A REMARK ON GROUND STATE OF BOUNDARY IZERGIN-KOREPIN MODEL

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Abstract

We study the ground state of the boundary Izergin-Korepin model. The boundary Izergin-Korepin model is defined by so-called R-matrix and K-matrix for $U_q(A_2^{(2)})$ which satisfy Yang-Baxter equation and boundary Yang-Baxter equation. The ground state associated with identity K-matrix $\bar{K}(z) = id$ was constructed by W.-L.Yang and Y.-Z.Zhang in earlier study. We construct the free field realization of the ground state associated with nontrivial diagonal K-matrix.

1 Introduction

There have been many developments in the field of exactly solvable models. Various methods were invented to solve models. The free field approach [1] provides a powerful method to study exactly solvable models. This paper is devoted to the free field approach to boundary problem of exactly solvable statistical mechanics [2]. Exactly solvable boundary model [2, 3] is defined by the solutions of the Yang-Baxter equation and the boundary Yang-Baxter equation

$$K_2(z_2)R_{2,1}(z_1z_2)K_1(z_1)R_{1,2}(z_1/z_2) = R_{2,1}(z_1/z_2)K_1(z_1)R_{1,2}(z_1z_2)K_2(z_2).$$

In this paper we are going to study the boundary Izergin-Korepin model defined by the solutions of the Yang-Baxter equation [4] and the boundary Yang-Baxter equation [5, 6] for the quantum group $U_q(A_2^{(2)})$. We are going to diagonalize the infinite transfer matrix $T_\epsilon(z)$ of the boundary Izergin-Korepin model, by means of the free field approach. For better understanding of the model that we are going to study, we give comments on the solutions of the boundary Yang-Baxter equation. The R-matrix associated with non-exceptional affine symmetry except for $D_n^{(2)}$ commute with each other

$$[\widehat{R}(z_1), \widehat{R}(z_2)] = 0,$$

where $\widehat{R}(z) = PR(z)$ and $P(a \otimes b) = b \otimes a$. Hence we know that the identity K-matrix

$$\bar{K}(z) = id$$

is a particular solution of the boundary Yang-Baxter equation [3] for $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ and $A_n^{(2)}$. There exist more general solutions of the boundary Yang-Baxter equation. The diagonal solutions of the boundary Yang-Baxter equation for $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ and $A_n^{(2)}$ are classified in [5]. For $A_n^{(1)}$ there exists the diagonal K-matrix that has one continuous free parameter. However for $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ and $A_n^{(2)}$ there exist only discrete solutions. For example, for $A_2^{(2)}$ there exist three isolated solutions $\bar{K}(z)$.

$$\bar{K}(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} z^2 & 0 & 0 \\ 0 & \dfrac{\pm\sqrt{-1}q^{\frac{3}{2}} + z}{\pm\sqrt{-1}q^{\frac{3}{2}} + z^{-1}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In earlier study [7] W.-L. Yang and Y.-Z. Zhang constructed the free field realization of the ground state associated with identity K-matrix $\bar{K}(z) = id$ for $A_2^{(2)}$. In this paper we construct the free field realization of the ground state associated with nontrivial diagonal K-matrix $\bar{K}(z)$ for $A_2^{(2)}$. This realization is the first example associated with the discrete solutions of the boundary Yang-Baxter equation. It is thought that the free field approach to the boundary problem works for every discrete K-matrix of the affine symmetry. Contrary to $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ and $A_n^{(2)}$ case, there have been many papers on the free field approach to boundary problem for $A_n^{(1)}$ symmetry. For example, the higher-rank generalization [8, 10] and the elliptic deformation [9, 10] have been solved.

The plan of this paper is as follows. In section 2 we give physical picture of our problem. We introduce the boundary Izergin-Korepin model and introduce the transfer matrix $T_\epsilon(z)$. In section 3 we translate physical picture of our problem into mathematical picture. We construct the free field realization of the ground state $|B\rangle_\epsilon$ and give the diagonalization of the transfer matrix $T_\epsilon(z)$.

2 Boundary Izergin-Korepin model

In this section we formulate physical picture of our problem.

2.1 R-matrix and K-matrix

In this section we introduce R-matrix $R(z)$ and K-matrix $K(z)$ associated with the quantum group $U_q(A_2^{(2)})$. We choose q and z such that $-1 < q^{\frac{1}{2}} < 0$ and $|q^2| < |z| < |q^{-2}|$. Let $\{v_+, v_0, v_-\}$ denotes the natural basis of $V = \mathbb{C}^3$. We introduce the R-matrix $R(z) \in \mathrm{End}(V \otimes V)$ [4].

$$R(z) = \frac{1}{\kappa(z)} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(z) & 0 & c(z) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d(z) & 0 & e(z) & 0 & f(z) & 0 & 0 \\ 0 & zc(z) & 0 & b(z) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -q^2 ze(z) & 0 & j(z) & 0 & e(z) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(z) & 0 & c(z) & 0 \\ 0 & 0 & n(z) & 0 & -q^2 ze(z) & 0 & d(z) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & zc(z) & 0 & b(z) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.1}$$

Here we have set the elements

$$\begin{aligned} b(z) &= \frac{q(z-1)}{q^2 z - 1}, \; c(z) = \frac{q^2-1}{q^2 z - 1}, \\ d(z) &= \frac{q^2(z-1)(qz+1)}{(q^2z-1)(q^3z+1)}, \; e(z) = \frac{q^{\frac{1}{2}}(z-1)(q^2-1)}{(q^2z-1)(q^3z+1)}, \\ f(z) &= \frac{(q^2-1)\{(q^3+q)z-(q-1)\}z}{(q^2z-1)(q^3z+1)}, \; n(z) = \frac{(q^2-1)\{(q^3-q^2)z+(q^2+1)\}}{(q^2z-1)(q^3z+1)}, \\ j(z) &= \frac{q^4z^2+(q^5-q^4-q^3+q^2+q-1)z-q}{(q^2z-1)(q^3z+1)}, \end{aligned}$$

and the normalizing function

$$\kappa(z) = z\frac{(q^6z;q^6)_\infty (q^2/z;q^6)_\infty (-q^5z;q^6)_\infty (-q^3/z;q^6)_\infty}{(q^6/z;q^6)_\infty (q^2z;q^6)_\infty (-q^5/z;q^6)_\infty (-q^3z;q^6)_\infty}. \tag{2.2}$$

Here we have used the abbreviation

$$(z;p)_\infty = \prod_{m=0}^{\infty} (1-p^m z).$$

The matrix elements of $R(z)$ are given by $R(z)v_{j_1} \otimes v_{j_2} = \sum_{k_1,k_2=\pm,0} v_{k_1} \otimes v_{k_2} R(z)_{k_1,k_2}^{j_1,j_2}$, where the ordering of the index is given by $v_+ \otimes v_+, v_+ \otimes v_0, v_+ \otimes v_-,\; v_0 \otimes v_+, v_0 \otimes v_0, v_0 \otimes v_-,$

$v_-\otimes v_+, v_-\otimes v_0, v_-\otimes v_-$. The R-matrix $R(z)$ satisfies the Yang-Baxter equation

$$R_{1,2}(z_1/z_2)R_{1,3}(z_1/z_3)R_{2,3}(z_2/z_3)=R_{2,3}(z_2/z_3)R_{1,3}(z_1/z_3)R_{1,2}(z_1/z_2), \tag{2.3}$$

the unitarity

$$R_{1,2}(z_1/z_2)R_{2,1}(z_2/z_1)=id, \tag{2.4}$$

and the crossing symmetry

$$R(z)_{k_1,k_2}^{j_1,j_2}=q^{\frac{j_2-k_2}{2}}R(-q^{-3}z^{-1})_{-j_2,k_1}^{-k_2,j_1}. \tag{2.5}$$

We have set the normalizing function $\kappa(z)$ such that the minimal eigenvalue of the corner transfer matrix becomes 1 [11, 12].

We introduce K-matrix $K(z)\in\mathrm{End}(V)$ representing an interaction at the boundary, which satisfies the boundary Yang-Baxter equation

$$K_2(z_2)R_{2,1}(z_1z_2)K_1(z_1)R_{1,2}(z_1/z_2)=R_{2,1}(z_1/z_2)K_1(z_1)R_{1,2}(z_1z_2)K_2(z_2). \tag{2.6}$$

We consider only the diagonal solutions $K(z)=K_\epsilon(z), (\epsilon=\pm,0)$ [5, 6].

$$K_0(z)=\frac{\varphi_0(z)}{\varphi_0(z^{-1})}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\qquad K_\pm(z)=\frac{\varphi_\pm(z)}{\varphi_\pm(z^{-1})}\begin{pmatrix}z^2&0&0\\0&\frac{\pm\sqrt{-1}q^{\frac{3}{2}}+z}{\pm\sqrt{-1}q^{\frac{3}{2}}+z^{-1}}&0\\0&0&1\end{pmatrix}. \tag{2.7}$$

Here we have set the normalizing function

$$\varphi_\epsilon(z)=\frac{(q^8z;q^{12})_\infty(-q^9z^2;q^{12})_\infty}{(q^{12}z;q^{12})_\infty(-q^5z^2;q^{12})_\infty}\times\begin{cases}1 & (\epsilon=0)\\ \frac{(\pm\sqrt{-1}q^{\frac{9}{2}}z;q^6)_\infty(\mp\sqrt{-1}q^{\frac{7}{2}}z;q^6)_\infty}{(\pm\sqrt{-1}q^{\frac{1}{2}}z;q^6)_\infty(\mp\sqrt{-1}q^{\frac{3}{2}}z;q^6)_\infty} & (\epsilon=\pm)\end{cases}. \tag{2.8}$$

The matrix elements of $K(z)$ are defined by $K(z)v_j=\sum_{k=\pm,0}v_kK(z)_k^j$, where the ordering of the index is given by v_+,v_0,v_-. The K-matrix $K(z)\in\mathrm{End}(V)$ satisfies the boundary unitarity

$$K(z)K(z^{-1})=id, \tag{2.9}$$

and the boundary crossing symmetry

$$K(z)_{k_1}^{k_2}=\sum_{j,j_2=\pm,0}q^{\frac{1}{2}(k_1-j_1)}R(-q^{-3}z^{-2})_{j_2,-j_1}^{-k_1,k_2}K(-q^{-3}z^{-1})_{j_1}^{j_2}. \tag{2.10}$$

To put it another way we have chosen the normalizing function $\varphi_\epsilon(z)$ such that the transfer matrix $T_\epsilon(z)$ (2.16) acts on the ground state as 1 (2.24).

2.2 Physical picture

In this section, following [3, 2], we introduce the transfer matrix $T_\epsilon(z)$ that is generating function of the Hamiltonian H_ϵ of our problem. In order to consider physical problem, it is convenient to introduce graphical interpretation of R-matrix and K-matrix. We present the R-matrix $R(z)_{k_1,k_2}^{j_1,j_2}$ in Fig.1.

$$R(z_1/z_2)_{k_1,k_2}^{j_1,j_2} =$$

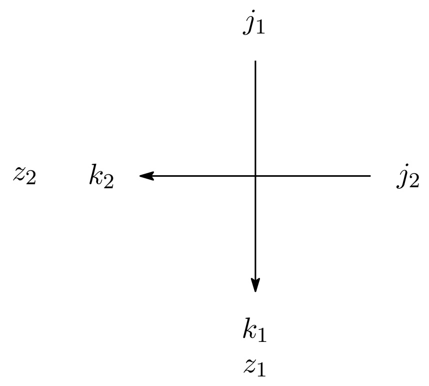

Fig.1: R-matrix

We present the K-matrix $K(z)_k^j$ in Fig.2.

$$K(z)_k^j =$$

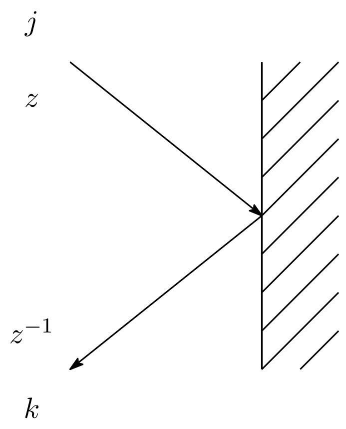

Fig.2: K-matrix

Let us consider the half-infinite spin chain $\cdots \otimes V \otimes V \otimes V$. Let us introduce the subspace $\mathcal{H}$ of the half-infinite spin chain by

$$\mathcal{H} = Span\{\cdots \otimes v_{p(N)} \otimes \cdots \otimes v_{p(2)} \otimes v_{p(1)} | p(s) = 0 \quad (s >> 0)\}. \tag{2.11}$$

We introduce the vertex operator $\Phi_j(z),(j=\pm,0)$ acting on the space $\mathcal{H}$ by Fig.3. To put it another way, the vertex operator $\Phi_j(z)$ is infinite-size matrix whose matrix elements are given by products of the R-matrix $R(z)_{k_1,k_2}^{j_1,j_2}$

$$(\Phi_j(z))_{\cdots p(N)\ \cdots p(2)\ p(1)}^{\cdots p(N)'\cdots p(2)'p(1)'}=\lim_{N\to\infty}\sum_{\nu(1),\nu(2),\cdots,\nu(N)=\pm,0}\prod_{j=1}^{N}R(z)_{\nu(j-1),p(j)}^{\nu(j)p(j)'}, \tag{2.12}$$

where $j=\nu(0)$. In order to avoid divergence we restrict our consideration to the subspace $\mathcal{H}$. We introduce the dual vertex operator $\Phi_j^*(z^{-1}),(j=\pm,0)$ acting on the space $\mathcal{H}$ by Fig.4.

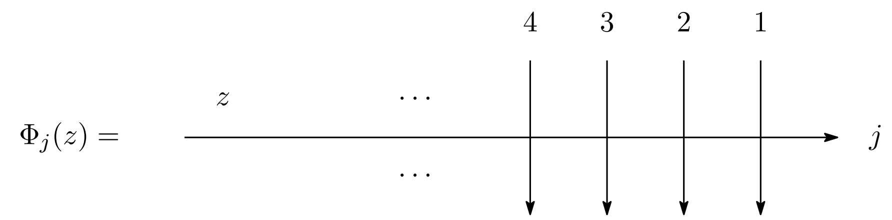

Fig.3: Vertex operator

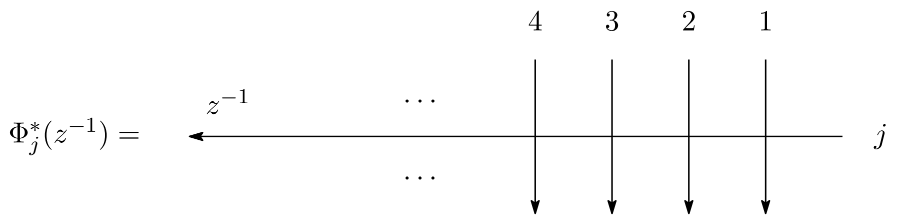

Fig.4: Dual vertex operator

From the Yang-Baxter equation, we have the commutation relation of the vertex operator $\Phi_j(z),(j=\pm,0)$

$$\Phi_{j_2}(z_2)\Phi_{j_1}(z_1)=\sum_{k_1,k_2=\pm,0}R(z_1/z_2)_{j_1,j_2}^{k_1,k_2}\Phi_{k_1}(z_1)\Phi_{k_2}(z_2). \tag{2.13}$$

From the unitarity and the crossing symmetry, we have the inversion relations

$$g\Phi_{j_1}(z)\Phi_{j_2}^*(z)=\delta_{j_1,j_2}id,\quad g\sum_{j=\pm,0}\Phi_j^*(z)\Phi_j(z)=id, \tag{2.14}$$

where we have set

$$g=\frac{1}{1+q}\frac{(q^2;q^6)_\infty(-q^3;q^6)_\infty}{(q^6;q^6)_\infty(-q^5;q^6)_\infty}.$$

From the crossing symmetry, we have

$$\Phi_j^*(z) = q^{-\frac{j}{2}}\Phi_{-j}(-q^{-3}z). \tag{2.15}$$

We introduce the transfer matrix $\overline{T}_\epsilon(z)$ acting on the space $\mathcal{H}$ by Fig.5. To put it another way we introduce the transfer matrix $\overline{T}_\epsilon(z)$ by product of the vertex operators. We define the "renormalized" transfer matrix $T_\epsilon(z)$ by

$$T_\epsilon(z) = g\overline{T}_\epsilon(z) = g\sum_{j,k=\pm,0}\Phi_j^*(z^{-1})K_\epsilon(z)_j^k\Phi_k(z), \quad (\epsilon = \pm, 0). \tag{2.16}$$

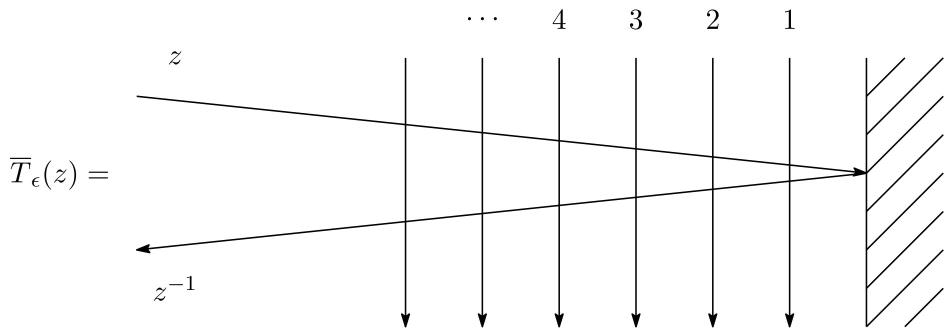

Fig.5: Boundary transfer matrix

From the commutation relations of the vertex operators $\Phi_j(z), \Phi_j^*(z)$ and the boundary Yang-Baxter equation (2.6), we have the commutativity relation of the transfer matrix $T_\epsilon(z)$, $(\epsilon = \pm, 0)$.

$$[T_\epsilon(z_1), T_\epsilon(z_2)] = 0, \quad \text{(for any } z_1, z_2). \tag{2.17}$$

The commutativity of the transfer matrix ensures that, if the transfer matrices $T_\epsilon(z)$ are diagonalizable, the transfer matrices $T_\epsilon(z)$ are diagonalized by the basis that is independent of the spectral parameter z. From the unitarity and the crossing symmetry, we have

$$T_\epsilon(z)T_\epsilon(z^{-1}) = id, \quad T_\epsilon(-q^{-3}z^{-1}) = T_\epsilon(z). \tag{2.18}$$

The Hamiltonian H_{IK} of the boundary Izergin-Korepin model is obtained by

$$\frac{-1}{4(q-q^{-1})(q^{\frac{3}{2}}+q^{-\frac{3}{2}})}\, H_{IK} = \frac{d}{dz}T_\epsilon(z)\bigg|_{z=1} + const. \tag{2.19}$$

The Hamiltonian H_{IK} is written as

$$H_{IK} = H_1^b + \sum_{j=1}^{\infty} H_{j+1,j}, \tag{2.20}$$

where we have set $H^b \in \mathrm{End}(V)$ and $H \in \mathrm{End}(V \otimes V)$ by

$$H^b = \begin{cases} 0, & (\epsilon = 0) \\ 4(q - q^{-1})\{-(q^{\frac{3}{2}} + q^{-\frac{3}{2}})\lambda_3 - \frac{\sqrt{3}}{3}(q^{\frac{3}{2}} - q^{-\frac{3}{2}} \pm 2\sqrt{-1})\lambda_8\}, & (\epsilon = \pm) \end{cases}, \tag{2.21}$$

$$\begin{aligned} H &= (q^{\frac{1}{2}} + q^{-\frac{1}{2}})(q^2 + q^{-2})(\lambda_1 \otimes \lambda_1 + \lambda_2 \otimes \lambda_2) \\ &+ (q^{\frac{1}{2}} + q^{-\frac{1}{2}})(q^2 - q^{-2})\sqrt{-1}(-\lambda_1 \otimes \lambda_2 + \lambda_2 \otimes \lambda_1) \\ &+ 2(q^{\frac{1}{2}} + q^{-\frac{1}{2}})\lambda_3 \otimes \lambda_3 \\ &+ (q^{\frac{3}{2}} + q^{-\frac{3}{2}})(q + q^{-1})(\lambda_4 \otimes \lambda_4 + \lambda_5 \otimes \lambda_5 + \lambda_6 \otimes \lambda_6 + \lambda_7 \otimes \lambda_7) \\ &+ (q^{\frac{3}{2}} + q^{-\frac{3}{2}})(q - q^{-1})\sqrt{-1}(\lambda_4 \otimes \lambda_5 - \lambda_5 \otimes \lambda_4 + \lambda_6 \otimes \lambda_7 - \lambda_7 \otimes \lambda_6) \\ &+ (q - q^{-1})^2(\lambda_4 \otimes \lambda_6 + \lambda_6 \otimes \lambda_4 - \lambda_5 \otimes \lambda_7 - \lambda_7 \otimes \lambda_5) \\ &+ (q^2 - q^{-2})\sqrt{-1}(-\lambda_4 \otimes \lambda_7 + \lambda_7 \otimes \lambda_4 - \lambda_5 \otimes \lambda_6 + \lambda_6 \otimes \lambda_5) \\ &+ \frac{2}{3}(-(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) + 2(q^{\frac{3}{2}} + q^{-\frac{3}{2}}) + 2(q^{\frac{5}{2}} + q^{-\frac{5}{2}}))\lambda_8 \otimes \lambda_8 \\ &+ \frac{1}{3\sqrt{3}}(-(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) + 2(q^{\frac{3}{2}} + q^{-\frac{3}{2}}) - (q^{\frac{5}{2}} + q^{-\frac{5}{2}}))(\lambda_8 \otimes id + id \otimes \lambda_8). \end{aligned} \tag{2.22}$$

Here we have used Gell-Mann matrices $\lambda_1, \lambda_2, \cdots, \lambda_8$ satisfying $\mathrm{Tr}(\lambda_j \lambda_k) = 2\delta_{j,k}$.

$$\lambda_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \lambda_2 = \begin{pmatrix} 0 & 0 & \sqrt{-1} \\ 0 & 0 & 0 \\ -\sqrt{-1} & 0 & 0 \end{pmatrix}, \ \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda_5 = \begin{pmatrix} 0 & -\sqrt{-1} & 0 \\ \sqrt{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{-1} \\ 0 & -\sqrt{-1} & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The diagonalization of the Hamiltonian H_{IK} is reduced to those of the transfer matrix $T_\epsilon(z)$. Let us consider the eigenvector problem

$$T_\epsilon(z)|v\rangle = t_\epsilon(z)|v\rangle. \tag{2.23}$$

We have chosen the normalizing function $\varphi_\epsilon(z)$ (2.8) such that the ground state $|B\rangle_\epsilon$ satisfies

$$T_\epsilon(z)|B\rangle_\epsilon = |B\rangle_\epsilon, \quad (\epsilon = \pm, 0). \tag{2.24}$$

We would like to construct the ground state $|B\rangle_\epsilon$ and would like to diagonalize the Hamiltonian H_{IK}.

3 Free field realization

In this section we give mathematical formulation of our problem. We construct the free field realization of the ground state $|B\rangle_\epsilon$, and give the diagonalization of the transfer matrix $T_\epsilon(z)$.

3.1 Mathematical picture

In order to diagonalize the transfer matrix $T_\epsilon(z)$, we follows the strategy proposed in [1, 2]. The corner transfer matrix method [11, 12] suggests that we identify $\mathcal{H}$ with the integrable highest-weight representation $V(\Lambda_1)$ of the quantum group $U_q(A_2^{(2)})$, because the characters of $\mathcal{H}$ and $V(\Lambda_1)$ coincide. We note that the representation $V(\Lambda_1)$ is the only one level-1 integrable highest-weight representation of $U_q(A_2^{(2)})$. We identify the line $\Phi_j(z)$ in Fig.3 with the components $\widetilde{\Phi}_j(z)$ of the vertex operator $\widetilde{\Phi}(z)$, and the line $\Phi_j^*(z)$ in Fig.4 with the components $\widetilde{\Phi}_j^*(z)$ of the dual vertex operator $\widetilde{\Phi}^*(z)$.

$$\widetilde{\Phi}(z) : V(\Lambda_1) \to V(\Lambda_1) \otimes V_z, \quad \widetilde{\Phi}(z) = \sum_{j=\pm,0} \widetilde{\Phi}_j(z) \otimes v_j, \tag{3.1}$$

$$\widetilde{\Phi}^*(z) : V(\Lambda_1) \otimes V_z \to V(\Lambda_1), \quad \widetilde{\Phi}_j^*(z)|v\rangle = \widetilde{\Phi}^*(z)(|v\rangle \otimes v_j). \tag{3.2}$$

Here V_z is the evaluation representation. The vertex operator $\widetilde{\Phi}_j(z)$ for $U_q(A_2^{(2)})$ satisfies exactly the same functional relations as those of $\Phi_j(z)$.

$$\widetilde{\Phi}_{j_2}(z_2)\widetilde{\Phi}_{j_1}(z_1) = \sum_{k_1,k_2=\pm,0} R(z_1/z_2)_{j_1,j_2}^{k_1,k_2} \widetilde{\Phi}_{k_1}(z_1)\widetilde{\Phi}_{k_2}(z_2), \tag{3.3}$$

$$g\widetilde{\Phi}_{j_1}(z)\widetilde{\Phi}_{j_2}^*(z) = \delta_{j_1,j_2} id, \quad \widetilde{\Phi}_j^*(z) = q^{-\frac{j}{2}}\widetilde{\Phi}_{-j}(-q^{-3}z). \tag{3.4}$$

Let us set the transfer matrix $\widetilde{T}_\epsilon(z), (\epsilon = \pm, 0)$ by

$$\widetilde{T}_\epsilon(z) = g \sum_{j,k=\pm,0} \widetilde{\Phi}_j^*(z^{-1}) K_\epsilon(z)_j^k \widetilde{\Phi}_k(z), \quad (\epsilon = \pm, 0). \tag{3.5}$$

We have the commutativity of the transfer matrix, the unitarity and the crossing symmetry.

$$[\widetilde{T}_\epsilon(z_1), \widetilde{T}_\epsilon(z_2)] = 0, \quad (\text{for any } z_1, z_2), \tag{3.6}$$

$$\widetilde{T}_\epsilon(z)\widetilde{T}_\epsilon(z^{-1}) = id, \quad \widetilde{T}_\epsilon(-q^{-3}z^{-1}) = \widetilde{T}_\epsilon(z). \tag{3.7}$$

Let us set the ground state $|\widetilde{B}\rangle_\epsilon \in V(\Lambda_1)$ by

$$\widetilde{T}_\epsilon(z)|\widetilde{B}\rangle_\epsilon = |\widetilde{B}\rangle_\epsilon, \quad (\epsilon = \pm, 0). \tag{3.8}$$

Following the strategy proposed in [1, 2], we consider our problem upon the following identification.

$$\mathcal{H} = V(\Lambda_1), \quad \Phi_j(z) = \widetilde{\Phi}_j(z), \quad \Phi_j^*(z) = \widetilde{\Phi}_j^*(z), \quad T_\epsilon(z) = \widetilde{T}_\epsilon(z), \quad |B\rangle_\epsilon = |\widetilde{B}\rangle_\epsilon. \tag{3.9}$$

In order to study the excitations we introduce type-II vertex operator $\widetilde{\Psi}_\mu^*(z), (\mu = \pm, 0)$.

$$\widetilde{\Psi}^*(z) : V_z \otimes V(\Lambda_1) \to V(\Lambda_1), \quad \widetilde{\Psi}_\mu^*(z)|v\rangle = \widetilde{\Psi}^*(z)(v_\mu \otimes |v\rangle). \tag{3.10}$$

Type-II vertex operator $\widetilde{\Psi}_\mu^*(\xi)$ satisfies

$$\widetilde{\Phi}_j(z)\widetilde{\Psi}_\mu^*(\xi) = \tau(z/\xi)\widetilde{\Psi}_\mu^*(\xi)\widetilde{\Phi}_j(z), \quad (j, \mu = \pm, 0). \tag{3.11}$$

Here we have set

$$\tau(z) = z^{-1}\frac{\Theta_{q^6}(q^5 z)\Theta_{q^6}(-q^4 z)}{\Theta_{q^6}(q^5 z^{-1})\Theta_{q^6}(-q^4 z^{-1})}, \quad \Theta_p(z) = (p;p)_\infty(z;p)_\infty(pz^{-1};p)_\infty. \tag{3.12}$$

Let us set the vectors $|\xi_1, \xi_2, \cdots, \xi_N\rangle_{\mu_1,\mu_2,\cdots,\mu_N,\epsilon} \in V(\Lambda_1)$, $(\mu_1, \mu_2, \cdots, \mu_N, \epsilon = \pm, 0)$ by

$$|\xi_1, \xi_2, \cdots, \xi_N\rangle_{\mu_1,\mu_2,\cdots,\mu_N,\epsilon} = \widetilde{\Psi}_{\mu_1}^*(\xi_1)\widetilde{\Psi}_{\mu_2}^*(\xi_2)\cdots\widetilde{\Psi}_{\mu_N}^*(\xi_N)|B\rangle_\epsilon. \tag{3.13}$$

From the commutation relation (3.11) we have

$$\begin{aligned} & \widetilde{T}_\epsilon(z)|\xi_1, \xi_2, \cdots, \xi_N\rangle_{\mu_1,\mu_2,\cdots,\mu_N,\epsilon} \\ = & \prod_{j=1}^N \tau(z/\xi_j)\tau(-1/q^3 z\xi_j)\ |\xi_1, \xi_2, \cdots, \xi_N\rangle_{\mu_1,\mu_2,\cdots,\mu_N,\epsilon}. \end{aligned} \tag{3.14}$$

Comparing with the Bethe ansatz calculation [13] we conclude that the vectors $|\xi_1, \xi_2, \cdots, \xi_N\rangle_{\mu_1,\mu_2,\cdots,\mu_N,\epsilon}$ are the basis of the space of state of the Izergin-Korepin model. In order to construct the ground state $|B\rangle_\epsilon \in V(\Lambda_1)$, it is convenient to introduce the free field realization.

3.2 Vertex operator

In this section we give the free field realization of the vertex operator $\widetilde{\Phi}_j(z), (j = \pm, 0)$ [14, 16, 17]. Let us introduce the bosons $a_m, (m \in \mathbb{Z}_{\neq 0})$ as following [14, 15, 16, 17].

$$[a_m, a_n] = \delta_{m+n}\frac{[m]_q}{m}([2m]_q - (-1)^m[m]_q). \tag{3.15}$$

Here we have used q-integer

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{3.16}$$

Let us set the zero-mode operators P, Q by

$$[a_m, P] = [a_m, Q] = 0, \ \ [P, Q] = 1. \tag{3.17}$$

The integrable highest weight representation $V(\Lambda_1)$ of $U_q(A_2^{(2)})$ is realized by

$$V(\Lambda_1) = \mathbb{C}[a_{-1}, a_{-2}, \cdots] \oplus_{n \in \mathbb{Z}} e^{nQ}|\Lambda_1\rangle, \quad |\Lambda_1\rangle = e^{\frac{Q}{2}}|0\rangle. \tag{3.18}$$

The vacuum vector $|0\rangle$ is characterized by

$$a_m|0\rangle = 0, \ (m > 0), \quad P|0\rangle = 0. \tag{3.19}$$

We give the free field realizations of the vertex operators. Let us set $\epsilon(q) = ([2]_{q^{\frac{1}{2}}})^{\frac{1}{2}}$. The highest elements of the vertex operators are given by

$$\widetilde{\Phi}_-(z) = \frac{1}{\epsilon(q)} e^{P(z)} e^{Q(z)} e^{Q} (-zq^4)^{P+\frac{1}{2}}, \tag{3.20}$$

$$\widetilde{\Psi}^*_+(z) = \frac{1}{\epsilon(q)} e^{P^*(z)} e^{Q^*(z)} e^{-Q} (-qz)^{-P+\frac{1}{2}}. \tag{3.21}$$

Here we have set the auxiliary operators $P(z), Q(z), P^*(z), Q^*(z)$ by

$$P(z) = \sum_{m>0} \frac{a_{-m}}{[2m]_q - (-1)^m [m]_q} q^{\frac{9m}{2}} z^m, \tag{3.22}$$

$$Q(z) = -\sum_{m>0} \frac{a_m}{[2m]_q - (-1)^m [m]_q} q^{-\frac{7m}{2}} z^{-m}, \tag{3.23}$$

$$P^*(z) = -\sum_{m>0} \frac{a_{-m}}{[2m]_q - (-1)^m [m]_q} q^{\frac{m}{2}} z^m, \tag{3.24}$$

$$Q^*(z) = \sum_{m>0} \frac{a_m}{[2m]_q - (-1)^m [m]_q} q^{-\frac{3m}{2}} z^{-m}. \tag{3.25}$$

The other elements of the vertex operators are given by the intertwining relations (3.1) and (3.10).

$$\widetilde{\Phi}_0(z) = \widetilde{\Phi}_-(z) x_0^- - q x_0^- \widetilde{\Phi}_-(z), \tag{3.26}$$

$$\widetilde{\Phi}_+(z) = q^{\frac{1}{2}} (\widetilde{\Phi}_0(z) x_0^- - x_0^- \widetilde{\Phi}_0(z)), \tag{3.27}$$

$$\widetilde{\Psi}^*_0(z) = x_0^+ \widetilde{\Psi}^*_+(z) - q^{-1} \widetilde{\Psi}^*_+(z) x_0^+, \tag{3.28}$$

$$\widetilde{\Psi}^*_-(z) = q^{-\frac{1}{2}} (x_0^+ \widetilde{\Psi}^*_0(z) - \widetilde{\Psi}^*_0(z) x_0^+). \tag{3.29}$$

The elements $x_m^\pm, (m \in \mathbb{Z})$ are given by integral of the currents $x^\pm(w) = \sum_{m\in\mathbb{Z}} x_m^\pm w^{-m}$,

$$x_m^\pm = \oint \frac{dw}{2\pi\sqrt{-1}} w^{m-1} x^\pm(w), \quad (m \in \mathbb{Z}). \tag{3.30}$$

Here the free field realizations of the currents $x^\pm(w)$ are given by

$$x^\pm(w) = \epsilon(q) e^{R^\pm(w)} e^{S^\pm(w)} e^{\pm Q} w^{\pm P + \frac{1}{2}}, \tag{3.31}$$

where we have set the auxiliary operators $R^\pm(w), S^\pm(w)$ by

$$R^\pm(w) = \pm \sum_{m>0} \frac{a_{-m}}{[m]_q} q^{\mp\frac{m}{2}} w^m, \tag{3.32}$$

$$S^\pm(w) = \mp \sum_{m>0} \frac{a_m}{[m]_q} q^{\mp\frac{m}{2}} w^{-m}. \tag{3.33}$$

3.3 Ground state

In this section we give the free field realization of the ground state $|B\rangle_\epsilon$ which satisfies

$$T_\epsilon(z)|B\rangle_\epsilon = |B\rangle_\epsilon.$$

Theorem 3.1 *The free field realizations of the ground states $|B\rangle_\epsilon$ are given by*

$$|B\rangle_\epsilon = e^{F(\epsilon)}|\Lambda_1\rangle, \quad (\epsilon = \pm, 0). \tag{3.34}$$

Here we have set

$$\begin{aligned} F(\epsilon) &= -\frac{1}{2}\sum_{m>0} \frac{mq^{8m}}{[2m]_q - (-1)^m [m]_q} a_{-m}^2 \\ &+ \sum_{m>0} \left\{ \theta_m \left(\frac{(q^{\frac{m}{2}} - q^{-\frac{m}{2}} - (\sqrt{-1})^m) q^{4m}}{[2m]_q - (-1)^m [m]_q} \right) - \frac{(\epsilon\sqrt{-1})^m q^{3m}}{[2m]_q - (-1)^m [m]_q} \right\} a_{-m} - Q, \end{aligned} \tag{3.35}$$

where

$$\theta_m(x) = \begin{cases} x, & m : \text{even} \\ 0, & m : \text{odd} \end{cases}.$$

Multiplying the type-II vertex operators $\widetilde{\Psi}_\mu^*(\xi)$ to the ground state $|B\rangle_\epsilon$, we get the diagonalization of the transfer matrix $T_\epsilon(z)$ on the space of state.

Proof. Let us multiply the vertex operator $\Phi_j(z)$ to the relation $T_\epsilon(z)|B\rangle_\epsilon = |B\rangle_\epsilon$ from the left and use the inversion relation (2.14). Then we have

$$K_\epsilon(z)_j^j \Phi_j(z)|B\rangle_\epsilon = \Phi_j(z^{-1})|B\rangle_\epsilon, \quad (j, \epsilon = \pm, 0). \tag{3.36}$$

We would like to calculate the action of the vertex operator $\Phi_j(z)$ on the vector $|B\rangle_\epsilon$. Using the normal orderings

$$\Phi_-(z)x^-(x^4w) = :\Phi_-(z)x^-(x^4w): \frac{-1}{z(1-qw/z)}, \quad (|qw|<|z|), \tag{3.37}$$

$$x^-(x^4w)\Phi_-(z) = :x^-(x^4w)\Phi_-(z): \frac{1}{w(1-qz/w)}, \quad (|qz|<|w|), \tag{3.38}$$

$$x^-(w_1)x^-(w_2) = :x^-(w_1)x^-(w_2): \frac{w_1(1-q^2w_2/w_1)(1-w_2/w_1)}{(1+qw_2/w_1)}, \ (|w_2|<|w_1|), \tag{3.39}$$

we have following realizations of the vertex operators $\Phi_j(z)$.

$$\Phi_-(z) = \frac{1}{\epsilon(q)}e^{P(z)}e^{Q(z)}e^{Q}(-zq^4)^{P+\frac{1}{2}}, \tag{3.40}$$

$$\Phi_0(z) = \oint_{C_1}\frac{dw}{2\pi\sqrt{-1}w}\frac{(q^2-1)}{q^4z(1-qw/z)(1-qz/w)}:\Phi_-(z)x^-(q^4w):, \tag{3.41}$$

$$\begin{aligned}\Phi_+(z) &= \oint\oint_{C_2}\frac{dw_1}{2\pi\sqrt{-1}w_1}\frac{dw_2}{2\pi\sqrt{-1}w_2}q^{-\frac{5}{2}}(1-q^2)^2\\ &\times \frac{(1-w_1/w_2)(1-w_2/w_1)}{(1+qw_1/w_2)(1+qw_2/w_1)}\frac{z^{-2}\{(q+q^{-1})z-(w_1+w_2)\}}{\prod_{j=1}^{2}(1-qw_j/z)(1-qz/w_j)}\\ &\times :\Phi_-(z)x^-(q^4w_1)x^-(q^4w_2):.\end{aligned} \tag{3.42}$$

The integration contour C_1 encircles $w=0,qz$ but not $w=q^{-1}z$. The integration contour C_2 encircles $w_1=0,qz,qw_2$ but not $w_1=q^{-1}z,q^{-1}w_2$, and encircles $w_2=0,qz,qw_1$ but not $w_2=q^{-1}z,q^{-1}w_1$.

The actions of the basic operators $e^{S^-(w)}, e^{Q(z)}$ on the vector $|B\rangle_\epsilon$ have the following formulae.

$$e^{Q(z)}|B\rangle_\epsilon = \varphi_\epsilon(z^{-1})e^{P(z^{-1})}|B\rangle_\epsilon, \tag{3.43}$$

$$e^{S^-(q^4w)}|B\rangle_\epsilon = g_\epsilon(w)e^{R^-(q^4/w)}|B\rangle_\epsilon, \tag{3.44}$$

where $\varphi_\epsilon(z)$ is given in (2.8) and $g_\epsilon(w)$ is given by

$$g_\epsilon(w) = \left\{\begin{array}{cc}(1-w^{-2}), & (\epsilon=0)\\ (1-w^{-2})(1\mp\sqrt{-1}q^{-\frac{1}{2}}w^{-1}), & (\epsilon=\pm)\end{array}\right. . \tag{3.45}$$

We have the action of the vertex operators $\Phi_j(z)$ on the vector $|B\rangle_\epsilon$ as following.

$$\Phi_-(z)|B\rangle_\epsilon = \frac{1}{\epsilon(q)}\varphi_\epsilon(z^{-1})e^{P(z)+P(z^{-1})}e^{Q}e^{F(\epsilon)}|\Lambda_1\rangle, \tag{3.46}$$

$$\Phi_0(z)|B\rangle_\epsilon = (1-q^2)\varphi_\epsilon(z^{-1})\oint_{\widetilde{C}_1}\frac{dw}{2\pi\sqrt{-1}w}\frac{z^{-1}wg_\epsilon(w)}{(1-qw/z)(1-qz/w)(1-q/zw)}$$

$$\begin{aligned}
& \times \quad e^{P(z)+P(z^{-1})+R^-(q^4w)+R^-(q^4/w)}e^{F(\epsilon)}|\Lambda_1\rangle, \qquad (3.47)\\
\Phi_+(z)|B\rangle_\epsilon \quad = \quad & (1-q^2)^2q^{-\frac{5}{2}}\epsilon(q)\varphi_\epsilon(z^{-1})\oint\oint_{\widetilde{C}_2}\frac{dw_1}{2\pi\sqrt{-1}w_1}\frac{dw_2}{2\pi\sqrt{-1}w_2}\\
& \times \quad \frac{(1-w_1/w_2)(1-w_2/w_1)(1-1/w_1w_2)(1-q^2/w_1w_2)}{(1+qw_1/w_2)(1+qw_2/w_1)(1+q/w_1w_2)}\prod_{j=1}^2 w_jg_\epsilon(w_j)\\
& \times \quad \frac{z^{-2}\{(q+q^{-1})z-(w_1+w_2)\}}{\prod_{j=1}^2(1-qw_j/z)(1-qz/w_j)(1-q/zw_j)} \qquad (3.48)\\
& \times \quad e^{P(z)+P(z^{-1})+R^-(q^4w_1)+R^-(q^4/w_1)+R^-(q^4w_2)+R^-(q^4/w_2)}e^{-Q}e^{F(\epsilon)}|\Lambda_1\rangle.
\end{aligned}$$

The integration contour $\widetilde{C}_1$ encircles $w=0,qz,qz^{-1}$ but not $w=q^{-1}z$. The integration contour $\widetilde{C}_2$ encircles $w_1=0,qz,qz^{-1},qw_2,qw_2^{-1}$ but not $w_1=q^{-1}z,q^{-1}w_2$, and encircles $w_2=0,qz,qz^{-1},qw_1,qw_1^{-1}$ but not $w_2=q^{-1}z,q^{-1}w_1$.

For simplicity we summarize the case $\epsilon=\pm$. The relation (3.36) is equivalent to the following three relations.

$$\varphi_\pm(z)\Phi_-(z)|B\rangle_\pm = \varphi_\pm(z^{-1})\Phi_-(z^{-1})|B\rangle_\pm, \qquad (3.49)$$

$$\varphi_\pm(z)(1\mp\sqrt{-1}q^{-\frac{3}{2}}z)\Phi_0(z)|B\rangle_\pm = \varphi_\pm(z^{-1})(1\mp\sqrt{-1}q^{-\frac{3}{2}}z^{-1})\Phi_0(z^{-1})|B\rangle_\pm, \qquad (3.50)$$

$$\varphi_\pm(z)z\Phi_+(z)|B\rangle_\pm = \varphi_\pm(z^{-1})z^{-1}\Phi_+(z^{-1})|B\rangle_\pm, \qquad (3.51)$$

- The relation (3.49). Using formula (3.46), we have

$$(LHS)=\frac{1}{\epsilon(q)}\varphi_\pm(z)\varphi_\pm(z^{-1})e^{P(z)+P(z^{-1})}e^{Q}e^{F(\epsilon)}|\Lambda_1\rangle=(RHS). \qquad (3.52)$$

- The relation (3.50). Using formula (3.47), we have

$$\begin{aligned}
&(LHS)-(RHS)=(1-q^2)\varphi_\pm(z)\varphi_\pm(z^{-1})(z^{-1}-z) \qquad (3.53)\\
\times \quad &\oint_{\widehat{C}_1}\frac{dw}{2\pi\sqrt{-1}w}I_1(w)\frac{e^{P(z)+P(z^{-1})+R^-(q^4w)+R^-(q^4w^{-1})}}{(1-qw/z)(1-qz/w)(1-q/wz)(1-qwz)}e^{F(\epsilon)}|\Lambda_1\rangle,
\end{aligned}$$

where we have set $I_1(w)=(w-w^{-1})(1\mp\sqrt{-1}q^{-\frac{1}{2}}w)(1\mp\sqrt{-1}q^{-\frac{1}{2}}w^{-1})$. Here the integration contour $\widehat{C}_1$ encircles $w=0,qz,qz^{-1}$ but not $w=q^{-1}z,q^{-1}z^{-1}$. The integration contour $\widehat{C}_1$ and the integrand

$$\frac{e^{P(z)+P(z^{-1})+R^-(q^4w)+R^-(q^4w^{-1})}}{(1-qw/z)(1-qz/w)(1-q/wz)(1-qwz)}$$

are invariant under $w\to w^{-1}$. Hence the relation $I_1(w)+I_1(w^{-1})=0$ ensures $(LHS)-(RHS)=0$.

- The relation (3.51). Using formula (3.48), we have

$$(LHS)-(RHS) \quad = \quad q^{-\frac{5}{2}}(1-q^2)^2\epsilon(q)\varphi_\pm(z)\varphi_\pm(z^{-1})\oint\oint_{\widehat{C}_2}\frac{dw_1}{2\pi\sqrt{-1}w_1}\frac{dw_2}{2\pi\sqrt{-1}w_2}$$

$$
\begin{aligned}
&\times \quad \frac{(1-w_1/w_2)(1-w_2/w_1)(1-w_1w_2)(1-1/w_1w_2)}{(1+qw_1/w_2)(1+qw_2/w_1)(1+qw_1w_2)(1+q/w_1w_2)}\\
&\times \quad \frac{q^2(z-z^{-1})I_2(w_1,w_2)}{\displaystyle\prod_{j=1}^{2}(1-qz/w_j)(1-qw_j/z)(1-qzw_j)(1-q/zw_j)} \qquad (3.54)\\
&\times \quad e^{P(z)+P(z^{-1})+R^-(q^4w_1)+R^-(q^4/w_1)+R^-(q^4w_2)+R^-(q^4/w_2)}e^{-Q}e^{F(\epsilon)}|\Lambda_1\rangle,
\end{aligned}
$$

where we have set

$$
\begin{aligned}
I_2(w_1,w_2) &= \{-(q+q^{-1})(z+z^{-1})w_1w_2+(w_1+w_2)(w_1w_2+1)\}\\
&\times \frac{(1+qw_1w_2)}{(1-w_1w_2)}(1-q^2/w_1w_2)\prod_{j=1}^{2}(w_j-w_j^{-1})(1\mp\sqrt{-1}q^{-\frac{1}{2}}w_j^{-1}). \qquad (3.55)
\end{aligned}
$$

Here the integration contour $\widehat{C}_2$ encircles $w_1=0,qz,qz^{-1}$, qw_2,qw_2^{-1} but not $w_1=q^{-1}z,q^{-1}z^{-1}$, $q^{-1}w_2,q^{-1}w_2^{-1}$, and encircles $w_2=0,qz,qz^{-1}$, qw_1,qw_1^{-1} but not $w_2=q^{-1}z,q^{-1}z^{-1}$, $q^{-1}w_1,q^{-1}w_1^{-1}$. The integration contour $\widehat{C}_2$ and the integrand

$$
\begin{aligned}
&\frac{(1-w_1/w_2)(1-w_2/w_1)(1-w_1w_2)(1-1/w_1w_2)}{(1+qw_1/w_2)(1+qw_2/w_1)(1+qw_1w_2)(1+q/w_1w_2)}\\
\times\ &\frac{e^{P(z)+P(z^{-1})+R^-(q^4w_1)+R^-(q^4/w_1)+R^-(q^4w_2)+R^-(q^4/w_2)}}{\displaystyle\prod_{j=1}^{2}(1-qz/w_j)(1-qw_j/z)(1-qzw_j)(1-q/zw_j)}
\end{aligned}
$$

are invariant under $(w_1,w_2)\to(w_1^{-1},w_2),(w_1,w_2^{-1}),(w_1^{-1},w_2^{-1})$. Hence the relation

$$
I_2(w_1,w_2)+I_2(w_1^{-1},w_2)+I_2(w_1,w_2^{-1})+I_2(w_1^{-1},w_2^{-1})=0
$$

ensures $(LHS)-(RHS)=0$.
Q.E.D.

3.4 Dual ground state

In this section we give the free field realizations of the dual ground state ${}_\epsilon\langle B|\in V(\Lambda_1)^*,(\epsilon=\pm,0)$, which satisfies

$$
{}_\epsilon\langle B|T_\epsilon(z)={}_\epsilon\langle B|,\quad (\epsilon=\pm,0). \qquad (3.56)
$$

The dual integrable highest weight representation $V(\Lambda_1)^*$ of $U_q(A_2^{(2)})$ is realized by

$$
V(\Lambda_1)^*=\langle\Lambda_1|\oplus_{n\in\mathbb{Z}}e^{nQ}\mathbb{C}[a_1,a_2,\cdots],\quad \langle\Lambda_1|=\langle 0|e^{-\frac{Q}{2}}. \qquad (3.57)
$$

The vacuum vector $\langle 0|$ is characterized by

$$
\langle 0|a_{-m}=0,\ (m>0),\quad \langle 0|P=0. \qquad (3.58)
$$

Theorem 3.2 *The free field realizations of the dual ground states* ${}_{\epsilon}\langle B|$ *are given by*

$$ {}_{\epsilon}\langle B| = \langle \Lambda_1 | e^{G(\epsilon)}, \quad (\epsilon = \pm, 0). \tag{3.59} $$

Here we have set

$$ \begin{aligned} G(\epsilon) &= -\frac{1}{2}\sum_{m>0} \frac{mq^{-2m}}{[2m]_q - (-1)^m [m]_q} a_m^2 \\ &+ \sum_{m>0} \left\{ -\theta_m \left(\frac{(q^{\frac{m}{2}} - q^{-\frac{m}{2}} - (\sqrt{-1})^m) q^{-m}}{[2m]_q - (-1)^m [m]_q} \right) + \frac{(-\epsilon\sqrt{-1})^m q^{-2m}}{[2m]_q - (-1)^m [m]_q} \right\} a_m - Q. \end{aligned} \tag{3.60} $$

The proof of (3.56) is given as the same way as those of (2.24). The following relations are useful for proof.

$$ {}_{\epsilon}\langle B| e^{P(-q^{-3}z^{-1})} = \varphi_{\epsilon}(z^{-1}) \, {}_{\epsilon}\langle B| e^{Q(-q^{-3}z)}, \tag{3.61} $$

$$ {}_{\epsilon}\langle B| e^{R^{-}(qw)} = g_{\epsilon}^{*}(w) \, {}_{\epsilon}\langle B| e^{S^{-}(qw^{-1})}, \tag{3.62} $$

where $\varphi_{\epsilon}(z)$ is given in (2.8) and $g_{\epsilon}^{*}(w)$ is given by

$$ g_{\epsilon}^{*}(w) = \begin{cases} (1-w^2), & (\epsilon = 0) \\ (1-w^2)(1 \pm \sqrt{-1} q^{-\frac{1}{2}} w), & (\epsilon = \pm) \end{cases} . \tag{3.63} $$

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References

[1] M.Jimbo and T.Miwa, *Algebraic Analysis of Solvable Lattice Models*, (CBMS Regional Conference Series **85**, American Mathematical Society 1995).

[2] M.Jimbo, R.Kedem, T.Kojima, H.Konno and T.Miwa, *Nucl.Phys.***B441** 437 (1995).

[3] E.K.Sklyanin, *J.Phys.***A21** 2375 (1988).

[4] A.G.Izergin and V.E.Korepin, *Commun.Math.Phys.***79** 303 (1981).

[5] M.T.Batchelor, V.Fridkin, A.Kuniba and Y.K.Zhou, *Phys.Lett.***B376** 266 (1996).

[6] L.Mezincescu and R.I.Nepomechie, *Int.J.Mod.Phys.***A6** 5231 (1991).

[7] W.-L.Yang and Y.-Z.Zhang, *Nucl.Phys.*__B596__ 495 (2001).

[8] H.Furutsu and T.Kojima, *J.Math.Phys.*__41__ 4413 (2000).

[9] T.Miwa and R.Weston, *Nucl.Phys.*__B486__ 517 (1997).

[10] T.Kojima, *J.Math.Phys.*__52__ 01351 (2011).

[11] R.J.Baxter, *Exactly Solved Models in Statistical Mechanics*, (Academic Press, London, 1982).

[12] S.J.Kang, M.Kashiwara, K.Misra, T.Miwa, T.Nakashima and A.Nakayashiki, *Int.J.Mod.Phys.*__A7__ 449 (1992).

[13] V.I.Vichirko and N.Yu.Reshetikhin, *Theor.Math.Phys.*__56__ 805 (1983).

[14] Y.Matsuno, Thesis (Kyoto University) 1997.

[15] N.H.Jing, *Invent.Math.*__102__ 663 (1990).

[16] N.H.Jing and K.C.Misra, *Trans.Amer.Math.Soc.* **351** 1663 (1999).

[17] B.-Y.Hou, W.-L.Yang and Y.-Z.Zhang, *Nucl.Phys.* **B556** 485 (1999).