Free field approach to diagonalization of boundary transfer matrix : recent advances

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Abstract. We diagonalize infinitely many commuting operators $T_B(z)$. We call these operators $T_B(z)$ the boundary transfer matrix associated with the quantum group and the elliptic quantum group. The boundary transfer matrix is related to the solvable model with a boundary. When we diagonalize the boundary transfer matrix, we can calculate the correlation functions for the solvable model with a boundary. We review the free field approach to diagonalization of the boundary transfer matrix $T_B(z)$ associated with $U_q(A_2^{(2)})$ and $U_{q,p}(\widehat{sl_N})$. We construct the free field realizations of the eigenvectors of the boundary transfer matrix $T_B(z)$. This paper includes new unpublished formula of the eigenvector for $U_q(A_2^{(2)})$. It is thought that this diagonalization method can be extended to more general quantum group $U_q(g)$ and elliptic quantum group $U_{q,p}(g)$.

1. Introduction

We study infinitely many commuting operators $T_B(z)$ that we call the boundary transfer matrix. The boundary transfer matrix is related to the solvable model with a boundary. There have been many developments in solvable models in the last 30 years. Various models were found to be exactly solvable and various methods were invented to solve these models. The Free field approach is a powerful method to study exactly solvable models [1]. This paper is devoted to the free field approach to diagonalization of the boundary transfer matrix $T_B(z)$. When we diagonalize the boundary transfer matrix, we can calculate the correlation functions for the solvable model with a boundary. The first paper on this subject was devoted to the XXZ chain with a boundary [2], in which the boundary transfer matrix $T_B(z)$ acts on the highest representation of the quantum group $U_q(\widehat{sl_2})$. It is thought that this basic theory for the quantum group $U_q(\widehat{sl_2})$ can be extended to the quantum group $U_q(g)$ for arbitrary affine Lie algebra g. It is thought that the theory on the quantum group $U_q(g)$ can be generalized to those on the elliptic quantum group $U_{q,p}(g)$. In this paper we summarize the generalization on this direction. This paper includes a review on free field approach to the boundary transfer matrix [2, 3, 4, 5, 6, 7, 8] and new unpublished formula of the boundary state for the quantum group $U_q(A_2^{(2)})$. The plan of this paper is as follows. In section 2 we summarize the results for the quantum group $U_q(A_2^{(2)})$ [4] and give new unpublished formulae of the boundary state associated with nontrivial K-matrix $K_\pm(z)$. In section 3 we review the results for the elliptic quantum group $U_{q,p}(\widehat{sl_N})$, which gives a generalization of the papers [2, 3, 6, 7].

2. Quantum group $U_q(A_2^{(2)})$

In this section we diagonalize the boundary transfer matrix $T_B(z)$ for quantum group $U_q(A_2^{(2)})$.

2.1. Boundary transfer matrix $T_B(z)$

We fix q and z such that $0<|q|<1$ and $|q^2|<|z|<|q^{-2}|$. Let us set the q-integers

$$[a]_q=\frac{q^a-q^{-a}}{q-q^{-1}}.$$

We use the abbreviation.

$$(z;p_1,p_2,\cdots,p_M)_\infty=\prod_{k_1,k_2,\cdots,k_M=0}^{\infty}(1-p_1^{k_1}p_2^{k_2}\cdots p_M^{k_M}z).$$

The R-matrix $R(\zeta)$ for the twisted quantum group $U_q(A_2^{(2)})$ is given by following [9].

$$R(z)=\frac{1}{\kappa(z)}\begin{pmatrix}
1&&&&&&&&\\
&b(z)&&c(z)&&&&&\\
&&d(z)&&e(z)&&f(z)&&\\
&zc(z)&&b(z)&&&&&\\
&&-q^2ze(z)&&j(z)&&e(z)&&\\
&&&&&b(z)&&c(z)&\\
&&n(z)&&-q^2ze(z)&&d(z)&&\\
&&&&&zc(z)&&b(z)&\\
&&&&&&&&1
\end{pmatrix}. \qquad (1)$$

Here we have set

$$b(z)=\frac{q(z-1)}{q^2z-1},\ c(z)=\frac{q^2-1}{q^2z-1},\ d(z)=\frac{q^2(z-1)(qz+1)}{(q^2z-1)(q^3z+1)},\ e(z)=\frac{q^{\frac{1}{2}}(z-1)(q^2-1)}{(q^2z-1)(q^3z+1)},$$
$$f(z)=\frac{(q^2-1)((q^3+q)z-(q-1))}{(q^2z-1)(q^3z+1)},\ n(z)=\frac{(q^2-1)((q^3-q)z+(q^2+1))}{(q^2z-1)(q^3z+1)},$$
$$j(z)=\frac{q^4z^2+(q^5-q^4-q^3+q^2+q-1)z-q}{(q^2z-1)(q^3z+1)},$$
$$\kappa(z)=z\frac{(q^6z;q^6)_\infty(q^2/z;q^6)_\infty(-q^5z;q^6)_\infty(-q^3/z;q^6)_\infty}{(q^6/z;q^6)_\infty(q^2z;q^6)_\infty(-q^5/z;q^6)_\infty(-q^3z;q^6)_\infty}.$$

Let $\{v_+,v_0,v_-\}$ denote the natural basis of $V=\mathbf{C}^3$. When viewed as an operator on $V\otimes V$, the matrix element of $R(z)$ are defined by $R(z)v_{k_1}\otimes v_{k_2}=\sum_{j_1,j_2=\pm,0}v_{j_1}\otimes v_{j_2}R(z)_{j_1,j_2}^{k_1,k_2}$, where the ordering of the index is given by $(+,+),(+,0),(+,-),(0,+),(0,0),(0,-),(-,+),(-,0),(-,-)$. The R-matrix $R(z)$ satisfies the Yang-Baxter equation.

$$R_{1,2}(z_1/z_2)R_{1,3}(z_1/z_3)R_{2,3}(z_2/z_3)=R_{2,3}(z_2/z_3)R_{1,3}(z_1/z_3)R_{1,2}(z_1/z_2). \qquad (2)$$

The R-matrix $R(z)$ is characterized by the intertwiner as the quantum group $U_q(A_2^{(2)})$. We set the normalization function $\kappa(z)$ such that the minimal eigenvalue of the corner transfer matrix becomes 1 [20]. There exist three diagonal solutions of the K-matrix for the quantum group $U_q(A_2^{(2)})$. The K-matrix $K_\epsilon(z)$, $(\epsilon=\pm,0)$ are given by following [12].

$$K_0(z)=\frac{\varphi_0(z)}{\varphi_0(z^{-1})}\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix},\quad K_\pm(z)=\frac{\varphi_\pm(z)}{\varphi_\pm(z^{-1})}\begin{pmatrix}z^2&&\\&\frac{\pm\sqrt{-1}q^{\frac{3}{2}}+z}{\pm\sqrt{-1}q^{\frac{3}{2}}+z^{-1}}&\\&&1\end{pmatrix}. \qquad (3)$$

where

$$\varphi_0(z)=\frac{(q^8z;q^{12})_\infty(-q^9z^2;q^{12})_\infty}{(q^{12}z;q^{12})_\infty(-q^5z^2;q^{12})_\infty},\varphi_\pm(z)=\frac{(\pm\sqrt{-1}q^{\frac{9}{2}}z;q^6)_\infty(\mp\sqrt{-1}q^{\frac{7}{2}}z;q^6)_\infty}{(\pm\sqrt{-1}q^{\frac{1}{2}}z;q^6)_\infty(\mp\sqrt{-1}q^{\frac{3}{2}}z;q^6)_\infty}\,\varphi_0(z). \tag{4}$$

The K-matrix $K(z)=K_\epsilon(z)$ satisfies the boundary Yang-Baxter equation in $\mathrm{End}(V\otimes V)$ [10].

$$K_2(z_2)R_{2,1}(z_1z_2)K_1(z_1)R_{1,2}(z_1/z_2)=R_{2,1}(z_1/z_2)K_1(z_1)R_{1,2}(z_1z_2)K_2(z_2). \tag{5}$$

We set the normalization function $\varphi_\epsilon(z)$, $(\epsilon=\pm,0)$ such that the minimal eigenvalue of the boundary transfer matrix $T_B(z)$ becomes 1. The Izergin-Korepin model associated with the identity solution $\bar{K}_0(z)=id$ was studied in [4]. In this paper we give the free field realization of the boundary state for nontrivial solutions $K_\pm(z)$. The Izergin-Korepin model with the nontrivial solutions $K_\epsilon(z)$, $(\epsilon=\pm,0)$ was studied in [5].

Let us introduce the vertex operators $\Phi_\epsilon(z)$, $(\epsilon=\pm,0)$ [13, 15, 16] that satisfy the following commutation relation

$$\Phi_{\epsilon_2}(z_2)\Phi_{\epsilon_1}(z_1)=\sum_{\epsilon_1',\epsilon_2'=\pm,0}R_{\epsilon_1\epsilon_2}^{\epsilon_1'\epsilon_2'}(z_1/z_2)\Phi_{\epsilon_1'}(z_1)\Phi_{\epsilon_2'}(z_2). \tag{6}$$

The vertex operator $\Phi_\epsilon(z)$ and its dual $\Phi_\epsilon^*(z)=q^{-\frac{\epsilon}{2}}\Phi_{-\epsilon}(-q^{-3}z)$, $(\epsilon=\pm,0)$ satisfy the inversion relation

$$g\Phi_{\epsilon_1}(z)\Phi^*_{\epsilon_2}(z)=\delta_{\epsilon_1,\epsilon_2}id, \tag{7}$$

where $g=\frac{1}{1+q}\frac{(q^2;q^6)_\infty(-q^3;q^6)_\infty}{(q^6;q^6)_\infty(-q^5;q^6)_\infty}$. Let us introduce the boundary transfer matrix $T_B(z)$ for $U_q(A_2^{(2)})$. The boundary transfer matrix $T_B(z)$ is given by

$$T_B(z)=g\sum_{\epsilon,\epsilon'=\pm,0}\Phi_\epsilon^*(z^{-1})K_\epsilon^{\epsilon'}(z)\Phi_{\epsilon'}(z). \tag{8}$$

The boundary transfer matrix $T_B(z)$ is related to the boundary Izergin-Korepin model [4, 5]. From the commutation relations of the vertex operators (6) and the boundary Yang-Baxter equation (5), the commutativity of the boundary transfer matrix $T_B(z)$ is ensured.

$$[T_B(z_1),T_B(z_2)]=0,\quad \text{for any}\quad z_1,z_2. \tag{9}$$

We call the eigenvector $|B\rangle_\epsilon$,$(\epsilon=\pm,0)$ with eigenvalue 1 the boundary state.

$$T_B(z)|B\rangle_\epsilon=|B\rangle_\epsilon. \tag{10}$$

From the definition of $\varphi_\epsilon(z)$, the boundary state $|B\rangle$ is the eigenvector with the minimal eigenvalue. In the following section we give the free field realization of the boundary state $|B\rangle_\epsilon$ for the diagonal boundary K-matrix $K_\epsilon(z)$, $(\epsilon=\pm,0)$. The boundary state $|B\rangle_\epsilon$ for the identity K-matrix $K_0(z)$ was constructed in [4]. Let us introduce the type-II vertex operators $\Psi_\mu^*(\xi)$, $(\mu=\pm,0)$ that satisfy the following commutation relation.

$$\Phi_\epsilon(z)\Psi_\mu^*(\xi)=\tau(z/\xi)\Psi^*_\mu(\xi)\Phi_\epsilon(z),\quad(\epsilon,\mu=\pm,0), \tag{11}$$

where

$$\tau(z)=z^{-1}\frac{\Theta_{q^6}(q^5z)\Theta_{q^6}(-q^4z)}{\Theta_{q^6}(q^5z^{-1})\Theta_{q^6}(-q^4z^{-1})},\quad \Theta_p(z)=(p;p)_\infty(z;p)_\infty(pz^{-1};p)_\infty.$$

Multiplying the type-II vertex operators $\Psi^*_\mu(\xi)$ to the boundary state $|B\rangle_\epsilon$,

$$|\xi_1,\xi_2,\cdots,\xi_N\rangle_{\mu_1,\mu_2,\cdots,\mu_N}=\Psi^*_{\mu_1}(\xi_1)\Psi^*_{\mu_2}(\xi_2)\cdots\Psi^*_{\mu_N}(\xi_N)|B\rangle_\epsilon, \tag{12}$$

we have many eigenvectors.

$$T_B(z)|\xi_1,\xi_2,\cdots,\xi_N\rangle_{\mu_1,\mu_2,\cdots,\mu_N}=\prod_{s=1}^N \tau(z/\xi_s)\tau(-1/q^3z\xi_s)|\xi_1,\xi_2,\cdots,\xi_N\rangle_{\mu_1,\mu_2,\cdots,\mu_N}.$$

The vectors $\{|\xi_1,\cdots,\xi_N\rangle_{\mu_1,\cdots,\mu_N}\}$ are the basis of the space of the state of the boundary Izergin-Korepin model.

2.2. Free field realization
In this section we give the free field realization of the boundary state $|B\rangle_\epsilon$. Let us introduce bosons $a_m,(m\in\mathbf{Z}_{\neq 0})$ as following [13, 14, 15, 16].

$$[a_m,a_n]=\delta_{m+n}\frac{[m]_q}{m}([2m]_q-(-1)^m[m]_q). \tag{13}$$

Let us set the zero-mode operators P,Q by

$$[a_m,P]=[a_m,Q]=0,\ \ [P,Q]=1. \tag{14}$$

Level-1 irreducible highest representation $V(\Lambda_1)$ of $U_q(A_2^{(2)})$ is realized by

$$V(\Lambda_1)=\mathbf{C}[a_{-1},a_{-2},\cdots]\oplus_{n\in\mathbf{Z}}e^{nQ}|\Lambda_1\rangle,\quad |\Lambda_1\rangle=e^{\frac{Q}{2}}|0\rangle.$$

The vacuum vector $|0\rangle$ is characterized by

$$a_m|0\rangle=0,\ (m>0),\quad P|0\rangle=0.$$

Let us set the auxiliary operators $P(z),Q(z),R^-(w),S^-(w)$ by

$$P(z)=\sum_{m>0}\frac{a_{-m}q^{\frac{9m}{2}}z^m}{[2m]_q-(-1)^m[m]_q},\qquad Q(z)=-\sum_{m>0}\frac{a_mq^{-\frac{7m}{2}}z^{-m}}{[2m]_q-(-1)^m[m]_q},$$
$$R^-(w)=-\sum_{m>0}\frac{a_{-m}}{[m]_q}q^{\frac{m}{2}}w^m,\qquad S^-(w)=\sum_{m>0}\frac{a_m}{[m]_q}q^{\frac{m}{2}}w^{-m}.$$

Let us set $\epsilon(q)=([2]_{q^{\frac{1}{2}}})^{\frac{1}{2}}$. Let us set the current,

$$X^-(w)=\epsilon(q)e^{R^-(w)}e^{S^-(w)}e^{-Q}w^{-P+\frac{1}{2}}.$$

The free field realizations of the vertex operators $\Phi_\epsilon(z)$ [13, 15, 16] are given by

$$\begin{aligned}
\Phi_-(z)&=\frac{1}{\epsilon(q)}e^{P(z)}e^{Q(z)}e^{Q}(-zq^4)^{P+\frac{1}{2}},\\
\Phi_0(z)&=\oint_{C_1}\frac{dw}{2\pi\sqrt{-1}w}\frac{(q^2-1)}{q^4z(1-qw/z)(1-qz/w)}:\Phi_-(z)X^-(q^4w):,\\
\Phi_+(z)&=\oint\oint_{C_2}\frac{dw_1}{2\pi\sqrt{-1}w_1}\frac{dw_2}{2\pi\sqrt{-1}w_2}\frac{q^{1/2}(1-q^2)^2}{q^4z^2w_1w_2}\\
&\times\frac{(w_1-w_2)^2(q(w_1+w_2)-(1+q^2)z)}{(1+qw_1/w_2)(1+qw_2/w_1)(1-qw_1/z)(1-qz/w_1)(1-qw_2/z)(1-qz/w_2)}\\
&\times\ :\Phi_-(z)X^-(q^4w_1)X^-(q^4w_2):.
\end{aligned}$$

The integrand contour C_1 encircles $w = 0, qz$ but not $w = q^{-1}z$. The integrand contour C_2 encircles $w_1 = 0, qz, qw_2$ and $w_2 = 0, qz, qw_1$ but not $w_1 = q^{-1}z, q^{-1}w_2$ and $w_2 = q^{-1}z, q^{-1}w_1$. The free field realization of type-II vertex operators $\Psi_\mu^*(\xi)$ are given as similar way [13]. Now we have the free field realization of the boundary transfer matrix $T_B(z)$, using those of the vertex operators. We construct the free field realization of the boundary state $|B\rangle_\epsilon$, analyzing those of the boundary transfer matrix $T_B(z)$. The following is **main result** of this section. The free field realization of the boundary states $|B\rangle_\epsilon$, $(\epsilon = \pm, 0)$ are given by

$$|B\rangle_\epsilon = e^{F_\epsilon} e^{-\frac{Q}{2}} |0\rangle, \quad (\epsilon = \pm, 0). \tag{15}$$

Here we have set

$$\begin{aligned} F_\epsilon &= -\frac{1}{2} \sum_{m>0} \frac{m q^{8m}}{[2m]_q - (-1)^m [m]_q} a_{-m}^2 \\ &+ \sum_{m>0} \left\{ \theta_m \left(\frac{(q^{\frac{m}{2}} - q^{-\frac{m}{2}} - \sqrt{-1}^m) q^{4m}}{[2m]_q - (-1)^m [m]_q} \right) - \frac{(\epsilon\sqrt{-1})^m q^{3m}}{[2m]_q - (-1)^m [m]_q} \right\} a_{-m}. \end{aligned} \tag{16}$$

Here we have used $\theta_m(x) = \begin{cases} x, & m : \text{even} \\ 0, & m : \text{odd} \end{cases}$. The boundary state $|B\rangle_0$ for the identity K-matrix $\bar{K}_0(z) = id$ was constructed in [4]. The realizations of $|B\rangle_\epsilon$ for the nontrivial K-matrix $K_\epsilon(z)$ are new. Multiplying the type-II vertex operators $\Psi_\mu^*(\xi)$ to the boundary state $|B\rangle_\epsilon$, we get the diagonalization of the boundary transfer matrix $T_B(z)$. It is thought that this method can be extended to the case of the affine quantum group $U_q(g)$.

3. Elliptic quantum group $U_{q,p}(\widehat{sl_N})$

In this section we diagonalize the boundary transfer matrix $T_B(z)$ associated with the elliptic quantum group $U_{q,p}(\widehat{sl_N})$ [8]. It gives a generalization of the papers [2, 3, 6, 7].

3.1. Boundary transfer matrix

Let us set the integer $N = 2, 3, \cdots$. We assume that $0 < x < 1$ and $r \geq N + 2$ $(r \in \mathbf{Z})$. We set $z = x^{2u}, x = e^{-\pi i / r\tau}$. We set the elliptic theta function $[u]$ by

$$[u] = x^{\frac{u^2}{r} - u} \Theta_{x^{2r}}(x^{2u}), \quad \Theta_q(z) = (q; q)_\infty (z; q)_\infty (q/z; q)_\infty.$$

Let $\epsilon_\mu (1 \leq \mu \leq N)$ be the orthonormal basis of $\mathbf{R}^N$ with the inner product $(\epsilon_\mu | \epsilon_\nu) = \delta_{\mu,\nu}$. Let us set $\bar{\epsilon}_\mu = \epsilon_\mu - \epsilon$ where $\epsilon = \frac{1}{N} \sum_{\nu=1}^N \epsilon_\nu$. Note that $\sum_{\mu=1}^N \bar{\epsilon}_\mu = 0$. Let α_μ $(1 \leq \mu \leq N-1)$ the simple root : $\alpha_\mu = \bar{\epsilon}_\mu - \bar{\epsilon}_{\mu+1}$. Let ω_μ $(1 \leq \mu \leq N-1)$ be the fundamental weights, which satisfy

$$(\alpha_\mu | \omega_\nu) = \delta_{\mu,\nu}, \quad (1 \leq \mu, \nu \leq N-1).$$

Explicitly we set $\omega_\mu = \sum_{\nu=1}^\mu \bar{\epsilon}_\nu$. The type A_{N-1} weight lattice is the linear span of $\bar{\epsilon}_\mu$ or ω_μ.

$$P = \sum_{\mu=1}^{N-1} \mathbf{Z} \bar{\epsilon}_\mu = \sum_{\mu=1}^{N-1} \mathbf{Z} \omega_\mu.$$

For $a \in P$ we set a_μ and $a_{\mu,\nu}$ by

$$a_{\mu,\nu} = a_\mu - a_\nu, \quad a_\mu = (a + \rho | \bar{\epsilon}_\mu), \quad (\mu, \nu \in P).$$

Here we set $\rho=\sum_{\mu=1}^{N-1}\omega_\mu$. Let us set the restricted path P_{r-N}^+ by

$$P_{r-N}^+=\{a=\sum_{\mu=1}^{N-1}c_\mu\omega_\mu\in P|c_\mu\in\mathbf{Z},c_\mu\geq 0,\sum_{\mu=1}^{N-1}c_\mu\leq r-N\}.$$

For $a\in P_{r-N}^+$, condition $0<a_{\mu,\nu}<r,\ (1\leq\mu<\nu\leq N-1)$ holds.

We recall elliptic solutions of the Yang-Baxter equation of face type. An ordered pair $(b,a)\in P^2$ is called admissible if and only if there exists μ $(1\leq\mu\leq N)$ such that $b-a=\bar{\epsilon}_\mu$. An ordered set of four weights $(a,b,c,d)\in P^4$ is called an admissible configuration around a face if and only if the ordered pairs (b,a), (c,b), (d,a) and (c,d) are admissible. Let us set the Boltzmann weight functions $W\left(\begin{array}{cc}c & d\\ b & a\end{array}\middle| u\right)$ associated with admissible configuration $(a,b,c,d)\in P^4$ [11]. For $a\in P_{r-N}^+$, we set

$$W\left(\begin{array}{cc}a+2\bar{\epsilon}_\mu & a+\bar{\epsilon}_\mu\\ a+\bar{\epsilon}_\mu & a\end{array}\middle| u\right)=R(u), \tag{17}$$

$$W\left(\begin{array}{cc}a+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & a+\bar{\epsilon}_\mu\\ a+\bar{\epsilon}_\nu & a\end{array}\middle| u\right)=R(u)\frac{[u][a_{\mu,\nu}-1]}{[u-1][a_{\mu,\nu}]}, \tag{18}$$

$$W\left(\begin{array}{cc}a+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & a+\bar{\epsilon}_\nu\\ a+\bar{\epsilon}_\nu & a\end{array}\middle| u\right)=R(u)\frac{[u-a_{\mu,\nu}][1]}{[u-1][a_{\mu,\nu}]}. \tag{19}$$

The normalizing function $R(u)$ is given by

$$R(u)\ =\ z^{\frac{r-1}{r}\frac{N-1}{N}}\frac{\varphi(z^{-1})}{\varphi(z)},\quad \varphi(z)=\frac{(x^2z;x^{2r},x^{2N})_\infty(x^{2r+2N-2}z;x^{2r},x^{2N})_\infty}{(x^{2r}z;x^{2r},x^{2N})_\infty(x^{2N}z;x^{2r},x^{2N})_\infty}.$$

Because $0<a_{\mu,\nu}<r$ $(1\leq\mu<\nu\leq N-1)$ holds for $a\in P_{r-N}^+$, the Boltzmann weight functions are well defined. The Boltzmann weight functions satisfy the Yang-Baxter equation of the face type.

$$\begin{aligned}&\sum_g W\left(\begin{array}{cc}d & e\\ c & g\end{array}\middle| u_1\right)W\left(\begin{array}{cc}c & g\\ b & a\end{array}\middle| u_2\right)W\left(\begin{array}{cc}e & f\\ g & a\end{array}\middle| u_1-u_2\right)\\ =\ &\sum_g W\left(\begin{array}{cc}g & f\\ b & a\end{array}\middle| u_1\right)W\left(\begin{array}{cc}d & e\\ g & f\end{array}\middle| u_2\right)W\left(\begin{array}{cc}d & g\\ c & b\end{array}\middle| u_1-u_2\right).\end{aligned} \tag{20}$$

We set the normalization function $\varphi(z)$ such that the minimal eigenvalue of the corner transfer matrix becomes 1 [20]. An order set of three weights $(a,b,g)\in P^3$ is called an admissible configuration at a boundary if and only if the ordered pairs (g,a) and (g,b) are admissible. Let us set the boundary Boltzmann weight functions $K\left(\begin{array}{cc} & a\\ g & \\ & b\end{array}\middle| u\right)$ for admissible weights (a,b,g) as following [12].

$$K\left(\begin{array}{cc} & a\\ a+\bar{\epsilon}_\mu & \\ & b\end{array}\middle| u\right)=z^{\frac{r-1}{r}\frac{N-1}{N}-\frac{2}{r}a_1}\frac{h(z)}{h(z^{-1})}\frac{[c-u][a_{1,\mu}+c+u]}{[c+u][a_{1,\mu}+c-u]}\delta_{a,b}. \tag{21}$$

In this paper, we consider the case of continuous parameter $0<c<1$. The normalization function $h(z)$ is given by following [8].

$$h(z)\ =\ \frac{(x^{2r+2N-2}/z^2;x^{2r},x^{4N})_\infty(x^{2N+2}/z^2;x^{2r},x^{4N})_\infty}{(x^{2r}/z^2;x^{2r},x^{4N})_\infty(x^{4N}/z^2;x^{2r},x^{4N})_\infty}$$

$$\times \frac{(x^{2N+2c}/z;x^{2r},x^{2N})_\infty(x^{2r-2c}/z;x^{2r},x^{2N})_\infty}{(x^{2N+2r-2c-2}/z;x^{2r},x^{2N})_\infty(x^{2c+2}/z;x^{2r},x^{2N})_\infty}$$
$$\times \prod_{j=2}^{N} \frac{(x^{2r+2N-2c-2a_{1,j}}/z;x^{2r},x^{2N})_\infty(x^{2c+2a_{1,j}}/z;x^{2r},x^{2N})_\infty}{(x^{2r+2N-2c-2a_{1,j}-2}/z;x^{2r},x^{2N})_\infty(x^{2c+2+2a_{1,j}}/z;x^{2r},x^{2N})_\infty}. \quad (22)$$

The boundary Boltzmann weight functions and the Boltzmann weight functions satisfy the Boundary Yang-Baxter equation.

$$\sum_{f,g} W\left(\begin{array}{cc} c & f \\ b & a \end{array}\middle| u_1-u_2\right) W\left(\begin{array}{cc} c & d \\ f & g \end{array}\middle| u_1+u_2\right) K\left(\begin{array}{cc} & g \\ f & \\ & a \end{array}\middle| u_1\right) K\left(\begin{array}{cc} & e \\ d & \\ & g \end{array}\middle| u_2\right)$$
$$= \sum_{f,g} W\left(\begin{array}{cc} c & d \\ f & e \end{array}\middle| u_1-u_2\right) W\left(\begin{array}{cc} c & f \\ b & g \end{array}\middle| u_1+u_2\right) K\left(\begin{array}{cc} & e \\ f & \\ & g \end{array}\middle| u_1\right) K\left(\begin{array}{cc} & g \\ b & \\ & a \end{array}\middle| u_2\right). \quad (23)$$

We set the normalization function $h(z)$ such that the minimal eigenvalue of the boundary transfer matrix $T_B(z)$ becomes 1.

The vertex operator $\Phi^{(b,a)}(z)$ and the dual vertex operator $\Phi^{*(a,b)}(z)$ associated with the elliptic quantum group $U_{q,p}(\widehat{sl_N})$, are the operators which satisfy the following commutation relations

$$\Phi^{(a,b)}(z_1)\Phi^{(b,c)}(z_2) = \sum_g W\left(\begin{array}{cc} a & g \\ b & c \end{array}\middle| u_2-u_1\right) \Phi^{(a,g)}(z_2)\Phi^{(g,c)}(z_1), \quad (24)$$
$$\Phi^{*(a,b)}(z_1)\Phi^{*(b,c)}(z_2) = \sum_g W\left(\begin{array}{cc} c & b \\ g & a \end{array}\middle| u_2-u_1\right) \Phi^{*(a,g)}(z_2)\Phi^{*(g,c)}(z_1), \quad (25)$$
$$\Phi^{(a,b)}(z_1)\Phi^{*(b,c)}(z_2) = \sum_g W\left(\begin{array}{cc} g & c \\ a & b \end{array}\middle| u_1-u_2\right) \Phi^{*(a,g)}(z_2)\Phi^{(g,c)}(z_1), \quad (26)$$

and the inversion relation

$$\Phi^{(a,g)}(z)\Phi^{*(g,b)}(z) = \delta_{a,b}. \quad (27)$$

We define the boundary transfer matrix $T_B(z)$ for the elliptic quantum group $U_{q,p}(\widehat{sl_N})$.

$$T_B(z) = \sum_{\mu=1}^{N} \Phi^{*(a,a+\bar{\epsilon}_\mu)}(z^{-1}) K\left(\begin{array}{cc} & a \\ a+\bar{\epsilon}_\mu & \\ & a \end{array}\middle| u\right) \Phi^{(a+\bar{\epsilon}_\mu,a)}(z). \quad (28)$$

From the commutation relations of the vertex operators (24), (25), (26), and the boundary Yang-Baxter equation (23), the boundary $T_B(z)$ commute with each other.

$$[T_B(z_1), T_B(z_2)] = 0, \quad \text{for any } z_1, z_2. \quad (29)$$

We call the eigenvector $|B\rangle$ with the eigenvalue 1 the boundary state.

$$T_B(z)|B\rangle = |B\rangle. \quad (30)$$

Let us introduce the type-II vertex operators $\Psi^{*(b,a)}(z)$ by

$$\Phi^{(d,c)}(z_1)\Psi^{*(b,a)}(z_2) = \chi(z_2/z_1)\Psi^{*(b,a)}(z_2)\Phi^{(d,c)}(z_1), \quad (31)$$
$$\Phi^{*(c,d)}(z_1)\Psi^{*(b,a)}(z_2) = \chi(z_1/z_2)\Psi^{*(b,a)}(z_2)\Phi^{*(c,d)}(z_1). \quad (32)$$

where we have set

$$\chi(z)=z^{-\frac{N-1}{N}}\frac{\Theta_{x^{2N}}(-xz)}{\Theta_{x^{2N}}(-xz^{-1})}.$$

We set the vectors $|\xi_1,\xi_2,\cdots,\xi_M\rangle_{\mu_1,\mu_2,\cdots,\mu_M}$ $(1\leq \mu_1,\mu_2,\cdots,\mu_M\leq N)$.

$$\begin{aligned}&|\xi_1,\xi_2,\cdots,\xi_M\rangle_{\mu_1,\mu_2,\cdots,\mu_M} \\ =\ &\Psi^{*(b+\bar{\epsilon}_{\mu_1}+\bar{\epsilon}_{\mu_2}+\cdots+\bar{\epsilon}_{\mu_M},b+\bar{\epsilon}_{\mu_2}+\cdots+\bar{\epsilon}_{\mu_M})}(\xi_1)\cdots\Psi^{*(b+\bar{\epsilon}_{\mu_{M-1}}+\bar{\epsilon}_{\mu_M},b+\bar{\epsilon}_{\mu_M})}(\xi_{M-1})\Psi^{*(b+\bar{\epsilon}_{\mu_M},b)}(\xi_M)|B\rangle.\end{aligned}\tag{33}$$

Now we have many eigenvectors of $T_B(z)$.

$$T_B(z)|\xi_1,\xi_2,\cdots,\xi_M\rangle_{\mu_1,\mu_2,\cdots,\mu_M}=\prod_{j=1}^{M}\chi(\xi_j/z)\chi(1/\xi_j z)\ |\xi_1,\xi_2,\cdots,\xi_M\rangle_{\mu_1,\mu_2,\cdots,\mu_M}.$$

The vectors $|\xi_1,\xi_2,\cdots,\xi_M\rangle_{\mu_1,\mu_2,\cdots,\mu_M}$ are the basis of the space of the state of the boundary $U_{q,p}(\widehat{sl_N})$ face model.

3.2. Free field realization

In this section we give the free field realizations of the boundary state $|B\rangle$. Let us introduce the bosons $\beta_m^i, (i=1,2,\cdots,N-1; m\in\mathbf{Z})$ as following [17].

$$[\beta_m^j,\beta_n^k]=\begin{cases} m\dfrac{[(r-1)m]_x}{[rm]_x}\dfrac{[(N-1)m]_x}{[Nm]_x}\delta_{m+n,0} & (j=k)\\ -mx^{Nm\ sgn(j-k)}\dfrac{[(r-1)m]_x}{[rm]_x}\dfrac{[m]_x}{[Nm]_x}\delta_{m+n,0} & (j\neq k).\end{cases}\tag{34}$$

Let us set β_m^N by $\sum_{j=1}^N x^{-2jm}\beta_m^j=0$. The above commutation relations are valid for all $1\leq j,k\leq N$. We also introduce the zero-mode operators P_α, Q_α, $(\alpha\in P)$ by

$$[\sqrt{-1}P_\alpha,Q_\beta]=(\alpha|\beta),\quad(\alpha,\beta\in P).\tag{35}$$

In what follows we deal with the bosonic Fock space $\mathcal{F}_{l,k}$, generated by $\beta_{-m}^j (m>0)$ over the vacuum vector $|l,k\rangle$, where $l=b+\rho,\ k=a+\rho$ for $a\in P_{r-N}^+, b\in P_{r-1-N}^+$.

$$\mathcal{F}_{l,k}=\mathbf{C}[\{\beta_{-1}^j,\beta_{-2}^j,\cdots\}_{j=1,\cdots,N-1}]|l,k\rangle,\quad |l,k\rangle=e^{\sqrt{-1}\sqrt{\frac{r}{r-1}}Q_l-\sqrt{-1}\sqrt{\frac{r-1}{r}}Q_k}|0,0\rangle.$$

where

$$\beta_m^j|l,k\rangle=0,\ (m>0),\ P_\alpha|l,k\rangle=\left(\alpha\left|\sqrt{\frac{r}{r-1}}l-\sqrt{\frac{r-1}{r}}k\right.\right)|l,k\rangle.$$

The commutation relation of bosons β_m^j is not symmetric. It is convenient to introduce new generators of bosons α_m^j $(m\in\mathbf{Z}_{\neq 0}; 1\leq j\leq N-1)$ by

$$\alpha_m^j=x^{-jm}(\beta_m^j-\beta_m^{j+1}).\tag{36}$$

They satisfy the following commutation relations.

$$[\alpha_m^j,\alpha_n^k]=m\frac{[(r-1)m]_x}{[rm]_x}\frac{[A_{j,k}m]_x}{[m]_x}\delta_{m+n,0},$$

where $A_{j,k}$ is a matrix element of the Cartan matrix of sl_N type. We give a free field realization of the vertex operators $\Phi^{(b,a)}(z)$. Let us set the operators $P_-(z), Q_-(z)$, $R_-^j(z), S_-^j(z)$, $(1\leq j\leq N-1)$ by

$$\begin{aligned}P_-(z) &= \sum_{m>0}\frac{1}{m}\beta_{-m}^1 z^m,\ Q_-(z)=-\sum_{m>0}\frac{1}{m}\beta_m^1 z^{-m},\\ R_-^j(z) &= -\sum_{m>0}\frac{1}{m}\alpha_{-m}^j z^m,\ S_-^j(z)=\sum_{m>0}\frac{1}{m}\alpha_m z^{-m}.\end{aligned}$$

Let us set the basic operators $U(z)$,$F_{\alpha_j}(z)$, $(1\leq j\leq N-1)$ on the Fock space $\mathcal{F}_{l,k}$.

$$\begin{aligned}U(z) &= z^{\frac{r-1}{2r}\frac{N-1}{N}}e^{-\sqrt{-1}\sqrt{\frac{r-1}{r}}Q_{\bar{\epsilon}_1}}z^{-\sqrt{\frac{r-1}{r}}P_{\bar{\epsilon}_1}}e^{P_-(z)}e^{Q_-(z)},\\ F_{\alpha_j}(z) &= z^{\frac{r-1}{r}}e^{\sqrt{-1}\sqrt{\frac{r-1}{r}}Q_{\alpha_j}}z^{\sqrt{\frac{r-1}{r}}P_{\alpha_j}}e^{R_-^j(z)}e^{S_-^j(z)}.\end{aligned}$$

In what follows we set $l=b+\rho, k=a+\rho$, $(a\in P_{r-N}^+, b\in P_{r-N-1}^+)$ and $\pi_\mu=\sqrt{r(r-1)}P_{\bar{\epsilon}_\mu}$, $\pi_{\mu,\nu}=\pi_\mu-\pi_\nu$. Then $\pi_{\mu\nu}$ acts on $\mathcal{F}_{l,k}$ as an integer $(\epsilon_\mu-\epsilon_\nu|rl-(r-1)k)$. We give the free field realization of the vertex operators $\Phi^{(a+\bar{\epsilon}_\mu,a)}(z)$, $(1\leq\mu\leq N-1)$ [17] by

$$\begin{aligned}\Phi^{(a+\bar{\epsilon}_1,a)}(z_0^{-1}) &= U(z_0),\\ \Phi^{(a+\bar{\epsilon}_\mu,a)}(z_0^{-1}) &= \oint\cdots\oint\prod_{j=1}^{\mu-1}\frac{dz_j}{2\pi i z_j}U(z_0)F_{\alpha_1}(z_1)F_{\alpha_2}(z_2)\cdots F_{\alpha_{\mu-1}}(z_{\mu-1})\\ &\times\prod_{j=1}^{\mu-1}\frac{[u_j-u_{j-1}+\frac{1}{2}-\pi_{j,\mu}]}{[u_j-u_{j-1}-\frac{1}{2}]}.\end{aligned}$$

Here we set $z_j=x^{2u_j}$. We take the integration contour to be simple closed curve that encircles $z_j=0, x^{1+2rs}z_{j-1}, (s\in\mathbf{N})$ but not $z_j=x^{-1-2rs}z_{j-1}, (s\in\mathbf{N})$ for $1\leq j\leq\mu-1$. The $\Phi^{(a+\bar{\epsilon}_\mu,a)}(z)$ is an operator such that $\Phi^{(a+\bar{\epsilon}_\mu,a)}(z):\mathcal{F}_{l,k}\to\mathcal{F}_{l,k+\bar{\epsilon}_\mu}$. The free field realization of the dual vertex operator $\Phi^{*(a,b)}(z)$ and the type-II vertex operator $\Psi^{*(a,b)}(z)$ are given by similar way [17, 18]. Now we have the free field realization of the boundary transfer matrix $T_B(z)$, using those of the vertex operators. We construct the free field realization of the boundary state $|B\rangle$, analyzing those of the transfer matrix $T_B(z)$. The following is **main result** of this section. The free field realization of the boundary state $|B\rangle$ is given as following [8].

$$|B\rangle=e^F|k,k\rangle. \tag{37}$$

Here we have set

$$F = -\frac{1}{2}\sum_{m>0}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1}\frac{1}{m}\frac{[rm]_x}{[(r-1)m]_x}I_{j,k}(m)\alpha_{-m}^j\alpha_{-m}^k+\sum_{m>0}\sum_{j=1}^{N-1}\frac{1}{m}D_j(m)\beta_{-m}^j, \tag{38}$$

where

$$\begin{aligned}D_j(m) &= -\theta_m\left(\frac{[(N-j)m/2]_x[rm/2]_x^+x^{\frac{(3j-N-1)m}{2}}}{[(r-1)m/2]_x}\right.\\ &+\frac{x^{(j-1)m}[(-r+2\pi_{1,j}+2c-j+2)m]_x}{[(r-1)m]_x}\\ &+\frac{[m]_x x^{(r-2c+2j-2)m}}{[(r-1)m]_x}\left(\sum_{k=j+1}^{N-1}x^{-2m\pi_{1,k}}\right)\\ &+\frac{x^{(2j-N)m}[(r-2\pi_{1,N}-2c+N-1)m]_x}{[(r-1)m]_x},\end{aligned} \tag{39}$$

and

$$I_{j,k}(m)=\frac{[jm]_x[(N-k)m]_x}{[m]_x[Nm]_x}=I_{k,j}(m)\quad(1\leq j\leq k\leq N-1). \tag{40}$$

Here we have used

$$[a]_x^+=x^a+x^{-a},\quad \theta_m(x)=\left\{\begin{array}{ll} x, & m:\ \text{even},\\ 0, & m:\ \text{odd}.\end{array}\right.$$

Multiplying the type-II vertex operators $\Psi_\mu^{*(a,b)}(\xi)$ to the boundary state $|B\rangle$, we get the diagonalization of the boundary transfer matrix $T_B(z)$ on the space of state of the boundary $U_{q,p}(\widehat{sl_N})$ face model. It is thought that this method can be extended to the case of the elliptic quantum group $U_{q,p}(g)$ for affine Lie algebra g [19].

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