1) \frac{\{z_2\}^* \{qw_1\}^* \{z_1\}^* \{qz_1\}^*}{\{qz_2\}^* \{w_1\}^* \{w_2\}^* \{qw_1\}^*} \left(\frac{z_2}{w_2} \right)^{\frac{1}{r^*}} \end{aligned}$$

$$+(z_1 \leftrightarrow z_2) = 0,$$

(2.39)

$$\begin{aligned}
& \left(F_M(z_1)F_{M+1}(w_1)F_M(z_2)F_{M-1}(w_2) \frac{\left\{ \frac{w_1}{qz_1} \right\} \left\{ \frac{z_2}{qw_1} \right\} \left\{ \frac{qw_2}{z_1} \right\} \left\{ \frac{qw_2}{z_2} \right\}}{\left\{ \frac{qw_1}{z_1} \right\} \left\{ \frac{qz_2}{w_1} \right\} \left\{ \frac{w_2}{qz_1} \right\} \left\{ \frac{w_2}{qz_2} \right\}} \left(\frac{z_2}{w_2} \right)^{\frac{1}{r}} \right. \\
& - q^{-1} F_M(z_1)F_{M+1}(w_1)F_{M-1}(w_2)F_M(z_2) \frac{\left\{ \frac{w_1}{qz_1} \right\} \left\{ \frac{qw_2}{z_1} \right\} \left\{ \frac{z_2}{qw_1} \right\} \left\{ \frac{qz_2}{w_2} \right\}}{\left\{ \frac{qw_1}{z_1} \right\} \left\{ \frac{w_2}{qz_1} \right\} \left\{ \frac{qz_2}{w_1} \right\} \left\{ \frac{z_2}{qw_2} \right\}} \\
& - q F_M(z_1)F_M(z_2)F_{M-1}(w_2)F_{M+1}(w_1) \frac{\left\{ \frac{qw_2}{z_1} \right\} \left\{ \frac{qw_2}{z_2} \right\} \left\{ \frac{w_1}{qz_1} \right\} \left\{ \frac{w_1}{qz_2} \right\}}{\left\{ \frac{w_2}{qz_2} \right\} \left\{ \frac{w_1}{qz_1} \right\} \left\{ \frac{qw_1}{z_2} \right\} \left\{ \frac{qz_2}{w_2} \right\}} \left(\frac{w_1}{w_2} \right)^{\frac{1}{r}} \\
& + F_M(z_1)F_{M-1}(w_2)F_M(z_2)F_{M+1}(w_1) \frac{\left\{ \frac{qw_2}{z_1} \right\} \left\{ \frac{qz_2}{z_1} \right\} \left\{ \frac{w_1}{qz_1} \right\} \left\{ \frac{w_1}{qz_2} \right\}}{\left\{ \frac{w_2}{qz_1} \right\} \left\{ \frac{z_2}{qz_1} \right\} \left\{ \frac{qw_1}{z_1} \right\} \left\{ \frac{qw_1}{z_2} \right\}} \left(\frac{w_1}{z_2} \right)^{\frac{1}{r}} \\
& + F_{M+1}(w_1)F_M(z_2)F_{M-1}(w_2)F_M(z_1) \frac{\left\{ \frac{z_2}{qw_1} \right\} \left\{ \frac{qw_2}{z_2} \right\} \left\{ \frac{z_1}{qw_1} \right\} \left\{ \frac{qz_1}{w_2} \right\}}{\left\{ \frac{qz_2}{w_1} \right\} \left\{ \frac{w_2}{qz_2} \right\} \left\{ \frac{qz_1}{w_1} \right\} \left\{ \frac{z_1}{qw_2} \right\}} \left(\frac{z_2}{w_1} \right)^{\frac{1}{r}} \\
& - q^{-1} F_{M+1}(w_1)F_{M-1}(w_2)F_M(z_2)F_M(z_1) \frac{\left\{ \frac{z_2}{qw_1} \right\} \left\{ \frac{qz_2}{w_2} \right\} \left\{ \frac{z_1}{qw_1} \right\} \left\{ \frac{qz_1}{w_2} \right\}}{\left\{ \frac{qz_2}{w_1} \right\} \left\{ \frac{z_2}{qw_2} \right\} \left\{ \frac{qz_1}{w_1} \right\} \left\{ \frac{z_1}{qw_2} \right\}} \left(\frac{w_2}{w_1} \right)^{\frac{1}{r}} \\
& - q F_M(z_2)F_{M-1}(w_2)F_{M+1}(w_1)F_M(z_1) \frac{\left\{ \frac{qw_2}{z_2} \right\} \left\{ \frac{qw_1}{z_2} \right\} \left\{ \frac{z_1}{qw_2} \right\} \left\{ \frac{z_1}{qw_1} \right\}}{\left\{ \frac{w_2}{qz_2} \right\} \left\{ \frac{w_1}{qz_2} \right\} \left\{ \frac{qz_1}{w_2} \right\} \left\{ \frac{qz_1}{w_1} \right\}} \\
& + F_{M-1}(w_2)F_M(z_2)F_{M+1}(w_1)F_M(z_1) \frac{\left\{ \frac{qz_2}{w_2} \right\} \left\{ \frac{w_1}{qz_1} \right\} \left\{ \frac{qz_1}{w_2} \right\} \left\{ \frac{z_1}{qw_1} \right\}}{\left\{ \frac{z_2}{qw_2} \right\} \left\{ \frac{qw_1}{z_1} \right\} \left\{ \frac{z_1}{qw_2} \right\} \left\{ \frac{qz_1}{w_1} \right\}} \left(\frac{w_2}{z_2} \right)^{\frac{1}{r}} \\
& \left. + (z_1 \leftrightarrow z_2) = 0. \right.
\end{aligned}$$

(2.40)

Here we have used the abbreviations

$$\{z\}^* = (p^*z; p^*)_\infty, \quad \{z\} = (pz; p)_\infty. \quad (2.41)$$

2.3 Dressing construction

In this section we construct $U_{q,p}(\widehat{sl}(M|N))$ from $U_q(\widehat{sl}(M|N))$ by developing the dressing procedure [5].

Definition 2.3 Let us introduce the dressing operators $u_j^\pm(z, p)$, $(1 \leq j \leq M + N - 1)$ by

$$u_j^+(z, p) = \exp \left(\sum_{m>0} \frac{1}{[r^*m]_q} a_{j,-m} (q^r z)^m \right), \quad (2.42)$$

$$u_j^-(z, p) = \exp \left(- \sum_{m>0} \frac{1}{[rm]_q} a_{j,m} (q^{-r} z)^{-m} \right). \quad (2.43)$$

Straightforward calculations show the following propositions.

Proposition 2.4 For $1 \leq i, j \leq M + N - 1$, we have

$$u_i^+(z_1, p)x_j^+(z_2) = \frac{(p^*q^{A_{i,j}}z_1/z_2 : p^*)_\infty}{(p^*q^{-A_{i,j}}z_1/z_2; p^*)_\infty} x_j^+(z_2)u_i^+(z_1, p), \quad (2.44)$$

$$u_i^+(z_1, p)x_j^-(z_2) = \frac{(p^*q^{-A_{i,j}+c}z_1/z_2 : p^*)_\infty}{(p^*q^{A_{i,j}+c}z_1/z_2; p^*)_\infty} x_j^-(z_2)u_i^+(z_1, p), \quad (2.45)$$

$$u_i^-(z_1, p)x_j^+(z_2) = \frac{(pq^{-A_{i,j}-c}z_1/z_2 : p)_\infty}{(pq^{A_{i,j}-c}z_1/z_2; p)_\infty} x_j^+(z_2)u_i^-(z_1, p), \quad (2.46)$$

$$u_i^-(z_1, p)x_j^-(z_2) = \frac{(pq^{A_{i,j}}z_1/z_2 : p)_\infty}{(pq^{-A_{i,j}}z_1/z_2; p)_\infty} x_j^-(z_2)u_i^-(z_1, p), \quad (2.47)$$

$$u_i^+(z_1, p)u_j^-(z_2, p) = \frac{(pq^{-A_{i,j}-c}z_1/z_2; p)_\infty (p^*q^{A_{i,j}+c}z_1/z_2; p^*)_\infty}{(pq^{A_{i,j}-c}z_1/z_2; p)_\infty (p^*q^{-A_{i,j}+c}z_1/z_2; p^*)_\infty} u_j^-(z_2, p)u_i^+(z_1, p). \quad (2.48)$$

Definition 2.5 We define the dressing currents $e_j(z, p), f_j(z, p), \psi_j^\pm(z, p)$, ($1 \leq j \leq M + N - 1$) by

$$e_j(z, p) = u_j^+(z, p)x_j^+(z), \quad (2.49)$$

$$f_j(z, p) = x_j^-(z)u_j^-(z, p), \quad (2.50)$$

$$\psi_j^+(z, p) = u_j^+(q^{\frac{c}{2}}z, p)\psi_j^+(z)u_j^-(q^{-\frac{c}{2}}z, p), \quad (2.51)$$

$$\psi_j^-(z, p) = u_j^+(q^{-\frac{c}{2}}z, p)\psi_j^-(z)u_j^-(q^{\frac{c}{2}}z, p). \quad (2.52)$$

Proposition 2.6 The currents $e_i(z, p), f_i(z, p)$ and $a_{i,n}, h_i, c$, ($1 \leq i \leq M + N - 1, n \in \mathbb{Z}_{\neq 0}$) satisfy the following relations

$$c : \text{central}, [h_i, a_{j,m}] = 0, \quad (2.53)$$

$$[a_{i,m}, a_{j,n}] = \frac{[A_{i,j}m]_q [cm]_q}{m} q^{-c|m|} \delta_{m+n,0}, \quad (2.54)$$

$$[h_i, e_j(z, p)] = A_{i,j}e_j(z, p), \quad [h_i, f_j(z, p)] = -A_{i,j}f_j(z, p), \quad (2.55)$$

$$[a_{i,m}, e_j(z, p)] = \frac{[A_{i,j}m]_q}{m} z^m e_j(z, p) \times \begin{cases} \frac{[rm]_q}{[r^*m]_q}, & (m > 0) \\ q^{cm}, & (m < 0) \end{cases}, \quad (2.56)$$

$$[a_{i,m}, f_j(z, p)] = -\frac{[A_{i,j}m]_q}{m} z^m f_j(z, p) \times \begin{cases} 1, & (m > 0) \\ \frac{[r^*m]_q}{[rm]_q} q^{cm}, & (m < 0) \end{cases}, \quad (2.57)$$

$$\begin{aligned} & z_1 \Theta_{p^*}(q^{A_{i,j}}z_2/z_1) e_i(z_1, p) e_j(z_2, p) \\ &= -z_2 \Theta_{p^*}(q^{A_{i,j}}z_2/z_1) e_j(z_2, p) e_i(z_1, p), \quad \text{for } |A_{i,j}| \neq 0, \end{aligned} \quad (2.58)$$

$$[e_i(z_1, p), e_j(z_2, p)] = 0, \quad \text{for } |A_{i,j}| = 0, (i, j) \neq (M, M), \quad (2.59)$$

$$\{e_M(z_1, p), e_M(z_2, p)\} = 0, \quad (2.60)$$

$$z_1 \Theta_p(q^{-A_{i,j}}z_2/z_1) f_i(z_1, p) f_j(z_2, p)$$

$$= -z_2 \Theta_p(q^{-A_{j,i}} z_2/z_1) f_j(z_2, p) f_i(z_1, p), \quad \text{for } |A_{i,j}| \neq 0, \quad (2.61)$$

$$[f_i(z_1, p), f_j(z_2, p)] = 0, \quad \text{for } |A_{i,j}| = 0, (i, j) \neq (M, M), \quad (2.62)$$

$$\{f_M(z_1, p), f_M(z_2, p)\} = 0, \quad (2.63)$$

$$[e_i(z_1, p), f_j(z_2, p)] = \frac{\delta_{i,j}}{(q - q^{-1})z_1 z_2} \left(\delta(q^{-c} z_1/z_2) \psi_i^+(q^{\frac{c}{2}} z_2, p) - \delta(q^c z_1/z_2) \psi_i^-(q^{-\frac{c}{2}} z_2, p) \right),$$

for $(i, j) \neq (M, M)$, (2.64)

$$\{e_M(z_1, p), f_M(z_2, p)\} = \frac{1}{(q - q^{-1})z_1 z_2} \left(\delta(q^{-c} z_1/z_2) \psi_M^+(q^{\frac{c}{2}} z_2, p) - \delta(q^c z_1/z_2) \psi_M^-(q^{-\frac{c}{2}} z_2, p) \right), \quad (2.65)$$

$$\begin{aligned} & \left(e_i(z_1, p) e_i(z_2, p) e_j(z, p) \frac{\{q^{A_{i,j}} \frac{z}{z_1}\}^* \{q^{A_{i,j}} \frac{z}{z_2}\}^*}{\{q^{-A_{i,j}} \frac{z}{z_1}\}^* \{q^{-A_{i,j}} \frac{z}{z_2}\}^*} \right. \\ & - (q + q^{-1}) e_i(z_1, p) e_j(z, p) e_i(z_2, p) \frac{\{q^{A_{i,j}} \frac{z}{z_1}\}^* \{q^{A_{i,j}} \frac{z_2}{z}\}^*}{\{q^{-A_{i,j}} \frac{z}{z_1}\}^* \{q^{-A_{i,j}} \frac{z_2}{z}\}^*} \\ & + e_j(z, p) e_i(z_1, p) e_i(z_2, p) \frac{\{q^{A_{i,j}} z_1/z\}^* \{q^{A_{i,j}} z_2/z\}^*}{\{q^{-A_{i,j}} \frac{z_1}{z}\}^* \{q^{-A_{i,j}} \frac{z_2}{z}\}^*} \left. \right) \frac{\{q^{A_{i,i}} \frac{z_2}{z_1}\}^*}{\{q^{-A_{i,i}} \frac{z_2}{z_1}\}^*} \\ & + (z_1 \leftrightarrow z_2) = 0, \quad \text{for } |A_{i,j}| = 1, i \neq M, \end{aligned} \quad (2.66)$$

$$\begin{aligned} & \left(f_i(z_1, p) f_i(z_2, p) f_j(z, p) \frac{\{q^{-A_{i,j}} \frac{z}{z_1}\} \{q^{-A_{i,j}} \frac{z}{z_2}\}}{\{q^{A_{i,j}} \frac{z}{z_1}\} \{q^{A_{i,j}} \frac{z}{z_2}\}} \right. \\ & - (q + q^{-1}) f_i(z_1, p) f_j(z, p) f_i(z_2, p) \frac{\{q^{-A_{i,j}} \frac{z}{z_1}\} \{q^{-A_{i,j}} \frac{z_2}{z}\}}{\{q^{A_{i,j}} \frac{z}{z_1}\} \{q^{A_{i,j}} \frac{z_2}{z}\}} \\ & + f_j(z, p) f_i(z_1, p) f_i(z_2, p) \frac{\{q^{-A_{i,j}} \frac{z_1}{z}\} \{q^{-A_{i,j}} \frac{z_2}{z}\}}{\{q^{A_{i,j}} \frac{z_1}{z}\} \{q^{A_{i,j}} \frac{z_2}{z}\}} \left. \right) \frac{\{q^{-A_{i,i}} \frac{z_2}{z_1}\}}{\{q^{A_{i,i}} \frac{z_2}{z_1}\}} \\ & + (z_1 \leftrightarrow z_2) = 0, \quad \text{for } |A_{i,j}| = 1, i \neq M, \end{aligned} \quad (2.67)$$

$$\begin{aligned} & \left(e_M(z_1, p) e_{M+1}(w_1, p) e_M(z_2, p) e_{M-1}(w_2, p) \frac{\{\frac{qw_1}{z_1}\}^* \{\frac{qz_2}{w_1}\}^* \{\frac{w_2}{qz_1}\}^* \{\frac{w_2}{qz_2}\}^*}{\{\frac{w_1}{qz_1}\}^* \{\frac{z_2}{qw_1}\}^* \{\frac{qw_2}{z_1}\}^* \{\frac{qw_2}{z_2}\}^*} \right. \\ & - q^{-1} e_M(z_1, p) e_{M+1}(w_1, p) e_{M-1}(w_2, p) e_M(z_2, p) \frac{\{\frac{qw_1}{z_1}\}^* \{\frac{w_2}{qz_1}\}^* \{\frac{qz_2}{w_1}\}^* \{\frac{z_2}{qw_2}\}^*}{\{\frac{w_1}{qz_1}\}^* \{\frac{qw_2}{z_1}\}^* \{\frac{z_2}{qw_1}\}^* \{\frac{qw_2}{w_2}\}^*} \\ & - q e_M(z_1, p) e_M(z_2, p) e_{M-1}(w_2, p) e_{M+1}(w_1, p) \frac{\{\frac{w_2}{qz_1}\}^* \{\frac{w_2}{qz_2}\}^* \{\frac{qw_1}{z_1}\}^* \{\frac{qw_1}{z_2}\}^*}{\{\frac{qw_2}{z_2}\}^* \{\frac{qw_1}{z_1}\}^* \{\frac{w_1}{qz_2}\}^* \{\frac{z_2}{qw_2}\}^*} \\ & + e_M(z_1, p) e_{M-1}(w_2, p) e_M(z_2, p) e_{M+1}(w_1, p) \frac{\{\frac{w_2}{qz_1}\}^* \{\frac{z_2}{qz_1}\}^* \{\frac{qw_1}{z_1}\}^* \{\frac{qw_1}{z_2}\}^*}{\{\frac{qw_2}{z_1}\}^* \{\frac{qz_2}{z_1}\}^* \{\frac{w_1}{qz_1}\}^* \{\frac{w_1}{qz_2}\}^*} \\ & + e_{M+1}(w_1, p) e_M(z_2, p) e_{M-1}(w_2, p) e_M(z_1, p) \frac{\{\frac{qz_2}{w_1}\}^* \{\frac{w_2}{qz_2}\}^* \{\frac{qz_1}{w_1}\}^* \{\frac{z_1}{qw_2}\}^*}{\{\frac{z_2}{qw_1}\}^* \{\frac{qw_2}{z_2}\}^* \{\frac{z_1}{qw_1}\}^* \{\frac{qz_1}{w_2}\}^*} \\ & - q^{-1} e_{M+1}(w_1, p) e_{M-1}(w_2, p) e_M(z_2, p) e_M(z_1, p) \frac{\{\frac{qz_2}{w_1}\}^* \{\frac{z_2}{qw_2}\}^* \{\frac{qz_1}{w_1}\}^* \{\frac{z_1}{qw_2}\}^*}{\{\frac{z_2}{qw_1}\}^* \{\frac{qz_2}{w_2}\}^* \{\frac{z_1}{qw_1}\}^* \{\frac{qz_1}{w_2}\}^*} \\ & - q e_M(z_2, p) e_{M-1}(w_2, p) e_{M+1}(w_1, p) e_M(z_1, p) \frac{\{\frac{w_2}{qz_2}\}^* \{\frac{w_1}{qz_2}\}^* \{\frac{qz_1}{w_2}\}^* \{\frac{qz_1}{w_1}\}^*}{\{\frac{qw_2}{z_2}\}^* \{\frac{qw_1}{z_2}\}^* \{\frac{z_1}{qw_2}\}^* \{\frac{z_1}{qw_1}\}^*} \end{aligned}$$

$$\begin{aligned}
& + e_{M-1}(w_2, p) e_M(z_2, p) e_{M+1}(w_1, p) e_M(z_1, p) \frac{\left\{ \frac{z_2}{qw_2} \right\}^* \left\{ \frac{qw_1}{z_1} \right\}^* \left\{ \frac{z_1}{qw_2} \right\}^* \left\{ \frac{qz_1}{w_1} \right\}^*}{\left\{ \frac{qz_2}{w_2} \right\}^* \left\{ \frac{w_1}{qz_1} \right\}^* \left\{ \frac{qz_1}{w_2} \right\}^* \left\{ \frac{z_1}{qw_1} \right\}^*} \\
& + (z_1 \leftrightarrow z_2) = 0,
\end{aligned} \tag{2.68}$$

$$\begin{aligned}
& \left(f_M(z_1, p) f_{M+1}(w_1, p) f_M(z_2, p) f_{M-1}(w_2, p) \frac{\left\{ \frac{w_1}{qz_1} \right\} \left\{ \frac{z_2}{qw_1} \right\} \left\{ \frac{qw_2}{z_1} \right\} \left\{ \frac{qw_2}{z_2} \right\}}{\left\{ \frac{qw_1}{z_1} \right\} \left\{ \frac{qz_2}{w_1} \right\} \left\{ \frac{w_2}{qz_1} \right\} \left\{ \frac{w_2}{qz_2} \right\}} \right. \\
& - q^{-1} f_M(z_1, p) f_{M+1}(w_1, p) f_{M-1}(w_2, p) f_M(z_2, p) \frac{\left\{ \frac{w_1}{qz_1} \right\} \left\{ \frac{qw_2}{z_1} \right\} \left\{ \frac{z_2}{qw_1} \right\} \left\{ \frac{qz_2}{w_2} \right\}}{\left\{ \frac{qw_1}{z_1} \right\} \left\{ \frac{w_2}{qz_1} \right\} \left\{ \frac{qz_2}{w_1} \right\} \left\{ \frac{z_2}{qw_2} \right\}} \\
& - q f_M(z_1, p) f_M(z_2, p) f_{M-1}(w_2, p) f_{M+1}(w_1, p) \frac{\left\{ \frac{qw_2}{z_1} \right\} \left\{ \frac{qw_2}{z_2} \right\} \left\{ \frac{w_1}{qz_1} \right\} \left\{ \frac{w_1}{qz_2} \right\}}{\left\{ \frac{w_2}{qz_2} \right\} \left\{ \frac{w_1}{qz_1} \right\} \left\{ \frac{qw_1}{z_2} \right\} \left\{ \frac{qz_2}{w_2} \right\}} \\
& + f_M(z_1, p) f_{M-1}(w_2, p) f_M(z_2, p) f_{M+1}(w_1, p) \frac{\left\{ \frac{qw_2}{z_1} \right\} \left\{ \frac{qz_2}{z_1} \right\} \left\{ \frac{w_1}{qz_1} \right\} \left\{ \frac{w_1}{qz_2} \right\}}{\left\{ \frac{w_2}{qz_1} \right\} \left\{ \frac{z_2}{qz_1} \right\} \left\{ \frac{qw_1}{z_1} \right\} \left\{ \frac{qw_1}{z_2} \right\}} \\
& + f_{M+1}(w_1, p) f_M(z_2, p) f_{M-1}(w_2, p) f_M(z_1, p) \frac{\left\{ \frac{z_2}{qw_1} \right\} \left\{ \frac{qw_2}{z_2} \right\} \left\{ \frac{z_1}{qw_1} \right\} \left\{ \frac{qz_1}{w_2} \right\}}{\left\{ \frac{qz_2}{w_1} \right\} \left\{ \frac{w_2}{qz_2} \right\} \left\{ \frac{qz_1}{w_1} \right\} \left\{ \frac{z_1}{qw_2} \right\}} \\
& - q^{-1} f_{M+1}(w_1, p) f_{M-1}(w_2, p) f_M(z_2, p) f_M(z_1, p) \frac{\left\{ \frac{z_2}{qw_1} \right\} \left\{ \frac{qz_2}{w_2} \right\} \left\{ \frac{z_1}{qw_1} \right\} \left\{ \frac{qz_1}{w_2} \right\}}{\left\{ \frac{qz_2}{w_1} \right\} \left\{ \frac{z_2}{qw_2} \right\} \left\{ \frac{qz_1}{w_1} \right\} \left\{ \frac{z_1}{qw_2} \right\}} \\
& - q f_M(z_2, p) f_{M-1}(w_2, p) f_{M+1}(w_1, p) f_M(z_1, p) \frac{\left\{ \frac{qw_2}{z_2} \right\} \left\{ \frac{qw_1}{z_2} \right\} \left\{ \frac{z_1}{qw_2} \right\} \left\{ \frac{z_1}{qw_1} \right\}}{\left\{ \frac{w_2}{qz_2} \right\} \left\{ \frac{w_1}{qz_2} \right\} \left\{ \frac{qz_1}{w_2} \right\} \left\{ \frac{qz_1}{w_1} \right\}} \\
& \left. + f_{M-1}(w_2, p) f_M(z_2, p) f_{M+1}(w_1, p) f_M(z_1, p) \frac{\left\{ \frac{qz_2}{w_2} \right\} \left\{ \frac{w_1}{qz_1} \right\} \left\{ \frac{qz_1}{w_2} \right\} \left\{ \frac{z_1}{qw_1} \right\}}{\left\{ \frac{z_2}{qw_2} \right\} \left\{ \frac{qw_1}{z_1} \right\} \left\{ \frac{z_1}{qw_2} \right\} \left\{ \frac{qz_1}{w_1} \right\}} \right) \\
& + (z_1 \leftrightarrow z_2) = 0.
\end{aligned} \tag{2.69}$$

We have used the abbreviations (2.41).

Proposition 2.7 The currents $\psi_j^\pm(z)$ ($1 \leq j \leq M+N-1$) have the following formulae.

$$\psi_j^\pm(q^{\mp(r-\frac{c}{2})}z, p) = q^{\mp h_j} : \exp \left(- \sum_{m \neq 0} \frac{B_{j,m}}{[r^*m]_q} z^{-m} \right) : . \tag{2.70}$$

Here we have set

$$B_{j,m} = \begin{cases} \frac{[r^*m]_q}{[rm]_q} a_{j,m}, & (m > 0) \\ q^{c|m|} a_{j,m}, & (m < 0) \end{cases} \quad (1 \leq j \leq M+N-1). \tag{2.71}$$

Definition 2.8 We define elliptic currents $E_j(z), F_j(z), H_j(z)$, ($1 \leq j \leq M+N-1$) by

$$E_j(z) = e_j(z, p) e^{2Q_j} z^{-\frac{1}{r^*} P_j}, \tag{2.72}$$

$$F_j(z) = f_j(z, p) z^{\frac{1}{r} (P_j + h_j)}, \tag{2.73}$$

$$H_j^\pm(z) = H_j(q^{\pm(r-\frac{c}{2})}z), \tag{2.74}$$

$$H_j(z) =: \exp \left(- \sum_{m \neq 0} \frac{B_{j,m}}{[r^*m]_q} z^{-m} \right) : e^{2Q_j} z^{-\frac{c}{rr^*} P_j + \frac{1}{r} h_j}. \tag{2.75}$$

Here we have used the zero-mode operators P_j, Q_j , ($1 \leq j \leq M + N - 1$).

$$[P_i, Q_j] = -\frac{A_{i,j}}{2}, \quad (1 \leq i, j \leq M + N - 1). \quad (2.76)$$

Proposition 2.9 *The currents $E_j(z), F_j(z), H_j(z)$ and $B_{j,n}, h_j, c$, ($1 \leq j \leq M + N - 1, n \in \mathbb{Z}_{\neq 0}$) satisfy the defining relations of elliptic superalgebra $U_{q,p}(\widehat{sl}(M|N))$ (2.25), (2.26), (2.27), (2.28), (2.29), (2.30), (2.31), (2.32), (2.33), (2.34), (2.35), (2.36). They satisfy the Serre relations (2.37), (2.38) and (2.39), (2.40) for $1 \leq i, j \leq M + N - 1, (i \neq M)$ such that $|A_{i,j}| = 1$.*

We have constructed the elliptic deformed superalgebra $U_{q,p}(\widehat{sl}(M|N))$ from the quantum superalgebra $U_q(\widehat{sl}(M|N))$.

3 Bosonization

In this section we give new bosonization of the superalgebra $U_q(\widehat{sl}(1|2)), U_{q,p}(\widehat{sl}(1|2))$ for an arbitrary level k , and their screening currents. Wakimoto [41] constructed bosonization of affine algebra $\widehat{sl}(2)$ for an arbitrary level k . We call this-type bosonization based on the flag manifold [43] the Wakimoto realization. Feigin-Frenkel [42] generalized the Wakimoto realization to the higher-rank affine algebra $\widehat{sl}(N)$. Shiraishi [44] constructed the Wakimoto realization of the quantum algebra $U_q(\widehat{sl}(2))$ and its screening currents. Awata-Otake-Shiraishi constructed the Wakimoto realization for the quantum algebra $U_q(\widehat{sl}(N))$ and its screening currents [45]. In the case of $U_q(\widehat{sl}(2|1))$ Awata-Otake-Shiraishi [46] constructed the Wakimoto realization and Zhang-Gould [47] constructed the screening currents.

3.1 $U_q(\widehat{sl}(1|2)), U_{q,p}(\widehat{sl}(1|2))$, Screening

In this section we give new bosonizations of $U_q(\widehat{sl}(1|2)), U_{q,p}(\widehat{sl}(1|2))$ and their screening currents. In this section we assume the central element $c = k \neq 1$. The Cartan matrix $(A_{i,j})_{0 \leq i,j \leq 2}$ of $\widehat{sl}(1|2)$ is given by

$$(A_{i,j})_{0 \leq i,j \leq 2} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -2 \end{pmatrix}. \quad (3.1)$$

The Cartan matrix of the classical part $sl(1|2)$ is written by

$$(A_{i,j})_{1 \leq i,j \leq 2} = ((\nu_i + \nu_{i+1})\delta_{i,j} - \nu_i\delta_{i,j+1} - \nu_{i+1}\delta_{i+1,j})_{1 \leq i,j \leq 2},$$

where we have set $\nu_1 = +, \nu_2 = \nu_3 = -$. Let us introduce the bosons and the zero-mode operators a_m^j, Q_a^j ($m \in \mathbb{Z}, j = 1, 2$) $b_m^{i,j}, Q_b^{i,j}, c_m^{i,j}, Q_c^{i,j}$ ($m \in \mathbb{Z}, 1 \leq i < j \leq 3$) by

$$[a_m^i, a_n^j] = \frac{[(k-1)m]_q [A_{i,j}m]_q}{m} \delta_{m+n,0}, \quad [a_m^i, Q_a^j] = (k-1)A_{i,j} \delta_{m,0}, \quad (3.2)$$

$$[b_m^{i,j}, b_n^{i',j'}] = -\nu_i \nu_j \frac{[m]_q^2}{m} \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, \quad [b_m^{i,j}, Q_b^{i',j'}] = -\nu_i \nu_j \delta_{i,i'} \delta_{j,j'} \delta_{m,0}, \quad (3.3)$$

$$[c_m^{i,j}, c_n^{i',j'}] = \nu_i \nu_j \frac{[m]_q^2}{m} \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, \quad [c_m^{i,j}, Q_c^{i',j'}] = \nu_i \nu_j \delta_{i,i'} \delta_{j,j'} \delta_{m,0}. \quad (3.4)$$

Let us set the bosonic fields $a(z)$, $a_{\pm}(z)$ and $\left(\frac{1}{\beta} a\right)(z|\alpha)$ as follows.

$$a(z) = - \sum_{m \neq 0} \frac{a_m}{[m]_q} z^{-m} + Q_a + a_0 \log z, \quad (3.5)$$

$$a_{\pm}(z) = \pm(q - q^{-1}) \sum_{m > 0} a_{\pm m} z^{\mp m} \pm a_0 \log q, \quad (3.6)$$

$$\left(\frac{1}{\beta} a\right)(z|\alpha) = - \sum_{m \neq 0} \frac{a_m}{[\beta m]} q^{-\alpha|m|} z^{-m} + \frac{1}{\beta} (Q_a + a_0 \log z). \quad (3.7)$$

We impose the cocycle condition to the zero-mode operator.

$$e^{Q_b^{1,2}} e^{Q_b^{1,3}} = -e^{Q_b^{1,3}} e^{Q_b^{1,2}}, \quad e^{Q_b^{1,2}} e^{Q_b^{2,3}} = e^{Q_b^{2,3}} e^{Q_b^{1,2}}, \quad e^{Q_b^{1,2}} e^{Q_b^{2,3}} = e^{Q_b^{2,3}} e^{Q_b^{1,3}}. \quad (3.8)$$

Straightforward OPE calculations show the following propositions.

Proposition 3.1 *Bosonization of the quantum superalgebra $U_{q,p}(\widehat{sl}(1|2))$ is given as follows.*

$$c = k, \quad h_1 = a_0^1 - b_0^{2,3} - b_0^{1,2}, \quad h_2 = a_0^2 + 2b_0^{2,3} + b_0^{1,3} - b_0^{1,2}, \quad (3.9)$$

$$a_{1,m} = a_m^1 q^{-\frac{k-1}{2}|m|} - b_m^{2,3} q^{-(k-1)|m|} - b_m^{1,3} q^{-(k-1)|m|}, \quad (3.10)$$

$$a_{2,m} = a_m^2 q^{-\frac{k-1}{2}|m|} + b_m^{2,3} q^{-(k-1)|m|} (q^m + q^{-m}) + b_m^{1,3} q^{-(k-2)|m|} - b_m^{1,2} q^{-(k-1)|m|}, \quad (3.11)$$

$$x_1^+(z) = c_{1,1}^+ x_{1,1}^+(z) + c_{1,2}^+ x_{1,2}^+(z), \quad (3.12)$$

$$x_2^+(z) = \frac{1}{(q - q^{-1})z} (c_{2,1}^+ x_{2,1}^+(z) - c_{2,2}^+ x_{2,2}^+(z)), \quad (3.13)$$

$$x_1^-(z) = \frac{1}{(q - q^{-1})z} (c_{1,1}^- x_{1,1}^-(z) - c_{1,2}^- x_{1,2}^-(z) - c_{1,3}^- x_{1,3}^-(z) + c_{1,4}^- x_{1,4}^-(z)), \quad (3.14)$$

$$x_2^-(z) = \frac{1}{(q - q^{-1})z} (c_{2,1}^- x_{2,1}^-(z) - c_{2,2}^- x_{2,2}^-(z)) + c_{2,3}^- x_{2,3}^-(z), \quad (3.15)$$

where we have set

$$x_{1,1}^+(z) =: e^{-(b^{2,3} + b^{1,3}) + (q^{-1}z) - b^{1,2}(q^{-1}z)} :, \quad (3.16)$$

$$x_{1,2}^+(z) =: e^{-(b+c)^{2,3}(z) - b^{1,3}(z)} :, \quad (3.17)$$

$$x_{2,1}^+(z) =: e^{b_+^{2,3}(z) + (b+c)^{2,3}(q^{-1}z)} :, \quad (3.18)$$

$$x_{2,2}^+(z) =: e^{b_-^{2,3}(z)+(b+c)^{2,3}(qz)} :, \quad (3.19)$$

$$x_{1,1}^-(z) =: e^{a_+^1(q^{\frac{k-1}{2}}z)+b_+^{1,2}(q^{k-1}z)} :, \quad (3.20)$$

$$x_{1,2}^-(z) =: e^{a_-^1(q^{-\frac{k-1}{2}}z)+b_-^{1,2}(q^{-k+1}z)} :, \quad (3.21)$$

$$x_{1,3}^-(z) =: e^{a_-^1(q^{-\frac{k-1}{2}}z)-b_-^{2,3}(q^{-k+1}z)+(b+c)^{2,3}(q^{-k}z)-b_-^{1,3}(q^{-k+1}z)+b_+^{1,3}(q^{-k}z)} :, \quad (3.22)$$

$$x_{1,4}^-(z) =: e^{a_-^1(q^{-\frac{k-1}{2}}z)-b_+^{2,3}(q^{-k+1}z)-b_-^{1,3}(q^{-k+1}z)+(b+c)^{2,3}(q^{-k+2}z)+b_+^{1,3}(q^{-k}z)} :, \quad (3.23)$$

$$x_{2,1}^-(z) =: e^{a_+^2(q^{\frac{k-1}{2}}z)+b_+^{2,3}(q^{k-2}z)-(b+c)^{2,3}(q^{k-1}z)+b_+^{1,3}(q^{k-2}z)-b_+^{1,2}(q^{k-1}z)} :, \quad (3.24)$$

$$x_{2,2}^-(z) =: e^{a_-^2(q^{-\frac{k-1}{2}}z)+b_-^{2,3}(q^{-k+2}z)-(b+c)^{2,3}(q^{-k+1}z)+b_-^{1,3}(q^{-k+2}z)-b_-^{1,2}(q^{-k+1}z)} :, \quad (3.25)$$

$$x_{2,3}^-(z) =: e^{a_+^2(q^{\frac{k-1}{2}}z)+b_+^{1,3}(q^{k-1}z)-b_+^{1,2}(q^{k-1}z)-b_+^{1,2}(q^{k-2}z)} :. \quad (3.26)$$

Here we have set the coefficients as follows.

$$(c_{1,1}^+, c_{1,2}^+, c_{2,1}^+, c_{2,2}^+) = (\alpha, \beta, \gamma, \gamma), \quad (3.27)$$

$$(c_{1,1}^-, c_{1,2}^-, c_{1,3}^-, c_{1,4}^-, c_{2,1}^-, c_{2,2}^-, c_{2,3}^-) = \left(\frac{1}{q\alpha}, \frac{1}{q\alpha}, \frac{1}{\beta}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\gamma}, \frac{q^{k-1}\alpha}{\beta\gamma} \right). \quad (3.28)$$

Here $\alpha, \beta, \gamma \neq 0$ are arbitrary parameters.

Next we give bosonization of the elliptic superalgebra $U_{q,p}(\widehat{sl}(1|2))$. Our construction is based on the dressing procedure of the quantum algebra developed in this paper.

Proposition 3.2 *Bosonization of the elliptic superalgebra $U_{q,p}(\widehat{sl}(1|2))$ is given as follows.*

$$c = k, \quad h_1 = a_0^1 - b_0^{2,3} - b_0^{1,2}, \quad h_2 = a_0^2 + 2b_0^{2,3} + b_0^{1,3} - b_0^{1,2}, \quad (3.29)$$

$$B_{j,m} = \begin{cases} \frac{[r^*m]_q}{[rm]_q} a_{j,m}, & (m > 0) \\ q^{k|m} a_{j,m}, & (m < 0) \end{cases}, \quad (j = 1, 2), \quad (3.30)$$

$$a_{1,m} = a_m^1 q^{-\frac{k-1}{2}|m|} - b_m^{2,3} q^{-(k-1)|m|} - b_m^{1,3} q^{-(k-1)|m|}, \quad (3.31)$$

$$a_{2,m} = a_m^2 q^{-\frac{k-1}{2}|m|} + b_m^{2,3} q^{-(k-1)|m|} (q^m + q^{-m}) + b_m^{1,3} q^{-(k-2)|m|} - b_m^{1,2} q^{-(k-1)|m|}, \quad (3.32)$$

$$E_j(z) = u_j^+(z, p) x_j^+(z) e^{2Q_j} z^{-\frac{1}{r^*} P_j}, \quad (j = 1, 2), \quad (3.33)$$

$$F_j(z) = x_j^-(z) u_j^-(z, p) z^{\frac{1}{r} (P_j + h_j)}, \quad (j = 1, 2), \quad (3.34)$$

$$H_j^\pm(z) = H_j(q^{\pm(r-\frac{c}{2})}z), \quad (j = 1, 2), \quad (3.35)$$

where we have used (3.12), (3.13), (3.14), (3.15) and

$$u_j^+(z, p) = \exp \left(\sum_{m>0} \frac{q^{r^*m}}{[r^*m]_q} B_{j,-m} z^m \right), \quad (j = 1, 2), \quad (3.36)$$

$$u_j^-(z, p) = \exp \left(- \sum_{m>0} \frac{q^{rm}}{[r^*m]_q} B_{j,m} z^{-m} \right), \quad (j = 1, 2), \quad (3.37)$$

$$H_j(z) =: \exp \left(- \sum_{m \neq 0} \frac{B_{j,m}}{[r^*m]_q} z^{-m} \right) : e^{2Q_j} z^{-\frac{c}{rr^*} P_j + \frac{1}{r} h_j}, \quad (j = 1, 2). \quad (3.38)$$

Here we have used the zero-mode operators

$$[P_i, Q_j] = -\frac{A_{i,j}}{2}, \quad (1 \leq i, j \leq 2). \quad (3.39)$$

Proposition 3.3 *The bosonic operators $s_j(z)$ ($j = 1, 2$) given below are the screening currents that commute with the quantum superalgebra $U_q(\widehat{\mathfrak{sl}}(1|2))$ modulo total difference.*

$$s_j(z) =: e^{-(\frac{1}{k-1}a^j)(z_1|\frac{k-1}{2})} \tilde{s}_j(z) : \quad (j = 1, 2). \quad (3.40)$$

Here we have set

$$\tilde{s}_1(z) = -c_{1,5} \tilde{s}_{1,5}(z), \quad (3.41)$$

$$\tilde{s}_2(z) = \frac{1}{(q - q^{-1})z} (-c_{2,3} \tilde{s}_{2,3}(z) + c_{2,4} \tilde{s}_{2,4}(z)) + c_{2,5} \tilde{s}_{2,5}(z), \quad (3.42)$$

where

$$\tilde{s}_{1,5}(z) = : e^{-b^{1,2}(z)} :, \quad (3.43)$$

$$\tilde{s}_{2,3}(z) = : e^{-b_+^{2,3}(qz) + (b+c)^{2,3}(q^2z) - b_-^{1,3}(qz) + b_-^{1,2}(z)} :, \quad (3.44)$$

$$\tilde{s}_{2,4}(z) = : e^{-b_-^{2,3}(qz) + (b+c)^{2,3}(z) - b_-^{1,3}(qz) + b_-^{1,2}(z)} :, \quad (3.45)$$

$$\tilde{s}_{2,5}(z) = : e^{-b^{1,3}(z) + b_+^{1,2}(z) + b^{1,2}(q^{-1}z)} :. \quad (3.46)$$

Here we have set the coefficients as follows.

$$(c_{1,5}, c_{2,3}, c_{2,4}, c_{2,5}) = \left(q\alpha, \gamma, \gamma, \frac{\beta\gamma}{q\alpha} \right), \quad (3.47)$$

where parameters $\alpha, \beta, \gamma \neq 0$ have been introduced in (3.27), (3.28) for the bosonizations of $U_q(\widehat{\mathfrak{sl}}(1|2))$. Explicitly the bosonic operators $s_1(z), s_2(z)$ and $x_1^\pm(z), x_2^\pm(z)$ satisfy the following relations.

$$[a_{i,m}, s_j(z_2)] = 0, \quad (3.48)$$

$$[x_i^+(z_1), s_j(z_2)] = 0, \quad (3.49)$$

$$\begin{aligned} [x_i^-(z_1), s_j(z_2)] &= \frac{\delta_{i,j}}{(q - q^{-1})z_1 z_2} (\delta(q^{k-1}z_2/z_1) - \delta(q^{-k+1}z_2/z_1)) \\ &\times : e^{-(\frac{1}{k-1}a^i)(z_1|-\frac{k-1}{2})} :, \end{aligned} \quad (3.50)$$

and

$$\{\tilde{s}_1(z_1), \tilde{s}_1(z_2)\} = 0, \quad (3.51)$$

$$(z_1 - q^{-A_{1,2}} z_2) \tilde{s}_1(z_1) \tilde{s}_2(z_2) = (q^{-A_{2,1}} z_1 - z_2) \tilde{s}_2(z_2) \tilde{s}_1(z_1), \quad (3.52)$$

$$(z_1 - q^{-A_{2,2}} z_2) \tilde{s}_2(z_1) \tilde{s}_2(z_2) = (q^{-A_{2,2}} z_1 - z_2) \tilde{s}_2(z_2) \tilde{s}_2(z_1). \quad (3.53)$$

By the commutation relation $[a_{i,m}, s_j(z_2)] = 0$ we conclude the following.

Proposition 3.4 *The bosonic operators $s_j(z)$ ($j = 1, 2$) given in (3.40) become the screening currents that commute with the elliptic algebra $U_{q,p}(\widehat{sl}(1|2))$ modulo total difference. Explicitly the bosonic operators $s_1(z), s_2(z)$ and $E_1(z), E_2(z), F_1(z), F_2(z)$ satisfy the following relations.*

$$[B_{i,m}, s_j(z_2)] = 0, \quad (3.54)$$

$$[E_i(z_1), s_j(z_2)] = 0, \quad (3.55)$$

$$\begin{aligned} [F_i(z_1), s_j(z_2)] &= \frac{\delta_{i,j}}{(q - q^{-1})z_1 z_2} (\delta(q^{k-1} z_2 / z_1) - \delta(q^{-k+1} z_2 / z_1)) \\ &\times : e^{-(\frac{1}{k-1} a^i)(z_1| - \frac{k-1}{2})} u_i^-(z_1, p) z_1^{\frac{1}{r}(P_i + h_i)} : . \end{aligned} \quad (3.56)$$

The Jackson integral with parameters p and $s \neq 0$ is defined by

$$\int_0^{s\infty} f(z) d_p z = s(1-p) \sum_{n \in \mathbb{Z}} f(sp^n) p^n. \quad (3.57)$$

From the above proposition we have

$$\left[\int_0^{s\infty} s_j(z) d_{q^{2(k-1)}} z, U_{q,p}(\widehat{sl}(1|2)) \right] = 0. \quad (3.58)$$

3.2 $U_q(\widehat{sl}(2|1)), U_{q,p}(\widehat{sl}(2|1)), \text{Screening}$

In this section we review known results on bosonization of $U_q(\widehat{sl}(2|1))$ [46] and its screening currents [47]. We give bosonizations of $U_{q,p}(\widehat{sl}(2|1))$ and its screenings. In this section we assume the central element $c = k \neq -1$. The Cartan matrix $(A_{i,j})_{0 \leq i,j \leq 2}$ of $\widehat{sl}(2|1)$ is given by

$$(A_{i,j})_{0 \leq i,j \leq 2} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix}. \quad (3.59)$$

The Cartan matrix of the classical part $sl(2|1)$ is written by

$$(A_{i,j})_{1 \leq i,j \leq 2} = ((\nu_i + \nu_{i+1})\delta_{i,j} - \nu_i \delta_{i,j+1} - \nu_{i+1} \delta_{i+1,j})_{1 \leq i,j \leq 2},$$

where we have set $\nu_1 = \nu_2 = +, \nu_3 = -$. Let us introduce the bosons and the zero-mode operators $a_m^j, Q_a^j, (m \in \mathbb{Z}, j = 1, 2)$ $b_m^{i,j}, Q_b^{i,j}, c_m^{i,j}, Q_c^{i,j}$ ($m \in \mathbb{Z}, 1 \leq i < j \leq 3$) by

$$[a_m^i, a_n^j] = \frac{[(k+1)m]_q [A_{i,j}m]_q}{m} \delta_{m+n,0}, \quad [a_m^i, Q_a^j] = (k+1)A_{i,j} \delta_{m,0}, \quad (3.60)$$

$$[b_m^{i,j}, b_n^{i',j'}] = -\nu_i \nu_j \frac{[m]_q^2}{m} \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, \quad [b_m^{i,j}, Q_b^{i',j'}] = -\nu_i \nu_j \delta_{i,i'} \delta_{j,j'} \delta_{m,0}, \quad (3.61)$$

$$[c_m^{i,j}, c_n^{i',j'}] = \nu_i \nu_j \frac{[m]_q^2}{m} \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, \quad [c_m^{i,j}, Q_c^{i',j'}] = \nu_i \nu_j \delta_{i,i'} \delta_{j,j'} \delta_{m,0}. \quad (3.62)$$

We impose the cocycle condition to the zero-mode operators.

$$e^{Q_b^{1,2}} e^{Q_b^{1,3}} = e^{Q_b^{1,3}} e^{Q_b^{1,2}}, \quad e^{Q_b^{1,2}} e^{Q_b^{2,3}} = e^{Q_b^{2,3}} e^{Q_b^{1,2}}, \quad e^{Q_b^{1,3}} e^{Q_b^{2,3}} = -e^{Q_b^{2,3}} e^{Q_b^{1,3}}. \quad (3.63)$$

Proposition 3.5 [46] *Bosonization of the quantum superalgebra $U_q(\widehat{sl}(2|1))$ is given as follows.*

$$c = k, \quad h_1 = a_0^1 + 2b_0^{1,2} + b_0^{1,3} - b_0^{2,3}, \quad h_2 = a_0^2 - b_0^{1,2} - b_0^{1,3}, \quad (3.64)$$

$$a_{1,m} = a_m^1 q^{-\frac{k+1}{2}|m|} + b_m^{1,2} q^{-(k+1)|m|} (q^m + q^{-m}) + b_m^{1,3} q^{-(k+2)|m|} - b_m^{2,3} q^{-(k+1)|m|}, \quad (3.65)$$

$$a_{2,m} = a_m^2 q^{-\frac{k+1}{2}|m|} - b_m^{1,2} q^{-(k+1)|m|} - b_m^{1,3} q^{-(k+1)|m|}, \quad (3.66)$$

$$x_1^+(z) = \frac{1}{(q - q^{-1})z} (c_{1,1}^+ x_{1,1}^+(z) - c_{1,2}^+ x_{1,2}^+(z)), \quad (3.67)$$

$$x_2^+(z) = c_{2,1}^+ x_{2,1}^+(z) + c_{2,2}^+ x_{2,2}^+(z), \quad (3.68)$$

$$x_1^-(z) = \frac{1}{(q - q^{-1})z} (c_{1,1}^- x_{1,1}^-(z) - c_{1,2}^- x_{1,2}^-(z)) + c_{1,3}^- x_{1,3}^-(z), \quad (3.69)$$

$$x_2^-(z) = \frac{1}{(q - q^{-1})z} (c_{2,1}^- x_{2,1}^-(z) - c_{2,2}^- x_{2,2}^-(z) - c_{2,3}^- x_{2,3}^-(z) + c_{2,4}^- x_{2,4}^-(z)), \quad (3.70)$$

where we have set

$$x_{1,1}^+(z) =: e^{b_+^{1,2}(z) - (b+c)^{1,2}(qz)} :, \quad (3.71)$$

$$x_{1,2}^+(z) =: e^{b_-^{1,2}(z) - (b+c)^{1,2}(q^{-1}z)} :, \quad (3.72)$$

$$x_{2,1}^+(z) =: e^{-b_+^{1,2}(qz) - b_+^{1,3}(qz) + b^{2,3}(qz)} :, \quad (3.73)$$

$$x_{2,2}^+(z) =: e^{(b+c)^{1,2}(z) + b^{1,3}(z)} :, \quad (3.74)$$

$$x_{1,1}^-(z) =: e^{a_+^1(q^{\frac{k+1}{2}}z) + b_+^{1,2}(q^{k+2}z) + (b+c)^{1,2}(q^{k+1}z) + b_+^{1,3}(q^{k+2}z) - b_+^{2,3}(q^{k+1}z)} :, \quad (3.75)$$

$$x_{1,2}^-(z) =: e^{a_-^1(q^{-\frac{k+1}{2}}z) + b_-^{1,2}(q^{-k-2}z) + (b+c)^{1,2}(q^{-k-1}z) + b_-^{1,3}(q^{-k-2}z) - b_-^{2,3}(q^{-k-1}z)} :, \quad (3.76)$$

$$x_{1,3}^-(z) =: e^{a_+^1(q^{\frac{k+1}{2}}z) - b_+^{2,3}(q^{k+1}z) - b^{1,3}(q^{k+1}z) + b^{2,3}(q^{k+1}z)} :, \quad (3.77)$$

$$x_{2,1}^-(z) =: e^{a_+^2(q^{\frac{k+1}{2}}z) - b^{2,3}(q^{k+1}z)} :, \quad (3.78)$$

$$x_{2,2}^-(z) =: e^{a_-^2(q^{-\frac{k+1}{2}}z) - b_-^{2,3}(q^{-k-1}z)} :, \quad (3.79)$$

$$x_{2,3}^-(z) =: e^{a_-^2(q^{-\frac{k+1}{2}}z) - b_-^{1,2}(q^{-k-1}z) - b_-^{1,3}(q^{-k-1}z) - (b+c)^{1,2}(q^{-k}z) - b^{1,3}(q^{-k}z)} :, \quad (3.80)$$

$$x_{2,4}^-(z) =: e^{a_-^2(q^{-\frac{k+1}{2}}z) - b_+^{1,2}(q^{-k-1}z) - b_-^{1,3}(q^{-k-1}z) - (b+c)^{1,2}(q^{-k-2}z) - b^{1,3}(q^{-k}z)} :. \quad (3.81)$$

Here we have set the coefficients as follows.

$$(c_{1,1}^+, c_{1,2}^+, c_{2,1}^+, c_{2,2}^+) = (\alpha, \alpha, \beta, \gamma), \quad (3.82)$$

$$(c_{1,1}^-, c_{1,2}^-, c_{1,3}^-, c_{2,1}^-, c_{2,2}^-, c_{2,3}^-, c_{2,4}^-) = \left(\frac{1}{\alpha}, \frac{1}{\alpha}, \frac{q^{k+1}\beta}{\alpha\gamma}, \frac{q}{\beta}, \frac{q}{\beta}, \frac{1}{\gamma}, \frac{1}{\gamma} \right). \quad (3.83)$$

Here $\alpha, \beta, \gamma \neq 0$ are arbitrary parameters.

Note. The coefficients of the currents $x_j^\pm(z)$ have 4 free parameters in [46]. In this paper we have only three free parameters α, β, γ , because we assume the commutation relations (3.102), (3.103), (3.104) with the screening currents.

Proposition 3.6 *Bosonization of the elliptic superalgebra $U_{q,p}(\widehat{sl}(2|1))$ is given as follows.*

$$c = k, \quad h_1 = a_0^1 + 2b_0^{1,2} + b_0^{1,3} - b_0^{2,3}, \quad h_2 = a_0^2 - b_0^{1,2} - b_0^{1,3}, \quad (3.84)$$

$$B_{j,m} = \begin{cases} \frac{[r^*m]_q}{[rm]_q} a_{j,m}, & (m > 0) \\ q^{k|m|} a_{j,m}, & (m < 0) \end{cases}, \quad (j = 1, 2), \quad (3.85)$$

$$a_{1,m} = a_m^1 q^{-\frac{k+1}{2}|m|} + b_m^{1,2} q^{-(k+1)|m|} (q^m + q^{-m}) + b_m^{1,3} q^{-(k+2)|m|} - b_m^{2,3} q^{-(k+1)|m|}, \quad (3.86)$$

$$a_{2,m} = a_m^2 q^{-\frac{k+1}{2}|m|} - b_m^{1,2} q^{-(k+1)|m|} - b_m^{1,3} q^{-(k+1)|m|}, \quad (3.87)$$

$$E_j(z) = u_j^+(z, p) x_j^+(z) e^{2Q_j z^{-\frac{1}{r^*} P_j}}, \quad (j = 1, 2), \quad (3.88)$$

$$F_j(z) = x_j^-(z) u_j^-(z, p) z^{\frac{1}{r} (P_j + h_j)}, \quad (j = 1, 2), \quad (3.89)$$

$$H_j^\pm(z) = H_j(q^{\pm(r-\frac{c}{2})}z), \quad (j = 1, 2), \quad (3.90)$$

where we have used (3.67), (3.68), (3.69), (3.70) and

$$u_j^+(z, p) = \exp \left(\sum_{m>0} \frac{q^{r^*m}}{[r^*m]_q} B_{j,-m} z^m \right), \quad (j = 1, 2), \quad (3.91)$$

$$u_j^-(z, p) = \exp \left(- \sum_{m>0} \frac{q^{rm}}{[rm]_q} B_{j,m} z^{-m} \right), \quad (j = 1, 2), \quad (3.92)$$

$$H_j(z) =: \exp \left(- \sum_{m \neq 0} \frac{B_{j,m}}{[r^*m]_q} z^{-m} \right) : e^{2Q_j z^{-\frac{c}{rr^*} P_j + \frac{1}{r} h_j}}, \quad (j = 1, 2). \quad (3.93)$$

Here we have used the zero-mode operators

$$[P_i, Q_j] = -\frac{A_{i,j}}{2}, \quad (1 \leq i, j \leq 2). \quad (3.94)$$

Proposition 3.7 [47] *The bosonic operators $s_1(z), s_2(z)$ given below are the screening currents that commute with the quantum superalgebra $U_q(\widehat{sl}(2|1))$ modulo total difference.*

$$s_j(z) = : e^{-(\frac{1}{k+1}a^j)(z|\frac{k+1}{2})} \tilde{s}_j(z) : \quad (j = 1, 2). \quad (3.95)$$

Here we have set

$$\tilde{s}_1(z) = \frac{1}{(q - q^{-1})z} (-c_{1,3}\tilde{s}_{1,3}(z) + c_{1,4}\tilde{s}_{1,4}(z)) + c_{1,5}\tilde{s}_{1,5}(z), \quad (3.96)$$

$$\tilde{s}_2(z) = -c_{2,5}\tilde{s}_{2,5}(z), \quad (3.97)$$

where

$$\tilde{s}_{1,5}(z) = : e^{b^{1,3}(z) - b^{2,3}(qz) + b_+^{2,3}(z)} :, \quad (3.98)$$

$$\tilde{s}_{1,4}(z) = : e^{-b_-^{1,2}(q^{-1}z) - (b+c)^{1,2}(z) + b_-^{2,3}(z) - b_-^{1,3}(q^{-1}z)} :, \quad (3.99)$$

$$\tilde{s}_{1,3}(z) = : e^{-b_+^{1,2}(q^{-1}z) - (b+c)^{1,2}(q^{-2}z) + b_-^{2,3}(z) - b_-^{1,3}(q^{-1}z)} :, \quad (3.100)$$

$$\tilde{s}_{2,5}(z) = : e^{b^{2,3}(z)} :.$$

Here we have set the coefficients as follows.

$$(c_{1,3}, c_{1,4}, c_{1,5}, c_{2,5}) = \left(\alpha, \alpha, \frac{q\alpha\beta}{\gamma}, \frac{\beta}{q} \right), \quad (3.101)$$

where parameters $\alpha, \beta, \gamma \neq 0$ have been introduced in (3.82), (3.83) for the bosonizations of $U_q(\widehat{sl}(2|1))$. Explicitly the bosonic operators $s_1(z), s_2(z)$ and $x_1^\pm(z), x_2^\pm(z)$ satisfy the following relations.

$$[a_{i,m}, s_j(z_2)] = 0, \quad (3.102)$$

$$[x_i^+(z_1), s_j(z_2)] = 0, \quad (3.103)$$

$$\begin{aligned} [x_i^-(z_1), s_j(z_2)] &= \frac{\delta_{i,j}}{(q - q^{-1})z_1 z_2} (\delta(q^{k+1}z_2/z_1) - \delta(q^{-k-1}z_2/z_1)) \\ &\times : e^{-(\frac{1}{k+1}a^i)(z_1|-\frac{k+1}{2})} :, \end{aligned} \quad (3.104)$$

and

$$(z_1 - q^{-A_{1,1}}z_2)\tilde{s}_1(z_1)\tilde{s}_1(z_2) = (q^{-A_{1,1}}z_1 - z_2)\tilde{s}_1(z_2)\tilde{s}_1(z_1), \quad (3.105)$$

$$(z_1 - q^{-A_{1,2}}z_2)\tilde{s}_1(z_1)\tilde{s}_2(z_2) = (q^{-A_{2,1}}z_1 - z_2)\tilde{s}_2(z_2)\tilde{s}_1(z_1), \quad (3.106)$$

$$\{\tilde{s}_2(z_1), \tilde{s}_2(z_2)\} = 0. \quad (3.107)$$

By the commutation relation $[a_{i,m}, s_j(z_2)] = 0$ we conclude the following.

Proposition 3.8 *The bosonic operators $s_j(z)$ ($j = 1, 2$) given in (3.95) become the screening currents that commute with the elliptic algebra $U_{q,p}(\widehat{sl}(2|1))$ modulo total difference. Explicitly the bosonic operators $s_1(z), s_2(z)$ and $E_1(z), E_2(z), F_1(z), F_2(z)$ satisfy the following relations.*

$$[B_{i,m}, s_j(z_2)] = 0, \quad (3.108)$$

$$[E_i(z_1), s_j(z_2)] = 0, \quad (3.109)$$

$$\begin{aligned} [F_i(z_1), s_j(z_2)] &= \frac{\delta_{i,j}}{(q - q^{-1})z_1 z_2} (\delta(q^{k+1}z_2/z_1) - \delta(q^{-k-1}z_2/z_1)) \\ &\times : e^{-(\frac{1}{k+1}a^i)(z_1| - \frac{k+1}{2})} u_i^-(z_1, p) z_1^{\frac{1}{r}(P_i + h_i)} : . \end{aligned} \quad (3.110)$$

From the above proposition we have

$$\left[\int_0^{s\infty} s_j(z) d_{q^{2(k-1)}z}, U_{q,p}(\widehat{sl}(2|1)) \right] = 0. \quad (3.111)$$

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A Bosonization

In appendix we summarize relations of bosonization for $U_q(\widehat{sl}(1|2))$ and its screening currents relating to the delta-function $\delta(z) = \sum_{m \in \mathbb{Z}} z^m$.

$$\{x_{1,1}^+(z_1), x_{1,1}^-(z_2)\} \quad (A.1)$$

$$= \frac{q}{z_1} \delta(q^k z_2/z_1) e^{a_+^1(q^{\frac{k-1}{2}} z_2) - b_+^{2,3}(q^{k-1} z_2) - b_+^{1,3}(q^{k-1} z_2)},$$

$$\{x_{1,1}^+(z_1), x_{1,2}^-(z_2)\} \quad (A.2)$$

$$= \frac{q}{z_1} \delta(q^{-k+2} z_2/z_1) : e^{a_-^1(q^{-\frac{k-1}{2}} z_2) - b_+^{2,3}(q^{-k+2} z_2) - b_+^{1,3}(q^{-k+2} z_2)} :,$$

$$[x_{1,1}^+(z_1), x_{2,1}^-(z_2)] \quad (A.3)$$

$$= -(q - q^{-1}) \delta(q^{k-1} z_2/z_1) : e^{a_+^2(q^{\frac{k-1}{2}} z_2) - (b+c)^{2,3}(q^{k-1} z_2) - b_+^{1,2}(q^{k-1} z_2) - b^{1,2}(q^{k-2} z_2)} :,$$

$$\{x_{1,2}^+(z_1), x_{1,3}^-(z_2)\} \quad (\text{A.4})$$

$$= \frac{1}{z_1} \delta(q^{-k} z_2 / z_1) e^{a_-^1 (q^{-\frac{k-1}{2}} z_2) - b_-^{2,3} (q^{-k+1} z_2) - b_-^{1,3} (q^{-k+1} z_2)},$$

$$\{x_{1,2}^+(z_1), x_{1,4}^-(z_2)\} \quad (\text{A.5})$$

$$= \frac{1}{z_1} \delta(q^{-k+2} z_2 / z_1) : e^{a_-^1 (q^{-\frac{k-1}{2}} z_2) - b_+^{2,3} (q^{-k+2} z_2) - b_+^{1,3} (q^{-k+2} z_2)} :,$$

$$[x_{1,2}^+(z_1), x_{2,3}^-(z_2)] \quad (\text{A.6})$$

$$= -(q - q^{-1}) \delta(q^{k-1} z_2 / z_1) : e^{a_+^2 (q^{\frac{k-1}{2}} z_2) - (b+c)^{2,3} (q^{k-1} z_2) - b_+^{1,2} (q^{k-1} z_2) - b^{1,2} (q^{k-2} z_2)} :,$$

$$[x_{2,1}^+(z_1), x_{1,4}^-(z_2)] \quad (\text{A.7})$$

$$= -(q - q^{-1}) \delta(q^{-k+1} z_2 / z_1) : e^{a_-^1 (q^{-\frac{k-1}{2}} z_2) + (b+c)^{2,3} (q^{-k} z_2) + (b+c)^{2,3} (q^{-k+2} z_2) - b_-^{1,3} (q^{-k+1} z_2) + b^{1,3} (q^{-k} z_2)} :,$$

$$[x_{2,1}^+(z_1), x_{2,1}^-(z_2)] \quad (\text{A.8})$$

$$= (q - q^{-1}) \delta(q^k z_2 / z_1) e^{a_+^2 (q^{\frac{k-1}{2}} z_2) + b_+^{2,3} (q^{k-2} z_2) + b_+^{2,3} (q^k z_2) + b_+^{1,3} (q^{k-2} z_2) - b_+^{1,2} (q^{k-1} z_2)},$$

$$[x_{2,2}^+(z_1), x_{1,3}^-(z_2)] \quad (\text{A.9})$$

$$= (q - q^{-1}) \delta(q^{-k+1} z_2 / z_1) : e^{a_-^1 (q^{-\frac{k-1}{2}} z_2) + (b+c)^{2,3} (q^{-k} z_2) + (b+c)^{2,3} (q^{-k+2} z_2) - b_-^{1,3} (q^{-k+1} z_2) + b^{1,3} (q^{-k} z_2)} :,$$

$$[x_{2,2}^+(z_1), x_{1,3}^-(z_2)] \quad (\text{A.10})$$

$$= -(q - q^{-1}) \delta(q^{-k} z_2 / z_1) e^{a_-^2 (q^{-\frac{k-1}{2}} z_2) + b_-^{2,3} (q^{-k} z_2) + b_-^{2,3} (q^{-k+2} z_2) + b_-^{1,3} (q^{-k+2} z_2) - b_-^{1,2} (q^{-k+1} z_2)}.$$

$$[x_{2,1}^+(z_1), s_{2,3}(z_2)] \quad (\text{A.11})$$

$$= -(q - q^{-1}) \delta(q z_2 / z_1) : e^{-(\frac{1}{k-1} a^2) (z_2 | \frac{k-1}{2}) + b_-^{1,2} (z) - b_-^{1,3} (q z) + (b+c)^{2,3} (z) + (b+c)^{2,3} (q^2 z)} :,$$

$$[x_{2,2}^+(z_1), s_{2,4}(z_2)] \quad (\text{A.12})$$

$$= (q - q^{-1}) \delta(q z_2 / z_1) : e^{-(\frac{1}{k-1} a^2) (z_2 | \frac{k-1}{2}) + b_-^{1,2} (z) - b_-^{1,3} (q z) + (b+c)^{2,3} (z) + (b+c)^{2,3} (q^2 z)} :,$$

$$[x_{1,1}^+(z_1), s_{2,5}(z_2)] \quad (\text{A.13})$$

$$= \frac{1}{z_2} \delta(q^2 z_2 / z_1) : e^{-(\frac{1}{k-1} a^2) (z_2 | \frac{k-1}{2}) + b_-^{1,2} (z_2) - b_+^{1,3} (q z_2) - b^{1,3} (z_2) - b_+^{2,3} (q z_2)} :,$$

$$[x_{1,2}^+(z_1), s_{2,3}(z_2)] \quad (\text{A.14})$$

$$= q^{-1} (q - q^{-1}) \delta(q^2 z_2 / z_1) : e^{-(\frac{1}{k-1} a^2) (z_2 | \frac{k-1}{2}) + b_-^{1,2} (z_2) - b_+^{1,3} (q z_2) - b^{1,3} (z_2) - b_+^{2,3} (q z_2)} :, \quad (\text{A.15})$$

$$\{x_{1,1}^-(z_1), s_{1,5}(z_2)\} = \frac{1}{z_2} \delta(q^{-k+1} z_2 / z_1) : e^{-(\frac{1}{k-1} a^1) (z_1 | -\frac{k-1}{2})} :, \quad (\text{A.16})$$

$$\{x_{1,2}^-(z_1), s_{1,5}(z_2)\} = \frac{1}{z_2} \delta(q^{k-1} z_2 / z_1) : e^{-(\frac{1}{k-1} a^1) (z_1 | -\frac{k-1}{2})} :, \quad (\text{A.17})$$

$$[x_{2,1}^-(z_1), s_{2,3}(z_2)] = (q - q^{-1}) \delta(q^{-k+3} z_2 / z_1)$$

$$\times : e^{a_+^2 (q^{\frac{k-1}{2}} z_1) - (\frac{1}{k-1} a^2) (q^{k-3} z_1 | \frac{k-1}{2}) + b_+^{1,3} (q^{k-2} z_1) - b_-^{1,3} (q^{k-2} z_1) - b_+^{1,2} (q^{k-1} z_1) + b_-^{1,2} (q^{k-3} z_1)} :, \quad (\text{A.18})$$

$$[x_{2,2}^-(z_1), s_{2,4}(z_2)] = -(q - q^{-1}) \delta(q^{k-1} z_2 / z_1) : e^{-(\frac{1}{k-1} a^2) (z_1 | -\frac{k-1}{2})} :, \quad (\text{A.19})$$

$$[x_{2,3}^-(z_1), s_{2,5}(z_2)] = \frac{-1}{(q - q^{-1}) q^{k-2} z_1 z_2}$$

$$\times (\delta(q^{-k+1}z_2/z_1) : e^{-(\frac{1}{k-1}a^2)(z_1|-\frac{k-1}{2})} : \quad (\text{A.20})$$

$$-\delta(q^{-k+3}z_2/z_1) : e^{a_+^2(q^{\frac{k-1}{2}}z_1)-(\frac{1}{k-1}a^2)(q^{k-3}z_1|\frac{k-1}{2})+b_+^{1,3}(q^{k-2}z_1)-b_-^{1,3}(q^{k-2}z_1)-b_+^{1,2}(q^{k-1}z_1)+b_-^{1,2}(q^{k-3}z_1)} : ,$$

$$[x_{1,2}^-(z_1), s_{2,3}(z_2)] \quad (\text{A.21})$$

$$= (q - q^{-1})\delta(q^k z_2/z_1) : e^{a_-^1(q^{\frac{k+1}{2}}z_2)-(\frac{1}{k-1}a^2)(z_2|\frac{k-1}{2})+b_-^{1,2}(z_2)+b^{1,2}(qz_2)-b_-^{1,3}(qz_2)-b_+^{2,3}(qz_2)+(b+c)^{2,3}(q^2z_2)} : ,$$

$$[x_{1,2}^-(z_1), s_{2,4}(z_2)] \quad (\text{A.22})$$

$$= (q - q^{-1})\delta(q^k z_2/z_1) : e^{a_-^1(q^{\frac{k+1}{2}}z_2)-(\frac{1}{k-1}a^2)(z_2|\frac{k-1}{2})+b_-^{1,2}(z_2)+b^{1,2}(qz_2)-b_-^{1,3}(qz_2)-b_-^{2,3}(qz_2)+(b+c)^{2,3}(q^2z_2)} : ,$$

$$[x_{1,3}^-(z_1), s_{2,3}(z_2)] = -q(q - q^{-1})\delta(q^k z_2/z_1) \quad (\text{A.23})$$

$$\times : e^{a_-^1(q^{\frac{k+1}{2}}z_2)-(\frac{1}{k-1}a^2)(z_2|\frac{k-1}{2})+b_-^{1,2}(z_2)-2b_-^{1,3}(qz_2)-b_-^{2,3}(qz_2)-b_+^{2,3}(qz_2)+(b+c)^{2,3}(z_2)+(b+c)^{2,3}(q^2z_2)} : ,$$

$$[x_{1,3}^-(z_1), s_{2,5}(z_2)] \quad (\text{A.24})$$

$$= \frac{1}{z_2}\delta(q^k z_2/z_1) : e^{a_-^1(q^{\frac{k+1}{2}}z_2)-(\frac{1}{k-1}a^2)(z_2|\frac{k-1}{2})+b_-^{1,2}(z_2)+b^{1,2}(qz_2)-b_-^{1,3}(qz_2)-b_-^{2,3}(qz_2)+(b+c)^{2,3}(q^2z_2)} : ,$$

$$[x_{1,4}^-(z_1), s_{2,4}(z_2)] = q(q - q^{-1})\delta(q^k z_2/z_1) \quad (\text{A.25})$$

$$\times : e^{a_-^1(q^{\frac{k+1}{2}}z_2)-(\frac{1}{k-1}a^2)(z_2|\frac{k-1}{2})+b_-^{1,2}(z_2)-2b_-^{1,3}(qz_2)-b_-^{2,3}(qz_2)-b_+^{2,3}(qz_2)+(b+c)^{2,3}(z_2)+(b+c)^{2,3}(q^2z_2)} : ,$$

$$[x_{1,4}^-(z_1), s_{2,5}(z_2)] \quad (\text{A.26})$$

$$= \frac{1}{z_2}\delta(q^k z_2/z_1) : e^{a_-^1(q^{\frac{k+1}{2}}z_2)-(\frac{1}{k-1}a^2)(z_2|\frac{k-1}{2})+b_-^{1,2}(z_2)+b^{1,2}(qz_2)-b_-^{1,3}(qz_2)-b_+^{2,3}(qz_2)+(b+c)^{2,3}(q^2z_2)} : .$$

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