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WAKIMOTO REALIZATION OF THE ELLIPTIC QUANTUM GROUP $U_{q,p}(\widehat{\mathrm{sl}_N})$*

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We construct a free field realization of the elliptic quantum algebra $U_{q,p}(\widehat{\mathrm{sl}_N})$ for arbitrary level $k \neq 0, -N$. We study Drinfeld current and the screening current associated with $U_{q,p}(\widehat{\mathrm{sl}_N})$ for arbitrary level k. In the limit $p \to 0$ this realization becomes q-Wakimoto realization for $U_q(\widehat{\mathrm{sl}_N})$.

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1. Introduction

The elliptic quantum group has been proposed in papers.[1–5] There are two types of elliptic quantum groups, the vertex type $\mathcal{A}_{q,p}(\widehat{\mathrm{sl}_N})$ and the face type $\mathcal{B}_{q,\lambda}(\mathfrak{g})$, where $\mathfrak{g}$ is a Kac–Moody algebra associated with a symmetrizable Cartan matrix. Not only the quantum group but also the elliptic quantum groups have the structure of quasitriangular quasi-Hopf algebras introduced by V. Drinfeld.[6] H. Konno[7,8] introduced the elliptic quantum algebra $U_{q,p}(\widehat{\mathrm{sl}_2})$ as an algebra of the elliptic analogue of Drinfeld current in the context of the fusion SOS model.[11] M. Jimbo *et al.*,[12] continued to study the elliptic quantum algebra $U_{q,p}(\widehat{\mathrm{sl}_2})$. They identified $U_{q,p}(\widehat{\mathrm{sl}_2})$ with the tensor product of $\mathcal{B}_{q,\lambda}(\widehat{\mathrm{sl}_2})$ and a Heisenberg algebra $\mathcal{H}$. The elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathrm{sl}_2})$ is a quasi-Hopf algebra. The intertwining relation of the vertex operator of $\mathcal{B}_{q,\lambda}(\widehat{\mathrm{sl}_2})$ is based on the quasi-Hopf structure of $\mathcal{B}_{q,\lambda}(\widehat{\mathrm{sl}_2})$. By the above isomorphism $U_{q,p}(\widehat{\mathrm{sl}_2}) \simeq \mathcal{B}_{q,\lambda}(\widehat{\mathrm{sl}_2}) \otimes \mathcal{H}$, we can understand "intertwining relation" of the vertex operator for the elliptic algebra $U_{q,p}(\widehat{\mathrm{sl}_2})$. Going along the isomorphism $U_{q,p}(\mathfrak{g}) \simeq \mathcal{B}_{q,\lambda}(\mathfrak{g}) \otimes \mathcal{H}$, the elliptic analogue of Drinfeld current of $U_{q,p}(\widehat{\mathrm{sl}_2})$ is extended to those of $U_{q,p}(\mathfrak{g})$ for nontwisted affine Lie algebra $\mathfrak{g}$.[12,13] We give a comment on Hopf-algebroid structure[10] of the elliptic algebra $U_{q,p}(\widehat{\mathrm{sl}_2})$. Recently the elliptic algebra $U_{q,p}(\widehat{\mathrm{sl}_2})$ has been understood as an Hopf algebroid by Konno.[9] The vertex operator derived as the Hopf algebroid intertwiner of $U_{q,p}(\widehat{\mathrm{sl}_2})$ coincides with those derived as the "intertwiner" of $\mathcal{B}_{q,\lambda}(\widehat{\mathrm{sl}_2}) \otimes \mathcal{H}$. In this paper, we are

*Dedicated to Professor Michio Jimbo on the occasion on the 60th birthday.

interested in higher-rank generalization of level k free field realization of the elliptic quantum algebra. For the elliptic algebra $U_{q,p}(\widehat{\mathrm{sl}_2})$, there exists free field realization for arbitrary level k.[7,12,20] In this paper we are interested in the higher-rank generalization of Wakimoto realization of the elliptic algebra $U_{q,p}(\widehat{\mathrm{sl}_2})$. We construct level k free field realization of Drinfeld current associated with the elliptic algebra $U_{q,p}(\widehat{\mathrm{sl}_N})$. This gives the higher-rank generalization of the author's previous work on $U_{q,p}(\widehat{\mathrm{sl}_3})$.[21] It is supposed that this free field realization can be applied for construction of the level k integrals of motion for the elliptic algebra $U_{q,p}(\widehat{\mathrm{sl}_N})$. For this purpose, see Refs. 23–25.

The organization of this paper is as follows. In Sec. 2, we set the notation and introduce bosons. In Sec. 3, we review the level k free field realization of the quantum group $U_q(\widehat{\mathrm{sl}_N})$.[18,19] In Sec. 4, we give the free field realization of the dressing operators $U^i(z)$ and $U^{*i}(z)$, which cause the elliptic deformation of Drinfeld current, and also study the screening current. In App. A, we explain a systematic way of construction of a free field realization of the dressing operators $U^i(z)$ and $U^{*i}(z)$. In App. B, we summarize the normal ordering of the basic operators.

After finishing this work, the author noticed a paper on $U_{q,p}(\widehat{\mathrm{sl}_N})$ by W. Chang and X. Ding,[20] which seemed to be submitted to arXiv. a day after the author submitted his paper on $U_{q,p}(\widehat{\mathrm{sl}_3})$.[21]

2. Bosons

The purpose of this section is to set up the basic notation and to introduce boson. In this paper, we fix three parameters q, k, $r \in \mathbb{C}$. Let us set $r^* = r - k$. We assume $k \neq 0, -N$ and $\mathrm{Re}(r) > 0$, $\mathrm{Re}(r^*) > 0$. We assume q is generic with $|q| < 1, q \neq 0$. Let us set a pair of parameters p and p^* by

$$p = q^{2r}, \qquad p^* = q^{2r^*}.$$

We use the standard symbol of q-integer $[n]$ by

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Following Refs. 18 and 19, we introduce free bosons a_n^i $(1 \leqq i \leqq N-1; n \in \mathbb{Z}_{\neq 0})$, $b_n^{i,j}$ $(1 \leqq i < j \leqq N; n \in \mathbb{Z}_{\neq 0})$ and $c_n^{i,j}$ $(1 \leqq i < j \leqq N; n \in \mathbb{Z}_{\neq 0})$, and the zero-mode operators a^i $(1 \leqq i \leqq N-1)$, $b^{i,j}$ $(1 \leqq i < j \leqq N)$ and $c^{i,j}$ $(1 \leqq i < j \leqq N)$.

$$[a_n^i, a_m^j] = \frac{[(k+N)n][A_{i,j}n]}{n}\delta_{n+m,0}, \quad [p_a^i, q_a^j] = (k+N)A_{i,j}, \tag{2.1}$$

$$[b_n^{i,j}, b_m^{k,l}] = -\frac{[n]^2}{n}\delta_{i,k}\delta_{j,l}\delta_{n+m,0}, \qquad [p_b^{i,j}, q_b^{k,l}] = -\delta_{i,k}\delta_{k,l}, \tag{2.2}$$

$$[c_n^{i,j}, c_m^{k,l}] = \frac{[n]^2}{n}\delta_{i,k}\delta_{j,l}\delta_{n+m,0}, \qquad [p_c^{i,j}, q_c^{k,l}] = \delta_{i,k}\delta_{j,l}. \tag{2.3}$$

Here the matrix $(A_{i,j})_{1\leqq i,j\leqq N-1}$ represents the Cartan matrix of classical sl_N. For parameters $a_i \in \mathbb{R}$ $(1 \leqq i \leqq N-1)$, $b_{i,j} \in \mathbb{R}$ $(1 \leqq i < j \leqq N)$ and $c_{i,j} \in \mathbb{R}$ $(1 \leqq i < j \leqq N)$, we set the vacuum vector $|a,b,c\rangle$ of the Fock space $\mathcal{F}_{a,b,c}$ as follows:

$$a_n^i|a,b,c\rangle = b_n^{j,k}|a,b,c\rangle = c_n^{j,k}|a,b,c\rangle = 0 \quad (n>0; 1\leqq i \leqq N-1; 1 \leqq j < k \leqq N)\,,$$

$$p_a^i|a,b,c\rangle = a_i|a,b,c\rangle\,, \quad p_b^{j,k}|a,b,c\rangle = b_{j,k}|a,b,c\rangle\,,$$

$$p_c^{j,k}|a,b,c\rangle = c_{j,k}|a,b,c\rangle \quad (1 \leqq i \leqq N-1; 1 \leqq j < k \leqq N)\,.$$

The Fock space $\mathcal{F}_{a,b,c}$ is generated by bosons a_{-n}^i, $b_{-n}^{j,k}$ and $c_{-n}^{j,k}$ for $n \in \mathbb{N}_{\neq 0}$. The dual Fock space $\mathcal{F}_{a,b,c}^*$ is defined in the same manner. In this paper, we construct the elliptic analogue of Drinfeld current for $U_{q,p}(\widehat{\mathrm{sl}_N})$ by these bosons a_n^i, $b_n^{j,k}$ and $c_n^{j,k}$ acting on the Fock space.

Let us set the elliptic theta function $\Theta_p(z)$ by

$$\Theta_p(z) = (z;p)_\infty (p/z;p)_\infty (p;p)_\infty\,, \qquad (z;p)_\infty = \prod_{n=0}^{\infty}(1-p^n z)\,.$$

It is convenient to work with the additive notation. We use the parametrization

$$\begin{aligned} q &= e^{-\pi\sqrt{-1}/r\tau}\,, \\ p &= e^{-2\pi\sqrt{-1}/\tau}\,, \\ p^* &= e^{-2\pi\sqrt{-1}/\tau^*} \quad (r\tau = r^*\tau^*)\,, \\ z &= q^{2u}\,. \end{aligned}$$

Let us set Jacobi elliptic theta function $[u]_r$ by

$$[u]_r = q^{\frac{u^2}{r}-u}\frac{\Theta_{q^{2r}}(z)}{(q^{2r};q^{2r})_\infty^3}\,.$$

The function $[u]_r$ has a zero at $u=0$, enjoys the quasiperiodicity property

$$[u+r]_r = -[u]_r\,, \quad [u+r\tau]_r = -e^{-\pi\sqrt{-1}\tau - \frac{2\pi\sqrt{-1}u}{r}}[u]_r\,.$$

Let us set the q-difference $({}_\alpha\partial_z f)(z)$ by

$$({}_\alpha\partial_z f)(z) = \frac{f(q^\alpha z) - f(q^{-\alpha}z)}{(q-q^{-1})z}\,.$$

Let us set the delta-function $\delta(z)$ as formal power series:

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n\,.$$

3. Free Field Realization of $U_q(\widehat{\mathrm{sl}_N})$

The purpose of this section is to give a review on the free field realization of the quantum affine algebra $U_q(\widehat{\mathrm{sl}_N})$,[19] which is a basis of the elliptic algebra $U_{q,p}(\widehat{\mathrm{sl}_N})$.

3.1. *Drinfeld current*

Let us set the bosonic operators $a_\pm^i(z)$, $a^i(z)$ $(1 \leqq i \leqq N-1)$, $b_\pm^{i,j}(z)$, $b^{i,j}(z)$, $c^{i\cdot j}(z)$ $(1 \leqq i < j \leqq N)$ by

$$a_\pm^i(z) = \pm(q-q^{-1})\sum_{n>0} a_{\pm n}^i z^{\mp n} \pm p_a^i \log q\,, \tag{3.1}$$

$$b_\pm^{i,j}(z) = \pm(q-q^{-1})\sum_{n>0} b_{\pm n}^{i,j} z^{\mp n} \pm p_b^{i,j} \log q\,, \tag{3.2}$$

$$a^i(z) = -\sum_{n\neq 0} \frac{a_n^i}{[(k+N)n]} q^{-\frac{k+N}{2}|n|} z^{-n} + \frac{1}{k+N}(q_a^i + p_a^i \log z)\,, \tag{3.3}$$

$$b^{i,j}(z) = -\sum_{n\neq 0} \frac{b_n^{i,j}}{[n]} z^{-n} + q_b^{i,j} + p_b^{i,j} \log z\,, \tag{3.4}$$

$$c^{i,j}(z) = -\sum_{n\neq 0} \frac{c_n^{i,j}}{[n]} z^{-n} + q_c^{i,j} + p_c^{i,j} \log z\,, \tag{3.5}$$

Let us set the auxiliary operators $\gamma^{i,j}(z)$, $\beta_1^{i,j}(z)$, $\beta_2^{i,j}(z)$, $\beta_3^{i,j}(z)$, $\beta_4^{i,j}(z)$ $(1 \leqq i < j \leqq N)$ by

$$\gamma^{i,j}(z) = -\sum_{n\neq 0} \frac{(b+c)_n^{i,j}}{[n]} z^{-n} + (q_b^{i,j} + q_c^{i,j}) + (p_b^{i,j} + p_c^{i,j}) \log(-z)\,, \tag{3.6}$$

$$\begin{aligned} \beta_1^{i,j}(z) &= b_+^{i,j}(z) - (b^{i,j} + c^{i,j})(qz)\,,\\ \beta_2^{i,j}(z) &= b_-^{i,j}(z) - (b^{i,j} + c^{i,j})(q^{-1}z)\,, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \beta_3^{i,j}(z) &= b_+^{i,j}(z) + (b^{i,j} + c^{i,j})(q^{-1}z)\,,\\ \beta_4^{i,j}(z) &= b_-^{i,j}(z) + (b^{i,j} + c^{i,j})(qz)\,. \end{aligned} \tag{3.8}$$

We give a free field realization of Drinfeld current for $U_q(\widehat{\mathrm{sl}_N})$.

Definition 3.1. Let us set the bosonic operators $E^{\pm,i}(z)$ $(1 \leqq i \leqq N-1)$ by

$$E^{+,i}(z) = \frac{-1}{(q-q^{-1})z}\sum_{j=1}^{i} E_j^{+,i}(z)\,, \tag{3.9}$$

$$E^{-,i}(z) = \frac{-1}{(q-q^{-1})z}\sum_{j=1}^{N-1} E_j^{-,i}(z)\,, \tag{3.10}$$

where we have set

$$\begin{aligned} E_j^{+,i}(z) = {}:& e^{\gamma^{j,i}(q^{j-1}z)}\left(e^{\beta_1^{j,i+1}(q^{j-1}z)} - e^{\beta_2^{j,i+1}(q^{j-1}z)}\right)\\ &\times e^{\sum_{l=1}^{j-1}\left(b_+^{l,i+1}(q^{l-1}z) - b_+^{l,i}(q^l z)\right)}:\,, \end{aligned} \tag{3.11}$$

$$E_j^{-,i}(z) = :e^{\gamma^{j,i+1}(q^{-(k+j)}z)}\left(e^{-\beta_4^{j,i}(q^{-(k+j)}z)} - e^{-\beta_3^{j,i}(q^{-(k+j)}z)}\right)$$
$$\times e^{\sum_{l=j+1}^{i}\left(b_-^{l,i+1}\left(q^{-(k+l-1)}z\right) - b_-^{l,i}\left(q^{-(k+l)}z\right)\right) + a_-^i\left(q^{-\frac{k+N}{2}}z\right) + \sum_{l=i+1}^{N}\left(b_-^{i,l}\left(q^{-(k+l)}z\right) - b_-^{i+1,l}\left(q^{-(k+l-1)}z\right)\right)}:,$$
$$\text{for} \quad 1 \leqq j \leqq i-1, \tag{3.12}$$

$$E_i^{-,i}(z) = :e^{\gamma^{i,i+1}\left(q^{-(k+i)}z\right) + a_-^i\left(q^{-\frac{k+N}{2}}z\right) + \sum_{l=i+1}^{N}\left(b_-^{i,l}(q^{-(k+l)}z) - b_-^{i+1,l}\left(q^{-(k+l-1)}z\right)\right)}:$$
$$- :e^{\gamma^{i,i+1}(q^{k+i}z) + a_+^i\left(q^{\frac{k+N}{2}}z\right) + \sum_{l=i+1}^{N}\left(b_+^{i,l}(q^{k+l}z) - b_+^{i+1,l}(q^{k+l-1}z)\right)}:, \tag{3.13}$$

$$E_j^{-,i}(z) = :e^{\gamma^{i,j+1}(q^{k+j}z)}\left(e^{\beta_2^{i+1,j+1}(q^{k+j}z)} - e^{\beta_1^{i+1,j+1}(q^{k+j}z)}\right)$$
$$\times e^{a_+^i\left(q^{\frac{k+N}{2}}z\right) + \sum_{l=j+1}^{N}\left(b_+^{i,l}(q^{k+l}z) - b_+^{i+1,l}(q^{k+l-1}z)\right)}:,$$
$$\text{for} \quad i+1 \leqq j \leqq N-1. \tag{3.14}$$

Let us set the bosonic operators $\psi_i^{\pm}(z)$ $(1 \leqq i \leqq N-1)$ by

$$\psi_{\pm}^i\left(q^{\pm\frac{k}{2}}z\right) = :e^{\sum_{j=1}^{i}\left(b_{\pm}^{j,i+1}(q^{\pm(k+j-1)}z) - b_{\pm}^{j,i}\left(q^{\pm(k+j)}z\right)\right) + a_{\pm}^i\left(q^{\pm\frac{k+N}{2}}z\right) + \sum_{j=i+1}^{N}\left(b_{\pm}^{i,j}(q^{\pm(k+j)}z) - b_{\pm}^{i+1,j}(q^{\pm(k+j-1)}z)\right)}:. \tag{3.15}$$

Let us set

$$h_i = \sum_{j=1}^{i}\left(p_b^{j,i+1} - p_b^{j,i}\right) + p_a^i + \sum_{j=i+1}^{N}\left(p_b^{i,j} - p_b^{i+1,j}\right). \tag{3.16}$$

Here the symbol $:\mathcal{O}:$ represents the normal ordering of $\mathcal{O}$. For example, we have

$$:b_k^{i,j}b_l^{i,j}: = \begin{cases} b_k^{i,j}b_l^{i,j}, & k<0, \\ b_l^{i,j}b_k^{i,j}, & k>0, \end{cases} \qquad :p_b^{i,j}q_b^{i,j}: = :q_b^{i,j}p_b^{i,j}: = q_b^{i,j}p_b^{i,j}.$$

Theorem 3.2. *The operators* $E^{\pm,i}(z)$, $\psi_{\pm}^i(z)$, h_i $(1 \leqq i \leqq N-1)$ *give a free field realization of* $U_q(\widehat{\mathrm{sl}_N})$ *for arbitrary level* $k \neq 0, -N$. *In other words, they satisfy the following commutation relations*:

$$[h_i, E^{\pm,j}(z)] = \pm A_{i,j}E^{\pm,j}(z), \tag{3.17}$$

$$(z_1 - q^{\pm A_{i,j}}z_2)E^{\pm,i}(z_1)E^{\pm,j}(z_2) = (q^{\pm A_{i,j}}z_1 - z_2)E^{\pm,j}(z_2)E^{\pm,i}(z_1), \tag{3.18}$$

$$[\psi_{\pm}^i(z_1), \psi_{\pm}^j(z_2)] = 0, \tag{3.19}$$

$$(z_1 - q^{A_{i,j}-k}z_2)(z_1 - q^{-A_{i,j}+k}z_2)\psi_+^i(z_1)\psi_-^j(z_2)$$
$$= (z_1 - q^{A_{i,j}+k}z_2)(z_1 - q^{-A_{i,j}-k}z_2)\psi_-^j(z_2)\psi_+^i(z_1), \tag{3.20}$$

$$\left(z_1 - q^{\pm\left(A_{i,j}-\frac{k}{2}\right)} z_2\right) \psi_+^i(z_1) E^{\pm,j}(z_2)$$
$$= \left(q^{\pm A_{i,j}} z_1 - q^{\mp\frac{k}{2}} z_2\right) E^{\pm,j}(z_2) \psi_+^i(z_1)\,, \tag{3.21}$$

$$\left(z_1 - q^{\pm\left(A_{i,j}-\frac{k}{2}\right)} z_2\right) E^{\pm,i}(z_1) \psi_-^j(z_2)$$
$$= \left(q^{\pm A_{i,j}} z_1 - q^{\mp\frac{k}{2}} z_2\right) \psi_-^j(z_2) E^{\pm,i}(z_1)\,, \tag{3.22}$$

$$\big\{E^{\pm,i}(z_1)E^{\pm,i}(z_2)E^{\pm,j}(z_3) - (q+q^{-1})E^{\pm,i}(z_1)E^{\pm,j}(z_3)E^{\pm,i}(z_2)$$
$$+ E^{\pm,i}(z_3)E^{\pm,i}(z_1)E^{\pm,j}(z_2)\big\} + \{z_1 \leftrightarrow z_2\} = 0\,, \quad \text{for} \quad A_{i,j} = -1\,, \tag{3.23}$$

$$[E^{+,i}(z_1), E^{-,j}(z_2)] = \frac{\delta_{i,j}}{(q-q^{-1})z_1 z_2}\left(\delta\left(q^{-k}\frac{z_1}{z_2}\right)\psi_+^i\left(q^{-\frac{k}{2}} z_1\right)\right.$$
$$\left. - \delta\left(q^{k}\frac{z_1}{z_2}\right)\psi_-^i\left(q^{-\frac{k}{2}} z_2\right)\right)\,. \tag{3.24}$$

When we take the limit $q \to 1$, we recover Wakimoto realization for $\widehat{\mathrm{sl}_N}$.[15]

3.2. *Screening current*

Following Ref. 19, we define the screening current $S^i(z)$, which commutes with $U_q(\widehat{\mathrm{sl}_N})$.

Definition 3.3. Let us introduce the bosonic operator $S^i(z)$ $(1 \leqq i \leqq N-1)$ by

$$S^i(z) = \frac{-1}{(q-q^{-1})z} :e^{-a^i(z)} \tilde{S}^i(z):\,, \tag{3.25}$$

where we have set

$$\tilde{S}^i(z) = \sum_{j=i+1}^{N} :e^{\gamma^{i+1,j}(q^{N-j}z)}\left(e^{-\beta_4^{i,j}(q^{N-j}z)} - e^{-\beta_3^{i,j}(q^{N-j}z)}\right)$$
$$\times\, e^{\sum_{l=j+1}^{N}\left(b_-^{i+1,l}(q^{N-l+1}z) - b_-^{i,l}(q^{N-l}z)\right)}:\,.$$

Proposition 3.4. *The bosonic operators* $S^i(z)$, $E^{\pm,i}(z)$ $(1 \leqq j \leqq N-1)$ *satisfy the following commutation relations*:

$$\left[u_1 - u_2 - \frac{A_{i,j}}{2}\right]_{k+N} S^i(z_1) S^j(z_2)$$
$$= \left[u_1 - u_2 + \frac{A_{i,j}}{2}\right]_{k+N} S^j(z_2) S^i(z_1) \sim \text{reg.}\,, \tag{3.26}$$

$$E^{+,i}(z_1) S^j(z_2) = S^j(z_2) E^{+,i}(z_1) \sim \text{reg.}\,, \tag{3.27}$$

$$E^{-,i}(z_1)S^j(z_2) = S^j(z_2)E^{-,i}(z_1) \sim \text{reg.} + \delta_{i,j} \times {}_{k+N}\partial_{z_2} \times \left(\frac{1}{z_1 - z_2} :e^{\sum_{n\neq 0} \frac{a_n^i}{[(k+N)n]} q^{\frac{k+N}{2}|n|} z_2^{-n} - \frac{1}{k+N}(q_a^i + p_a^i \log z_2)}: \right). \quad (3.28)$$

The symbol $\sim$ reg. *means equality modulo regular function.*

The equalities (3.17)–(3.27) hold in :$\sim$ reg.” sense. The exceptional cases are (3.24) and (3.28), which do not exist inside regular function. Note that the elliptic theta function $[u]_{k+N}$ has already appeared in trigonometric symmetry $U_q(\widehat{\mathrm{sl}_N})$.

4. Free Field Realization of $U_{q,p}(\widehat{\mathrm{sl}_N})$

The purpose of this section is to give a free field realization for the elliptic algebra $U_{q,p}(\widehat{\mathrm{sl}_N})$ for arbitrary level $k \neq 0, -N$.

4.1. *Drinfeld current*

Following Ref. 20, let us introduce the auxiliary operators $\mathcal{B}_{\pm}^{*i,j}(z)$, $\mathcal{B}_{\pm}^{i,j}(z)$ $(1 \leqq i < j \leqq N)$ by

$$\mathcal{B}_{\pm}^{*i,j}(z) = \exp\left(\pm \sum_{n>0} \frac{1}{[r^* n]} b_{-n}^{i,j} (q^{r^*-1} z)^n \right), \quad (4.1)$$

$$\mathcal{B}_{\pm}^{i,j}(z) = \exp\left(\pm \sum_{n>0} \frac{1}{[rn]} b_{n}^{i,j} (q^{-r^*+1} z)^{-n} \right). \quad (4.2)$$

Let us introduce the auxiliary operators $\mathcal{A}^{*i}(z)$, $\mathcal{A}^{i}(z)$ $(1 \leqq i \leqq N-1)$ by

$$\mathcal{A}^{*i}(z) = \exp\left(\sum_{n>0} \frac{1}{[r^* n]} a_{-n}^{i} (q^{r^*} z)^n \right), \quad (4.3)$$

$$\mathcal{A}^{i}(z) = \exp\left(- \sum_{n>0} \frac{1}{[rn]} a_{n}^{i} (q^{-r^*} z)^{-n} \right). \quad (4.4)$$

Definition 4.1. We define the dressing operators $U^{*i}(z)$, $U^i(z)$ $(1 \leqq i \leqq N-1)$,

$$U^{*i}(z) = \left(\prod_{j=1}^{i-1} \mathcal{B}_{+}^{*j,i+1}(q^{2-j}z)\mathcal{B}_{-}^{*j,i}(q^{1-j}z) \right) \mathcal{B}_{+}^{*i,i+1}(q^{2-i}z)\mathcal{B}_{+}^{*i,i+1}(q^{-i}z) \times \left(\prod_{j=i+2}^{N} \mathcal{B}_{+}^{*i,j}(q^{-j+1}z)\mathcal{B}_{-}^{*i+1,j}(q^{-j+2}z) \right) \mathcal{A}^{*i}\left(q^{\frac{k-N}{2}}z\right), \quad (4.5)$$

$$U^i(z)=\left(\prod_{j=1}^{i-1}\mathcal{B}_-^{j,i+1}(q^{-2+j}z)\mathcal{B}_+^{j,i}(q^{-1+j}z)\right)\mathcal{B}_-^{i,i+1}(q^{-2+i}z)\mathcal{B}_-^{i,i+1}(q^iz)$$
$$\times\left(\prod_{j=i+2}^{N}\mathcal{B}_-^{i,j}(q^{j-1}z)\mathcal{B}_+^{i+1,j}(q^{j-2}z)\right)\mathcal{A}^i\left(q^{\frac{-k+N}{2}}z\right). \quad (4.6)$$

Formulae (4.5) and (4.6) are the main result of this paper. In App. A, we explain a systematic construction of the dressing operators $U^{*i}(z)$, $U^i(z)$.

Definition 4.2. We define the elliptic deformation of Drinfeld current $e_i(z)$, $f_i(z)$, $\Psi_i^{\pm}(z)$ $(1\leqq i\leqq N-1)$, by

$$e_i(z)=U^{*i}(z)E^{+,i}(z)\,, \quad (4.7)$$

$$f_i(z)=E^{-,i}(z)U^i(z)\,, \quad (4.8)$$

$$\Psi_i^+(z)=U^{*i}\left(q^{\frac{k}{2}}z\right)\psi_i^+(z)U^i\left(q^{-\frac{k}{2}}z\right), \quad (4.9)$$

$$\Psi_i^-(z)=U^{*i}\left(q^{-\frac{k}{2}}z\right)\psi_i^-(z)U^i\left(q^{\frac{k}{2}}z\right). \quad (4.10)$$

Example 4.3. Upon specialization $N=3$, we recover the dressing operator of $U_{q,p}(\widehat{\mathrm{sl}_3})$:[21]

$$U^{*1}(z)=\mathcal{B}_+^{*1,2}(qz)\mathcal{B}_+^{*1,2}(q^{-1}z)\mathcal{B}_+^{*13}(q^{-2}z)\mathcal{B}_-^{*23}(q^{-1}z)\mathcal{A}^{*1}\left(q^{\frac{k-3}{2}}z\right), \quad (4.11)$$

$$U^{*2}(z)=\mathcal{B}_+^{*1,3}(qz)\mathcal{B}_-^{*1,2}(z)\mathcal{B}_+^{*2,3}(z)\mathcal{B}_+^{*2,3}(q^{-2}z)\mathcal{A}^{*2}\left(q^{\frac{k-3}{2}}z\right), \quad (4.12)$$

$$U^1(z)=\mathcal{B}_-^{1,2}(q^{-1}z)\mathcal{B}_-^{12}(qz)\mathcal{B}_-^{1,3}(q^2z)\mathcal{B}_+^{2,3}(qz)\mathcal{A}^1\left(q^{\frac{-k+3}{2}}z\right), \quad (4.13)$$

$$U^2(z)=\mathcal{B}_-^{1,3}(q^{-1}z)\mathcal{B}_+^{1,2}(z)\mathcal{B}_-^{2,3}(z)\mathcal{B}_-^{2,3}(q^2z)\mathcal{A}^2\left(q^{\frac{-k+3}{2}}z\right). \quad (4.14)$$

The notation of this paper is slightly different from those of Ref. 21. For example, $\mathcal{B}_\pm^{*1,2}(z)=\mathcal{B}_\pm^{*1}(q^{r^*-1}z)$, $\mathcal{B}_\pm^{*1,3}(z)=\mathcal{B}_\pm^{*2}(q^{r^*-1}z)$ and $\mathcal{B}_\pm^{*2,3}(z)=\mathcal{B}_\pm^{*,3}(q^{r^*-1}z)$.

Proposition 4.4. *The bosonic operators* $e_i(z)$, $f_i(z)$, $\Psi_i^{\pm}(z)$ $(1\leqq i\leqq N-1)$ *satisfy the following commutation relations*:

$$\Theta_{p^*}(q^{-A_{i,j}}z_1/z_2)e_i(z_1)e_j(z_2)=q^{-A_{i,j}}\Theta_{p^*}(q^{A_{i,j}}z_1/z_2)e_j(z_2)e_i(z_1)\,, \quad (4.15)$$

$$\Theta_p(q^{A_{i,j}}z_1/z_2)f_i(z_1)f_j(z_2)=q^{A_{i,j}}\Theta_p(q^{-A_{i,j}}z_1/z_2)f_j(z_2)f_i(z_1)\,, \quad (4.16)$$

$$\Theta_p(q^{A_{i,j}}z_1/z_2)\Theta_{p^*}(q^{-A_{i,j}}z_1/z_2)\Psi_i^{\pm}(z_1)\Psi_j^{\pm}(z_2)$$
$$=\Theta_p(q^{-A_{i,j}}z_1/z_2)\Theta_{p^*}(q^{A_{i,j}}z_1/z_2)\Psi_j^{\pm}(z_2)\Psi_i^{\pm}(z_1)\,, \quad (4.17)$$

$$\Theta_p(pq^{A_{i,j}-k}z_1/z_2)\Theta_{p^*}(p^*q^{-A_{i,j}+k}z_1/z_2)\Psi_i^{\pm}(z_1)\Psi_j^{\mp}(z_2)$$
$$=\Theta_p(pq^{-A_{i,j}-k}z_1/z_2)\Theta_{p^*}(p^*q^{A_{i,j}+k}z_1/z_2)\Psi_j^{\mp}(z_2)\Psi_i^{\pm}(z_1)\,, \quad (4.18)$$

$$\Theta_{p^*}\left(q^{-A_{i,j}\pm\frac{k}{2}}z_1/z_2\right)\Psi_i^{\pm}(z_1)e_j(z_2)=\Theta_{p^*}\left(q^{A_{i,j}\pm\frac{k}{2}}z_1/z_2\right)e_j(z_2)\Psi_i^{\pm}(z_1)\,, \quad (4.19)$$

$$\Theta_{p}\left(q^{A_{i,j}\mp\frac{k}{2}}z_1/z_2\right)\Psi_i^{\pm}(z_1)f_j(z_2)=\Theta_{p}\left(q^{-A_{i,j}\mp\frac{k}{2}}z_1/z_2\right)f_j(z_2)\Psi_i^{\pm}(z_1)\,, \quad (4.20)$$

$$[e_i(z_1),f_j(z_2)]=\frac{\delta_{i,j}}{(q-q^{-1})z_1z_2}\left(\delta\left(q^{-k}\frac{z_1}{z_2}\right)\Psi_i^{+}(q^{-k/2}z_1)\right.$$
$$\left.-\,\delta\left(q^{k}\frac{z_1}{z_2}\right)\Psi_i^{-}(q^{-k/2}z_2)\right). \quad (4.21)$$

They satisfy Serre relation:

$$(p^*q^2z_2/z_1:p^*)_\infty(p^*q^{-2}z_1/z_2;p^*)_\infty$$
$$\times\Big\{(p^*q^{-1}z/z_1;p^*)_\infty(p^*q^{-1}z/z_2;p^*)_\infty(p^*qz_1/z;p^*)_\infty(p^*qz_1/z;p^*)_\infty e_i(z_1)e_i(z_2)e_j(z)$$
$$-[2](p^*q^{-1}z/z_1;p^*)_\infty(p^*q^{-1}z_2/z;p^*)_\infty(p^*qz_1/z;p^*)_\infty(p^*qz/z_2;p^*)_\infty e_i(z_1)e_j(z)e_i(z_2)$$
$$+(p^*q^{-1}z_1/z;p^*)_\infty(p^*q^{-1}z_2/z;p^*)_\infty(p^*qz/z_1;p^*)_\infty(p^*qz/z_2;p^*)_\infty e_j(z)e_i(z_1)e_i(z_2)\Big\}$$
$$+(z_1\leftrightarrow z_2)=0\,,\quad \text{for}\quad A_{i,j}=-1\,, \quad (4.22)$$

$$(pq^{-2}z_2/z_1:p)_\infty(pq^2z_1/z_2;p)_\infty$$
$$\times\Big\{(pqz/z_1;p)_\infty(pqz/z_2;p)_\infty(pq^{-1}z_1/z;p)_\infty(pq^{-1}z_1/z;p)_\infty f_i(z_1)f_i(z_2)f_j(z)$$
$$-[2](pqz/z_1;p)_\infty(pqz_2/z;p)_\infty(pq^{-1}z_1/z;p)_\infty(pq^{-1}z/z_2;p)_\infty f_i(z_1)f_j(z)f_i(z_2)$$
$$+(pqz_1/z;p)_\infty(pqz_2/z;p)_\infty(pq^{-1}z/z_1;p)_\infty(pq^{-1}z/z_2;p)_\infty f_j(z)f_i(z_1)f_i(z_2)\Big\}$$
$$+(z_1\leftrightarrow z_2)=0\,,\quad \text{for}\quad A_{i,j}=-1\,. \quad (4.23)$$

Following Refs. 7 and 12, we introduce the Heisenberg algebra $\mathcal{H}$ generated by the following P_i, Q_i $(1\leqq i\leqq N-1)$:

$$[P_i,Q_j]=\frac{A_{i,j}}{2}\,. \quad (4.24)$$

Definition 4.5. Let us define the bosonic operators $E_i(z)$, $F_i(z)$, $H_i^{\pm}(z)\in U_q(\widehat{\mathrm{sl}_N})\otimes\mathcal{H}$ $(1\leqq i\leqq N-1)$ by

$$E_i(z)=e_1(z)e^{2Q_i}z^{-\frac{P_i-1}{r^*}}\,, \quad (4.25)$$

$$F_j(z)=f_1(z)z^{\frac{h_i+P_i-1}{r}}\,, \quad (4.26)$$

$$H_i^{\pm}(z)=\Psi_i^{\pm}(z)e^{2Q_i}q^{\mp h_i}\left(q^{\pm(r-\frac{k}{2})}z\right)^{\frac{h_i+P_i-1}{r}-\frac{P_i-1}{r^*}}\,. \quad (4.27)$$

Theorem 4.6. *The bosonic operators* $E_i(z)$, $F_i(z)$, $H_i^{\pm}(z)$ $(1 \leqq i \leqq N-1)$ *satisfy the following commutation relations*:

$$\left[u_1 - u_2 - \frac{A_{i,j}}{2}\right]_{r^*} E_i(z_1)E_j(z_2) = \left[u_1 - u_2 + \frac{A_{i,j}}{2}\right]_{r^*} E_j(z_2)E_i(z_1)\,, \quad (4.28)$$

$$\left[u_1 - u_2 + \frac{A_{i,j}}{2}\right]_{r} F_i(z_1)F_j(z_2) = \left[u_1 - u_2 - \frac{A_{i,j}}{2}\right]_{r} F_j(z_2)F_i(z_1)\,, \quad (4.29)$$

$$\begin{aligned}&\left[u_1 - u_2 + \frac{A_{i,j}}{2}\right]_{r}\left[u_1 - u_2 - \frac{A_{i,j}}{2}\right]_{r^*} H_i^{\pm}(z_1)H_j^{\pm}(z_2)\\ &\quad = \left[u_1 - u_2 - \frac{A_{i,j}}{2}\right]_{r}\left[u_1 - u_2 + \frac{A_{i,j}}{2}\right]_{r^*} H_j^{\pm}(z_2)H_i^{\pm}(z_1)\,,\end{aligned} \quad (4.30)$$

$$\begin{aligned}&\left[u_1 - u_2 + \frac{A_{i,j}}{2} - \frac{k}{2}\right]_{r}\left[u_1 - u_2 - \frac{A_{i,j}}{2} + \frac{k}{2}\right]_{r^*} H_i^{+}(z_1)H_j^{-}(z_2)\\ &\quad = \left[u_1 - u_2 - \frac{A_{i,j}}{2} - \frac{k}{2}\right]_{r}\left[u_1 - u_2 + \frac{A_{i,j}}{2} + \frac{k}{2}\right]_{r^*} H_j^{-}(z_2)H_i^{+}(z_1)\,,\end{aligned} \quad (4.31)$$

$$\begin{aligned}&\left[u_1 - u_2 \pm \frac{k}{4} - \frac{A_{i,j}}{2}\right]_{r^*} H_i^{\pm}(z_1)E_j(z_2)\\ &\quad = \left[u_1 - u_2 \pm \frac{k}{4} + \frac{A_{i,j}}{2}\right]_{r^*} E_j(z_2)H_i^{\pm}(z_1)\,,\end{aligned} \quad (4.32)$$

$$\begin{aligned}&\left[u_1 - u_2 \mp \frac{k}{4} + \frac{A_{i,j}}{2}\right]_{r} H_i^{\pm}(z_1)F_j(z_2)\\ &\quad = \left[u_1 - u_2 \mp \frac{k}{4} - \frac{A_{i,j}}{2}\right]_{r} F_j(z_2)H_i^{\pm}(z_1)\,,\end{aligned} \quad (4.33)$$

$$\begin{aligned}[E_i(z_1), F_j(z_2)] &= \frac{\delta_{i,j}}{(q-q^{-1})z_1z_2}\left(\delta\left(q^{-k}\frac{z_1}{z_2}\right)H_i^{+}\left(q^{-\frac{k}{2}}z_1\right)\right.\\ &\quad \left. - \delta\left(q^{k}\frac{z_1}{z_2}\right)H_i^{-}\left(q^{-\frac{k}{2}}z_2\right)\right).\end{aligned} \quad (4.34)$$

They satisfy Serre relation:

$$\begin{aligned}&z_1^{-\frac{1}{r^*}}(p^*q^2z_2/z_1 : p^*)_\infty(p^*q^{-2}z_1/z_2;p^*)_\infty\\ &\quad \times \left\{z_2^{\frac{1}{r^*}}z^{-\frac{1}{r^*}}(p^*q^{-1}z/z_1;p^*)_\infty(p^*q^{-1}z/z_2;p^*)_\infty\right.\\ &\quad \times (p^*qz_1/z;p^*)_\infty(p^*qz_1/z;p^*)_\infty E_i(z_1)E_i(z_2)E_j(z)\\ &\quad - [2](p^*q^{-1}z/z_1;p^*)_\infty(p^*q^{-1}z_2/z;p^*)_\infty\\ &\quad \times (p^*qz_1/z;p^*)_\infty(p^*qz/z_2;p^*)_\infty E_i(z_1)E_j(z)E_i(z_2)\end{aligned}$$

$$
\begin{aligned}
&+ z^{\frac{1}{r^*}} z_1^{-\frac{1}{r^*}} (p^*q^{-1}z_1/z;p^*)_\infty (p^*q^{-1}z_2/z;p^*)_\infty \\
&\times (p^*qz/z_1;p^*)_\infty (p^*qz/z_2;p^*)_\infty E_j(z)E_i(z_1)E_i(z_2)\Big\} \\
&+(z_1 \leftrightarrow z_2) = 0\,, \quad \text{for} \quad A_{i,j} = -1\,,
\end{aligned} \tag{4.35}
$$

$$
\begin{aligned}
z_1^{\frac{1}{r}} & (pq^{-2}z_2/z_1 : p)_\infty (pq^2 z_1/z_2;p)_\infty \\
&\times \Big\{ z^{\frac{1}{r}} z_2^{-\frac{1}{r}} (pqz/z_1;p)_\infty (pqz/z_2;p)_\infty \\
&\times (pq^{-1}z_1/z;p)_\infty (pq^{-1}z_1/z;p)_\infty F_i(z_1)F_i(z_2)F_j(z) \\
&- [2](pqz/z_1;p)_\infty (pqz_2/z;p)_\infty \\
&\times (pq^{-1}z_1/z;p)_\infty (pq^{-1}z/z_2;p)_\infty F_i(z_1)F_j(z)F_i(z_2) \\
&+ z_1^{\frac{1}{r}} z^{-\frac{1}{r}} (pqz_1/z;p)_\infty (pqz_2/z;p)_\infty \\
&\times (pq^{-1}z/z_1;p)_\infty (pq^{-1}z/z_2;p)_\infty F_j(z)F_i(z_1)F_i(z_2)\Big\} \\
&+ (z_1 \leftrightarrow z_2) = 0\,, \quad \text{for} \quad A_{i,j} = -1\,.
\end{aligned} \tag{4.36}
$$

Now we have constructed level k free field realization of Drinfeld current $E_i(z)$, $F_i(z)$ and $H_i^{\pm}(z)$ for the elliptic algebra $U_{q,p}(\widehat{\mathrm{sl}_N})$.[12,13]

4.2. *Screening current*

In this section, we study the screening current for $U_{q,p}(\widehat{\mathrm{sl}_N})$. In Ref. 7, it was recognized that the screening current of $U_{q,p}(\widehat{\mathrm{sl}_2})$ was exactly the same as those of $U_q(\widehat{\mathrm{sl}_2})$. Hence we select the same definition of screening current of $U_q(\widehat{\mathrm{sl}_N})$ (Ref. 19) as the screening current of $U_{q,p}(\widehat{\mathrm{sl}_N})$.

$$
\begin{aligned}
S_i(z) = \frac{-1}{(q-q^{-1})z} &:e^{-a^i(z)}: \\
&\times \sum_{j=i+1}^{N} :e^{\gamma^{i+1,j}(q^{N-j}z)} \left(e^{-\beta_4^{i,j}(q^{N-j}z)} - e^{-\beta_3^{i,j}(q^{N-j}z)} \right) \\
&\times e^{\sum_{l=j+1}^{N} \left(b_-^{i+1,l}(q^{N-l+1}z) - b_-^{i,l}(q^{N-l}z) \right)}:
\end{aligned}
$$

Proposition 4.7. *The bosonic operator $S_i(z)$, $E_i(z)$, $F_i(z)$ $(1 \leqq i \leqq N-1)$ satisfy the following commutation relations*:

$$
\begin{aligned}
&\left[u_1 - u_2 - \frac{A_{i,j}}{2} \right]_{k+N} S_i(z_1)S_j(z_2) \\
&\quad = \left[u_1 - u_2 + \frac{A_{i,j}}{2} \right]_{k+N} S_j(z_2)S_i(z_1) \sim \text{reg.}\,,
\end{aligned} \tag{4.37}
$$

$$\left[u_1 - u_2 - \frac{A_{i,j}}{2}\right]_{r-k} E_i(z_1)E_j(z_2) = \left[u_1 - u_2 + \frac{A_{i,j}}{2}\right]_{r-k} E_j(z_2)E_i(z_1) \sim \text{reg.}\,, \tag{4.38}$$

$$\left[u_1 - u_2 + \frac{A_{i,j}}{2}\right]_{r} F_i(z_1)F_j(z_2) = \left[u_1 - u_2 - \frac{A_{i,j}}{2}\right]_{r} F_j(z_2)F_i(z_1) \sim \text{reg.}\,, \tag{4.39}$$

$$E_i(z_1)S_j(z_2) = S_j(z_2)E_i(z_1) \sim \text{reg.}\,, \tag{4.40}$$

$$\begin{aligned} &F_i(z_1)S_j(z_2) \\ &\quad = S_j(z_2)F_i(z_1) \sim \text{reg.} + \delta_{i,j} \times {}_{k+N}\partial_{z_2} \\ &\qquad \times \left(\frac{1}{z_1 - z_2} : e^{\sum_{n\neq 0} \frac{a_n^i}{[(k+N)n]} q^{\frac{k+N}{2}|n|} z_2^{-n} - \frac{1}{k+N}(q_a^i + p_a^i \log z_2)} U^i(z_2) z_2^{\frac{h_i+P_i-1}{r}} :\right). \end{aligned} \tag{4.41}$$

The symbol $\sim$ reg. *means eqality modulo regular function.*

The equalities (4.28)–(4.38) hold in "$\sim$ reg. sense. The exceptional cases are (4.34) and (4.41), which do not exist inside regular function. It seems to be possible to construct three kind of infinitly many commutative operators, which are based on the commutation relations (4.37)–(4.39).[23–25]

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Appendix A. Construction of Dressing Operators $U^i(z)$, $U^{*i}(z)$

In this appendix, we explain a systematic way to find the dressing operators $U^i(z)$ and $U^{*i}(z)$ associated with the elliptic algebra $U_{q,p}(\widehat{\mathrm{sl}_N})$. Wakimoto realization is not symmetric with Cartan subalgebra. In other words, Wakimoto realization of Drinfeld current $E^{+,i}(z)$ is very different from those of $E^{-,i}(z)$. The realization of $E^{+,i}(z)$ is simpler than those of $E^{-,i}(z)$. Hence it is better to consider the dressing operator $U^{*i}(z)$ associated with $E^{+,i}(z)$ at first. We construct the dressing operator $U^{*i}(z)$ by products of the basic operator $\mathcal{B}_{\pm}^{*i,j}(z)$. The commutation relation between $E_j^{\pm i}(z)$ and $\mathcal{B}_{\pm}^{*k,l}(z)$ are complicated. Hence we prepare auxiliary operators $\tilde{\mathcal{B}}_{+}^{*i,j}(z)$ which commute with at most every Drinfeld current $E_l^{+,k}(z)$:

$$[\tilde{\mathcal{B}}_{+}^{*j,i+1}(z_1), E_j^{+,i}(z_2)] \neq 0\,, \quad [\tilde{\mathcal{B}}_{+}^{*j,i+1}(z_1), E_l^{+,k}(z_2)] = 0\,, \quad \text{for} \quad (k,l) \neq (i,j)\,. \tag{A.1}$$

For example, the explicit formulae of $\tilde{\mathcal{B}}_{+}^{*i,j}(z)$ for $U_{q,p}(\widehat{\mathrm{sl}_4})$ are given as following

$$\begin{aligned}
\tilde{\mathcal{B}}_{+}^{*1,2}(z) &= \mathcal{B}_{+}^{*1,2}(z)\mathcal{B}_{+}^{*1,3}(q^{-1}z)\mathcal{B}_{+}^{*1,4}(q^{-2}z)\,,\\
\tilde{\mathcal{B}}_{+}^{*1,3}(z) &= \mathcal{B}_{+}^{*1,3}(z)\mathcal{B}_{+}^{*1,4}(q^{-1}z)\mathcal{B}_{+}^{*2,3}(q^{-1}z)\mathcal{B}_{+}^{*2,4}(q^{-2}z)\mathcal{B}_{-}^{*2,3}(qz)\mathcal{B}_{-}^{*2,4}(z)\,,\\
\tilde{\mathcal{B}}_{+}^{*2,3}(z) &= \mathcal{B}_{+}^{*2,3}(z)\mathcal{B}_{+}^{*2,4}(q^{-1}z)\,,\\
\tilde{\mathcal{B}}_{+}^{*1,4}(z) &= \mathcal{B}_{+}^{*1,4}(z)\mathcal{B}_{+}^{*2,4}(q^{-1}z)\mathcal{B}_{+}^{*3,4}(q^{-2}z)\mathcal{B}_{-}^{*2,4}(qz)\mathcal{B}_{-}^{*3,4}(z)\,,\\
\tilde{\mathcal{B}}_{+}^{*2,4}(z) &= \mathcal{B}_{+}^{*2,4}(z)\mathcal{B}_{+}^{*3,4}(q^{-1}z)\mathcal{B}_{-}^{*3,4}(qz)\,,\\
\tilde{\mathcal{B}}_{+}^{*3,4}(z) &= \mathcal{B}_{+}^{*3,4}(z)\,.
\end{aligned}$$

1 The remaining noncommutative commutation relation is given by

$$E_j^{+i-1}(z_1)\mathcal{B}_{+}^{*i,j}(q^{j-1}z_1) = \frac{(p^*q^{-1}z_2/z_1;p^*)_\infty}{(p^*qz_2/z_1;p^*)_\infty}\mathcal{B}_{+}^{*i,j}(q^{j-1}z_1)E_j^{+j-1}(z_1)\,.$$

For simplicity, we demonstrate this construction in $U_{q,p}(\widehat{\mathrm{sl}_4})$ case. The commutation relation between $:e^{\beta_2^{i,j}(z_1)}:$ and $\mathcal{B}_{\pm}^{*i,j}(z_2)$ is exactly the same as those between $:e^{\beta_1^{i,j}(z_1)}:$ and $\mathcal{B}_{\pm}^{*i,j}(z_2)$. Hence, in what follows, we can regard

$$\begin{aligned}
E_1^{+,1}(z) &\sim\, :e^{\beta_1^{12}(z)}:\,,\\
E_1^{+,2}(z) &\sim\, :e^{\gamma^{12}(z)+\beta_1^{13}(z)}:\,,\\
E_2^{+,2}(z) &\sim\, :e^{\beta_1^{23}(qz)+b_+^{13}(z)-b_+^{12}(qz)}:\,,\\
E_1^{+,3}(z) &\sim\, :e^{\gamma^{13}(z)+\beta_1^{14}(z)}:\,,\\
E_2^{+,3}(z) &\sim\, :e^{\gamma^{23}(qz)+\beta_1^{24}(qz)+b_+^{14}(z)-b_+^{13}(qz)}:\,,\\
E_3^{+,3}(z) &\sim\, :e^{\beta_1^{34}(q^2z)+b_+^{14}(z)-b_+^{13}(qz)+b_+^{24}(qz)-b_+^{23}(q^2z)}:\,.
\end{aligned}$$

There exists lexicographical ordering structure for index (i,j) of $b_m^{i,j}$ inside $E^{+,i}(z)$. Hence we assume the formulae of $\tilde{\mathcal{B}}_{+}^{*i,j}(z)$ as follows:

$$\tilde{\mathcal{B}}_{+}^{*i,j}(z) = \mathcal{B}_{+}^{*i,j}(z) \times \prod_{\substack{(k,l)\\(i,j)<(k,l)}} \mathcal{B}_{+}^{*k,l}\big(q^{m_{k,l}^{+}}z\big)^{\epsilon_{k,l}^{+}}\mathcal{B}_{-}^{*k,l}\big(q^{m_{k,l}^{-}}z\big)^{\epsilon_{k,l}^{-}}\,. \tag{A.2}$$

Here $m_{k,l}^{\pm} \in \mathbb{Z}$ and $\epsilon_{k,l}^{\pm} \in \mathbb{N}$. Here $(i,j) < (k,l)$ means the lexicographical ordering, i.e. $(1,2) < (1,3) < (1,4) < (2,3) < (2,4) < (3,4)$.

- Let us determine $\tilde{\mathcal{B}}_+^{*12}(z) = \mathcal{B}_+^{*12}(z) \times \cdots$. In order to satisfy the commutativity $[\mathcal{B}_+^{*12}(z_1), E_1^{+,2}(z_2)] = 0$, the auxiliary operator should be $\tilde{\mathcal{B}}_+^{*12}(z) = \mathcal{B}_+^{*12}(z)\mathcal{B}_+^{*13}(q^{-1}z) \times \cdots$. Upon this assumption, the commutativity $[\mathcal{B}_+^{*12}(z_1), E_2^{+,2}(z_2)] = 0$ holds automatically. In order to satisfy the commutativity $[\mathcal{B}_+^{*12}(z_1), E_1^{+,3}(z_2)] = 0$, the auxiliary operator should be $\tilde{\mathcal{B}}_+^{*12}(z) = \mathcal{B}_+^{*12}(z)\mathcal{B}_+^{*13}(q^{-1}z)\mathcal{B}_+^{*14}(q^{-2}z) \times \cdots$. Upon this assumption, the commutation relation $[\mathcal{B}_+^{*12}(z_1), E_2^{+,3}(z_2)] = 0$ and $[\mathcal{B}_+^{*12}(z_1), E_3^{+,3}(z_2)] = 0$ hold automatically. Hence we conclude $\tilde{\mathcal{B}}_+^{*12}(z) = \mathcal{B}_+^{*12}(z)\mathcal{B}_+^{*13}(q^{-1}z)\mathcal{B}_+^{*14}(q^{-2}z)$. The auxiliary operator $\tilde{\mathcal{B}}_+^{*i,i+1}(z)$ is determined as the same manner.
- Let us determine $\tilde{\mathcal{B}}_+^{*13}(z) = \mathcal{B}_+^{*13}(z) \times \cdots$. Because of the assumption (A.2), commutativity $[\tilde{\mathcal{B}}_+^{*13}(z_1), E_1^{+,1}(z_2)] = 0$ holds. In order to satisfy the commutativity $[\tilde{\mathcal{B}}_+^{*13}(z_1), E_2^{+,2}(z_2)] = 0$, the auxiliary operator should be $\tilde{\mathcal{B}}_+^{*13}(z) = \mathcal{B}_+^{*13}(z)\mathcal{B}_+^{*23}(q^{-1}z)\mathcal{B}_-^{*23}(qz) \times \cdots$. In order to satisfy the commutativity $[\tilde{\mathcal{B}}_+^{*13}(z_1), E_1^{+,3}(z_2)] = 0$, the dressing operator should be $\tilde{\mathcal{B}}_+^{*13}(z) = \mathcal{B}_+^{*13}(z)\mathcal{B}_+^{*23}(q^{-1}z)\mathcal{B}_-^{*23}(qz)\mathcal{B}_+^{*14}(q^{-1}z) \times \cdots$. In order to satisfy the commutativity $[\tilde{\mathcal{B}}_+^{*13}(z_1), E_2^{+,3}(z_2)] = 0$, the auxiliary operator should be $\tilde{\mathcal{B}}_+^{*13}(z) = \mathcal{B}_+^{*13}(z)\mathcal{B}_+^{*23}(q^{-1}z)\mathcal{B}_-^{*23}(qz)\mathcal{B}_+^{*14}(q^{-1}z)\mathcal{B}_+^{*24}(q^{-2}z)\mathcal{B}_-^{*24}(z) \times \cdots$. Upon these assumptions, the commutativity $[\tilde{\mathcal{B}}^{*13}(z_1), E_3^{+,3}(z_2)] = 0$ holds, automatically. Hence we conclude $\tilde{\mathcal{B}}_+^{*13}(z) = \mathcal{B}_+^{*13}(z)\mathcal{B}_+^{*23}(q^{-1}z)\mathcal{B}_-^{*23}(qz)\mathcal{B}_+^{*14}(q^{-1}z)\mathcal{B}_+^{*24}(q^{-2}z)\mathcal{B}_-^{*24}(z)$. The auxiliary operator $\tilde{\mathcal{B}}_+^{*i,i+2}(z)$ is determined in the same manner.
- Let us determine $\tilde{\mathcal{B}}_+^{*14}(z) = \mathcal{B}_+^{*14}(z) \times \cdots$. Because of the assumption (A.2), the commutation relations $[\tilde{B}_+^{*14}(z_1), E^{+1}(z_2)] = [\tilde{B}_+^{*14}(z_1), E^{+2}(z_2)] = 0$ hold. In order to satisfy the commutativity $[\tilde{\mathcal{B}}_+^{*14}(z_1), E_2^{+,3}(z_2)] = 0$, the auxiliary operator should be $\tilde{\mathcal{B}}_+^{*14}(z) = \mathcal{B}_+^{*14}(z)\mathcal{B}_+^{*24}(q^{-1}z)\mathcal{B}_-^{*24}(qz) \times \cdots$. In order to satisfy the commutativity $[\tilde{\mathcal{B}}_+^{*14}(z_1), E_3^{+,3}(z_2)] = 0$, the dressing operator should be $\tilde{\mathcal{B}}_+^{*14}(z) = \mathcal{B}_+^{*14}(z)\mathcal{B}_+^{*24}(q^{-1}z)\mathcal{B}_-^{*24}(qz)\mathcal{B}_+^{*34}(q^{-2}z)\mathcal{B}_-^{*34}(z)$. The dressing operator $\tilde{\mathcal{B}}_+^{*i,i+3}(z)$ is determined as the same manner.

We have determined the auxiliary operators $\tilde{\mathcal{B}}_+^{i,j}(z)$ for $U_{q,p}(\widehat{\mathrm{sl}_4})$.

As you have seen the above, the lexicographical ordering structure inside $E^{+,i}(z)$ plays an important role in construction of the auxiliary operator $\tilde{\mathcal{B}}_+^{*ij}(z)$. As the same manner as the above, we have the explicit formulae of the auxiliary operator $\mathcal{B}_+^{*i,j}(z)$ for the elliptic algebra $U_{q,p}(\widehat{\mathrm{sl}_N})$ as follows:

$$\tilde{\mathcal{B}}_+^{*i,j}(z) = \prod_{s=i}^{j-1}\prod_{t=j}^{N} \mathcal{B}_+^{*s,t}(q^{i+j-s-t}z) \prod_{s=i+1}^{j-1}\prod_{t=j}^{N} \mathcal{B}_-^{*s,t}(q^{i+j+2-s-t}z)\,, \tag{A.3}$$

We have the commutation relation

$$\begin{aligned}&\Theta_{p^*}(q^{-1}z_1/z_2)E_j^{+,i}(z_1)\tilde{\mathcal{B}}_+^{*i+1,j}(z_1)E_j^{+,i}(z_2)\tilde{\mathcal{B}}^{*i+1,j}(z_2)\\ &\quad = q^{-1}\Theta_{p^*}(qz_1/z_2)E_j^{+,i}(z_2)\tilde{\mathcal{B}}_+^{*i+1,j}(z_2)E_j^{+,i}(z_1)\tilde{\mathcal{B}}^{*i+1,j}(z_1)\,.\end{aligned} \tag{A.4}$$

Let us set the auxiliary operator $\tilde{\mathcal{B}}^{*i,j}_{-}(z) = \prod_{s=i}^{j-1}\prod_{t=j}^{N}\mathcal{B}^{*s,t}_{-}(q^{i+j-s-t}z)\prod_{s=i+1}^{j-1}\prod_{t=j}^{N}\mathcal{B}^{*s,t}_{+}(q^{i+j+2-s-t}z)$. Considering Eq. (A.4) and the structure of Cartan matrix of the classical sl_N, we set the dressing operator

$$\tilde{U}^{*i}(z) = \prod_{j=1}^{i}\tilde{B}^{*j,i+1}_{+}(q^{j}z)\tilde{B}^{*j,i+1}_{+}(q^{j-2}z)\prod_{j=1}^{i-1}\tilde{B}^{*j,i}_{-}(q^{j-1}z)\prod_{j=1}^{i+1}\tilde{B}^{*j,i+1}_{-}(q^{j-1}z)\,. \quad \text{(A.5)}$$

Let us set $\tilde{e}_i(z) = \tilde{U}^{*i}(z)E^{+,i}(z)$ $(1 \leqq i \leqq N-1)$. We have the commutation relations

$$\Theta_{p^*}(q^{-A_{i,j}}z_1/z_2)\tilde{e}_i(z_1)\tilde{e}_j(z_2) = q^{-A_{i,j}}\Theta_{p^*}(q^{A_{i,j}}z_1/z_2)\tilde{e}_j(z_2)\tilde{e}_i(z_1)\,.$$

Clearing up overlap, we have

$$\tilde{U}^{*i}(z) = \left(\prod_{j=1}^{i-1}\mathcal{B}^{*j,i+1}_{+}(q^{2-j}z)\mathcal{B}^{*j,i}_{-}(q^{1-j}z)\right)\mathcal{B}^{*i,i+1}_{+}(q^{2-i}z)\mathcal{B}^{*i,i+1}_{+}(q^{-i}z)$$
$$\times\left(\prod_{j=i+2}^{N}\mathcal{B}^{*i,j}_{+}(q^{-j+1}z)\mathcal{B}^{*i+1,j}_{-}(q^{-j+2}z)\right).$$

Next we consider the dressing operator $U^i(z)$. The structure of $E^{-,i}(z)$ is more complicated than those of $E^{+,i}(z)$. It is difficult to use lexicographical ordering structure for $E^{-,i}(z)$. Now let us go back to the explicit formulae of the dressing operator for $U_{q,p}(\widehat{\mathrm{sl}_3})$, (4.11)–(4.14).[21] There exists "duality" relation $\mathcal{B}^{*i,j}_{\pm}(q^s z) \leftrightarrow \mathcal{B}^{i,j}_{\mp}(q^{-s}z)$ between the dressing operators $U^{*i}(z)$ and $U^i(z)$ for $U_{q,p}(\widehat{\mathrm{sl}_3})$. Hence we set

$$\tilde{U}^{i}(z) = \left(\prod_{j=1}^{i-1}\mathcal{B}^{j,i+1}_{-}(q^{-2+j}z)\mathcal{B}^{j,i}_{+}(q^{-1+j}z)\right)\mathcal{B}^{i,i+1}_{-}(q^{-2+i}z)\mathcal{B}^{i,i+1}_{-}(q^{i}z)$$
$$\times\left(\prod_{j=i+2}^{N}\mathcal{B}^{i,j}_{-}(q^{j-1}z)\mathcal{B}^{i+1,j}_{+}(q^{j-2}z)\right).$$

Let us set $e_i(z)$, $f_i(z)$, $\Psi^{\pm}_i(z)$ $(1 \leqq i \leqq N-1)$ by

$$e_i(z) = U^{*i}(z)E^{+,i}(z)\,,$$
$$f_i(z) = E^{-,i}(z)U^{i}(z)\,,$$
$$\Psi^{+}_i(z) = U^{*i}\left(q^{\frac{k}{2}}z\right)\psi^{+}_i(z)U^{i}\left(q^{-\frac{k}{2}}z\right),$$
$$\Psi^{-}_i(z) = U^{*i}\left(q^{-\frac{k}{2}}z\right)\psi^{-}_i(z)U^{i}\left(q^{\frac{k}{2}}z\right).$$

where we have set

$$U^{*i}(z) = \tilde{U}^{*i}(z)\mathcal{A}^{*i}(q^{s_i}z)\,, \quad U^{i}(z) = \tilde{U}^{i}(z)\mathcal{A}^{i}(q^{-s_i}z) \quad (s_i \in \mathbb{R})\,.$$

By necessary condition on commutation relations we get the parameters $s_i = \frac{k-N}{2}$. Now we have got conjecturous formulae of the dressing operators $U^{*i}(z)$ and $U^i(z)$. Using App. B, we can show every commutation relations of $e_i(z)$, $f_i(z)$ and $\Psi_i^{\pm}(z)$, by direct calculation. It seems that the method explained above can be applied to the elliptic algebra $U_{q,p}(\mathfrak{g})$ for arbitrary $\mathfrak{g}$.

Appendix B. Normal Ordering

In this appendix we summarize the normal ordering of the basic operators:

$$:e^{\beta_1^{i,j}(z_1)}:\mathcal{B}_+^{*i,j}(z_2) = :: \frac{(q^{2r^*-1}z_2/z_1;q^{2r^*})_\infty}{(q^{2r^*+1}z_2/z_1;q^{2r^*})_\infty},$$

$$:e^{\beta_2^{i,j}(z_1)}:\mathcal{B}_+^{*i,j}(z_2) = :: \frac{(q^{2r^*-1}z_2/z_1;q^{2r^*})_\infty}{(q^{2r^*+1}z_2/z_1;q^{2r^*})_\infty},$$

$$:e^{\beta_3^{i,j}(z_1)}:\mathcal{B}_+^{*i,j}(z_2) = :: \frac{(q^{2r^*-1}z_2/z_1;q^{2r^*})_\infty}{(q^{2r^*-3}z_2/z_1;q^{2r^*})_\infty},$$

$$:e^{\beta_4^{i,j}(z_1)}:\mathcal{B}_+^{*i,j}(z_2) = :: \frac{(q^{2r^*-1}z_2/z_1;q^{2r^*})_\infty}{(q^{2r^*-3}z_2/z_1;q^{2r^*})_\infty},$$

$$:e^{\gamma^{i,j}(z_1)}:\mathcal{B}_+^{*i,j}(z_2) = :: \frac{(q^{2r^*}z_2/z_1;q^{2r^*})_\infty}{(q^{2r^*-2}z_2/z_1;q^{2r^*})_\infty},$$

$$:e^{b_+^{i,j}(z_1)}:\mathcal{B}_+^{*i,j}(z_2) = :: \frac{(q^{2r^*-1}z_2/z_1;q^{2r^*})_\infty^2}{(q^{2r^*+1}z_2/z_1;q^{2r^*})_\infty(q^{2r^*-3}z_2/z_1;q^{2r^*})_\infty},$$

$$\mathcal{B}_-^{i,j}(z_1):e^{\beta_1^{i,j}(z_2)}: = :: \frac{(q^{2r-k-1}z_2/z_1;q^{2r})_\infty}{(q^{2r-k+1}z_2/z_1;q^{2r})_\infty},$$

$$\mathcal{B}_-^{i,j}(z_1):e^{\beta_2^{i,j}(z_2)}: = :: \frac{(q^{2r-k-1}z_2/z_1;q^{2r})_\infty}{(q^{2r-k+1}z_2/z_1;q^{2r})_\infty},$$

$$\mathcal{B}_-^{i,j}(z_1):e^{\beta_3^{i,j}(z_2)}: = :: \frac{(q^{2r-k-1}z_2/z_1;q^{2r})_\infty}{(q^{2r-k-3}z_2/z_1;q^{2r})_\infty},$$

$$\mathcal{B}_-^{i,j}(z_1):e^{\beta_4^{i,j}(z_2)}: = :: \frac{(q^{2r-k-1}z_2/z_1;q^{2r})_\infty}{(q^{2r-k-3}z_2/z_1;q^{2r})_\infty},$$

$$\mathcal{B}_-^{i,j}(z_1):e^{\gamma^{i,j}(z_2)}: = :: \frac{(q^{2r-k}z_2/z_1;q^{2r})_\infty}{(q^{2r-k-2}z_2/z_1;q^{2r})_\infty},$$

$$\mathcal{B}_-^{i,j}(z_1):e^{b_-^{i,j}(z_2)}: = :: \frac{(q^{2r-k-1}z_2/z_1;q^{2r})_\infty^2}{(q^{2r-k+1}z_2/z_1;q^{2r})_\infty(q^{2r-k-3}z_2/z_1;q^{2r})_\infty},$$

$$\mathcal{A}^i(z_1):e^{a_-^j(z_2)}: = :: \frac{(q^{2r+N+A_{i,j}}z_2/z_1;q^{2r})_\infty(q^{2r-2k-N-A_{i,j}}z_2/z_1;q^{2r})_\infty}{(q^{2r+N-A_{i,j}}z_2/z_1;q^{2r})_\infty(q^{2r-2k-N+A_{i,j}}z_2/z_1;q^{2r})_\infty},$$

$$:e^{a^i_+(z_1)}:\mathcal{A}^{*j}(z_2) = :: \frac{(q^{2r^*+N+A_{i,j}}z_2/z_1;q^{2r^*})_\infty (q^{2r^*-2k-N-A_{i,j}}z_2/z_1;q^{2r^*})_\infty}{(q^{2r^*+N-A_{i,j}}z_2/z_1;q^{2r^*})_\infty (q^{2r^*-2k-N+A_{i,j}}z_2/z_1;q^{2r^*})_\infty}\,,$$

$$\mathcal{B}^{i,j}_-(z_1)\mathcal{B}^{*i,j}_+(z_2) = :: \frac{(q^k z_2/z_1;q^{2k},p^*)^2_\infty}{(q^{k+2}z_2/z_1;q^{2k},q^{2r^*})_\infty (q^{k-2}z_2/z_1;q^{2k},q^{2r^*})_\infty} \times \frac{(q^{k+2}z_2/z_1;q^{2k},q^{2r})_\infty (q^{k-2}z_2/z_1;q^{2k},q^{2r})_\infty}{(q^k z_2/z_1;q^{2k},q^{2r})^2_\infty}\,,$$

$$\mathcal{A}^i(z_1)\mathcal{A}^{*j}(z_2) = :: \frac{(q^{2k+N+A_{i,j}}z_2/z_1;q^{2k},q^{2r^*})_\infty (q^{-N-A_{i,j}}z_2/z_1;q^{2k},q^{2r^*})_\infty}{(q^{2k+N-A_{i,j}}z_2/z_1;q^{2k},q^{2r^*})_\infty (q^{-N+A_{i,j}}z_2/z_1;q^{2k},q^{2r^*})_\infty} \times \frac{(q^{2k+N-A_{i,j}}z_2/z_1;q^{2k},q^{2r})_\infty (q^{-N+A_{i,j}}z_2/z_1;q^{2k},q^{2r})_\infty}{(q^{2k+N+A_{i,j}}z_2/z_1;q^{2k},q^{2r})_\infty (q^{-N-A_{i,j})}z_2/z_1;q^{2k},q^{2r})_\infty}\,.$$

Here we have used the notation

$$(z;p_1,p_2)_\infty = \prod_{n_1,n_2=0}^{\infty}(1-p_1^{n_1}p_2^{n_2}z)\,.$$

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