# The Intergals of Motion for the Deformed $W$-Algebra $W_{q,t}(\widehat{sl_N})$

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**Abstract**

We review the deformed $W$-algebra $W_{q,t}(\widehat{sl_N})$ and its screening currents. We explicitly construct the Local Integrals of Motion $\mathcal{I}_n$ $(n = 1, 2, \cdots)$ for this deformed $W$-algebra. We explicitly construct the Nonlocal Integrals of Motion $\mathcal{G}_n$ $(n = 1, 2, \cdots)$ by means of screening currents. The Integrals of Motion $\mathcal{I}_n$ and $\mathcal{G}_n$ commute with each other. Our Integrals of Motion give the elliptic version of those of the Conformal Field Theory for the Virasoro algebra $Vir$ [1] and those for the $W$-algebra $W(\widehat{sl_3})$ [2].

# 1 Introduction

V.Bazhanov, S.Lukyanov and B.Zamolodchikov [1] constructed an infinite set of commuting Hamiltonians for Conformal Field Theory, which have the form,

$$I_{2k-1}=\int_0^{2\pi}\frac{du}{2\pi}T_{2k}(u),\quad (k=1,2,\cdots). \tag{1.1}$$

Here the densities $T_{2k}(u)$ are differential polynomials of the energy momentum tensor of the Virasoro algebra $T(u)=-\frac{C_{CFT}}{24}+\sum_{n=-\infty}^{\infty}L_{-n}e^{\sqrt{-1}nu}$, where the operators $L_n$ satisfies the commutation relation, $[L_n,L_m]=(n-m)L_{n+m}+\frac{C_{CFT}}{12}(n^3-n)\delta_{n+m,0}$. The first few densities $T_{2k}(u)$ are written by

$$T_2(u)=T(u),\ T_4(u)=:T^2(u):,\ T_6(u)=:T^3(u):+\frac{C_{CFT}+2}{12}:(T'(u))^2:. \tag{1.2}$$

However they did not give explicit formulae of general densities $T_{2k}$, all densities $T_{2k}(u)$ are uniequely determined by requirement of the commutativity, $[I_{2k-1},I_{2l-1}]=0,\ (k,l=1,2,\cdots)$. We call the operators $I_{2k-1}$ the Local Integrals of Motion. V.Bazhanov, S.Lukyanov and B.Zamolodchikov [1] constructed another infinite set of commuting operators $G_k,\ (k=1,2,\cdots)$, by means of the screening currents, $F_1(u),F_2(u)$. In this case, they gave explicit formulae for every $G_k,\ (k=1,2,\cdots)$, satisfying the commutation relations $[G_k,G_l]=0,\ (k,l=1,2,\cdots)$.

$$\begin{aligned}G_k &= \int\cdots\int_{2\pi\geqq u_1\geqq u_2\geqq\cdots\geqq u_{2k}\geqq 0}(e^{2\pi\sqrt{-1}P}F_1(u_1)F_2(u_2)F_1(u_3)F_2(u_4)\cdots F_2(u_{2k})\\ &+ e^{-2\pi\sqrt{-1}P}F_2(u_1)F_1(u_2)F_2(u_3)F_1(u_4)\cdots F_1(u_{2k}))du_1du_2\cdots du_{2k},\end{aligned} \tag{1.3}$$

where $P$ is zero-mode operator. We call the operators $G_k$ the Nonlocal Integrals of Motion. They conjectured commutativity $[I_{2k-1},G_l]=0,\ (k,l=1,2,\cdots)$.

V.Bazhanov, S.Lukyanov and B. Zamolodchikov's theory [1] can be thought of as the quantum version of the KdV problem, as it reduces to the classical KdV problem in classical limit $C_{CFT}\to-\infty$, under the substitution

$$T(u)\to-\frac{C_{CFT}}{6}U(u),\quad [\ ,\ ]\to\frac{6\pi}{\sqrt{-1}C_{CFT}}\{\ ,\ \}. \tag{1.4}$$

The Poisson bracket structure $\{,\}$ gives the second Hamiltonian structure of the KdV equation. The Local Integrals of Motion $I_{2k-1}$ tend to $I_{2k-1}^{(cl)}$.

$$I_{2k-1}^{(cl)}=\int_u^{2\pi}\frac{du}{2\pi}T_{2k}^{(cl)}(u),\quad (k=1,2,\cdots), \tag{1.5}$$

where the first few densities are written by

$$T_2^{(cl)}=U(u),\quad T_4^{(cl)}=U^2(u),\quad T_6^{(cl)}=U^3(u)-\frac{1}{2}(U'(u))^2. \tag{1.6}$$

The KdV hierarchy are given by

$$\partial_{t_{2k-1}}U=\{I_{2k-1}^{(cl)},U\},\ (k=1,2,\cdots). \tag{1.7}$$

In this paper we study the elliptic deformation of the Conformal Field Theory given by V.Bazhanov, S.Lukyanov and B.Zamolodchikov [1], and V.Bazhanov, A.Hibberd and S.Khoroshkin [2]. We construct explicit formulae of both Local Integrals of Motion $\mathcal{I}_n$ and the Nonlocal Integrals of Motion $\mathcal{G}_m$, for the deformed $W$-algebra $W_{q,t}(\widehat{sl_N})$.

$$[\mathcal{I}_m,\mathcal{I}_n]=0,\quad [\mathcal{G}_m,\mathcal{G}_n]=0,\quad [\mathcal{I}_m,\mathcal{G}_n]=0,\quad (m,n=1,2,\cdots). \tag{1.8}$$

Our Integrals of Motion give the elliptic version of those of the Conformal Field Theory for the Virasoro algebra [1] and $W$-algebra $W(\widehat{sl_3})$ [2].

The organization of this paper is as follows. In section 2, we give basic definition, including bosons, screening currents. In section 3, we review the deformed $W$-algebra $W_{q,t}(\widehat{sl_N})$. In section 4, we construct explicit formulae for the Local Integrals of Motion $\mathcal{I}_n,\ (n=1,2,\cdots)$ for the deformed $W$-algebra $W_{q,t}(\widehat{sl_N})$. In section 5, we construct explicit formulae for the Nonlocal Integrals of Motion $\mathcal{G}_m,\ (m=1,2,\cdots)$.

# 2 Basic Definition

In this section we give the basic definition. Let us fix three parameters $0<x<1, r\in\mathbb{C}$ and $s\in\mathbb{C}$.

## 2.1 Bosons

Let $\epsilon_i(1\leqq i\leqq N)$ be an orthonormal basis in $\mathbb{R}^N$ relative to the standard basis in $\mathbb{R}^N$ relative to the standard inner product $(,)$. Let us set $\bar{\epsilon}_i=\epsilon_i-\epsilon, \epsilon=\frac{1}{N}\sum_{j=1}^N\epsilon_j$. We idetify $\epsilon_{N+1}=\epsilon_1$. Let $P=\sum_{i=1}^N\mathbb{Z}\bar{\epsilon}_i$ the weight lattice. Let us set $\alpha_i=\bar{\epsilon}_i-\bar{\epsilon}_{i+1}\in P$.

Let $\beta_m^j$ be the oscillators $(1\leqq j\leqq N-1, m\in\mathbb{Z}-\{0\})$ with the commutation relations

$$[\beta_m^i,\beta_n^j]=\left\{\begin{array}{ll} m\frac{[(r-1)m]}{[rm]}\frac{[(s-1)m]}{[sm]}\delta_{n+m,0} & (1\leqq i=j\leqq N)\\ -m\frac{[(r-1)m]}{[rm]}\frac{[m]}{[sm]}x^{sm\ \mathrm{sgn}(i-j)}\delta_{n+m,0} & (1\leqq i\neq j\leqq N)\end{array}\right. \tag{2.1}$$

Here the symbol $[a]$ stands for $\frac{x^a-x^{-a}}{x-x^{-1}}$.

We also introduce the zero mode operator $P_\lambda$, $(\lambda \in P)$. They are $\mathbb{Z}$-linear in $\lambda$ and satisfy

$$[iP_\lambda, Q_\mu] = (\lambda, \mu), \quad (\lambda, \mu \in P). \tag{2.2}$$

Let us intrduce the bosonic Fock space $\mathcal{F}_{l,k}(l, k \in P)$ generated by $\beta^j_{-m}(m > 0)$ over the vacuum vector $|l, k\rangle$ :

$$\mathcal{F}_{l,k} = \mathbb{C}[\{\beta^j_{-1}, \beta^j_{-2}, \cdots\}_{1 \leqq j \leqq N}]|l, k\rangle, \tag{2.3}$$

where

$$\beta^j_m|l, k\rangle = 0, (m > 0), \tag{2.4}$$

$$P_\alpha|l, k\rangle = \left(\alpha, \sqrt{\frac{r}{r-1}}l - \sqrt{\frac{r-1}{r}}k\right)|l, k\rangle, \tag{2.5}$$

$$|l, k\rangle = e^{i\sqrt{\frac{r}{r-1}}Q_l - i\sqrt{\frac{r-1}{r}}Q_k}|0, 0\rangle. \tag{2.6}$$

Let us set the Dynkin-diagram automorphism $\eta$ by

$$\eta(\beta^1_m) = x^{-\frac{2s}{N}m}\beta^2_m, \cdots, \eta(\beta^{N-1}_m) = x^{-\frac{2s}{N}m}\beta^N_m, \ \eta(\beta^N_m) = x^{\frac{2s}{N}(N-1)m}\beta^1_m, \tag{2.7}$$

and $\eta(\epsilon_i) = \epsilon_{i+1}, \ (1 \leqq i \leqq N)$.

## 2.2 Basic Operators

In this section we introduce the basic operators. Let us set $z = x^{2u}$.

**Definition 2.1** *We set the screening currents $F_j(z)(1 \leqq j \leqq N)$ by*

$$F_j(z) = e^{i\sqrt{\frac{r-1}{r}}Q_{\alpha_j}}(x^{(\frac{2s}{N}-1)j}z)^{\sqrt{\frac{r-1}{r}}P_{\alpha_j} + \frac{r-1}{r}} \times :\exp\left(\sum_{m \neq 0} \frac{1}{m}B^j_m z^{-m}\right):, \ (1 \leqq j \leqq N-1) \tag{2.8}$$

$$F_N(z) = e^{i\sqrt{\frac{r-1}{r}}Q_{\alpha_N}}(x^{2s-N}z)^{\sqrt{\frac{r-1}{r}}P_{\epsilon_N} + \frac{r-1}{2r}}(z)^{-\sqrt{\frac{r-1}{r}}P_{\bar{\epsilon}_1} + \frac{r-1}{2r}} \times :\exp\left(\sum_{m \neq 0} \frac{1}{m}B^N_m z^{-m}\right):. \tag{2.9}$$

*Here we set*

$$B^j_m = (\beta^j_m - \beta^{j+1}_m)x^{-\frac{2s}{N}jm}, \ (1 \leqq j \leqq N-1), \tag{2.10}$$

$$B^N_m = (x^{-2sm}\beta^N_m - \beta^1_m). \tag{2.11}$$

The screening currents $F_j(z)$, $(1 \leqq j \leqq N-1)$ are studied well in [7]. We introduce new current $F_N(z)$, which can be regarded as "affinization" of screenings $F_j(z)$, $(1 \leqq j \leqq N-1)$.

In what follows, the symbol $[u]_r$ stands for the theta function satisfying

$$[u+r]_r = -[u]_r = [-u]_r, \qquad (2.12)$$

$$[u+\tau]_r = -e^{\frac{2\pi i}{r}(u+\frac{\tau}{2})}[u]_r, \quad \text{where } \tau = \frac{\pi i}{\log x}. \qquad (2.13)$$

Explicitly it is given by

$$[u] = x^{u^2/r-u}\Theta_{x^{2r}}(x^{2u}), \qquad (2.14)$$

$$\Theta_q(z) = (z;q)_\infty (q/z;q)_\infty (q;q)_\infty, \qquad (2.15)$$

$$(z;q)_\infty = \prod_{j=0}^{\infty}(1-zq^j). \qquad (2.16)$$

**Proposition 2.1** *The screening currents $F_j(z)$, $(1 \leqq j \leqq N)$ satisfy the following commutation relations,*

$$\frac{1}{[u_1-u_2-\frac{s}{N}+1]_r}F_j(z_1)F_{j+1}(z_2) = \frac{1}{[u_2-u_1+\frac{s}{N}]_r}F_{j+1}(z_2)F_j(z_1), \quad (1 \leqq j \leqq N), \qquad (2.17)$$

$$\frac{[u_1-u_2]_r}{[u_1-u_2-1]_r}F_j(z_1)F_j(z_2) = \frac{[u_2-u_1]_r}{[u_2-u_1-1]_r}F_j(z_2)F_j(z_1), \quad (1 \leqq j \leqq N), \qquad (2.18)$$

*and*

$$F_i(z_1)F_j(z_2) = F_j(z_2)F_i(z_1), \quad (|i-j| \geqq 2). \qquad (2.19)$$

*We read $F_{N+1}(z) = F_1(z)$.*

**Proposition 2.2** *The action of $\eta$ on the screenings $F_j(z)$ is given by*

$$\eta(F_j(z)) = F_{j+1}(x^{1-\frac{2s}{N}}z), \quad (1 \leqq j \leqq N-2), \qquad (2.20)$$

$$\eta(F_{N-1}(z)) = F_N(x^{1-\frac{2s}{N}}z)x^{(2s-N)(-\sqrt{\frac{r-1}{r}}P_{\bar{\epsilon}_1}+\frac{r-1}{r})}, \qquad (2.21)$$

$$\eta(F_N(z)) = F_1(x^{1-\frac{2s}{N}}z)x^{(2s-N)(\sqrt{\frac{r-1}{r}}P_{\bar{\epsilon}_1}+\frac{r-1}{r})}. \qquad (2.22)$$

*We have $\eta(F_{N-1}(z_1)F_N(z_2)) = F_N(z_1)F_1(z_2)$.*

**Definition 2.2** *We set the fundamental operator* $\Lambda_j(z), (1 \leqq j \leqq N)$ *by*

$$\Lambda_j(z) = x^{-2\sqrt{r(r-1)}P_{\bar{\epsilon}_j}} : \exp\left(\sum_{m\neq 0} \frac{x^{rm}-x^{-rm}}{m}\beta_m^j z^{-m}\right) : \quad (1 \leqq j \leqq N). \quad (2.23)$$

**Proposition 2.3** *The action of* $\eta$ *on the screenings* $\Lambda_j(z)$ *is given by*

$$\eta(\Lambda_j(z)) = \Lambda_{j+1}(xz), \quad (1 \leqq j \leqq N-1), \quad \eta(\Lambda_N(z)) = \Lambda_1(x^{1-2s}z). \quad (2.24)$$

**Proposition 2.4** *The screening currents* $F_j(z)$, $(1 \leqq j \leqq N)$ *and the fundamental operators* $\Lambda_j(z)$, $(1 \leqq j \leqq N)$ *commute up to delta-function* $\delta(z) = \sum_{n\in\mathbb{Z}} z^m$.

$$[\Lambda_j(z_1), F_j(z_2)] = (-1+x^{-2r+2})\delta\left(x^{\frac{2s}{N}j-r}\frac{z_2}{z_1}\right) : \Lambda_j(z_1)F_j(z_2) :, \quad (1 \leqq j \leqq N), \quad (2.25)$$

$$[\Lambda_{j+1}(z_1), F_j(z_2)] = (1-x^{-2r+2})\delta\left(x^{\frac{2s}{N}j+r}\frac{z_2}{z_1}\right) : \Lambda_{j+1}(z_1)F_j(z_2) :, \quad (1 \leqq j \leqq N-1), \quad (2.26)$$

$$[\Lambda_1(z_1), F_N(z_2)] = (1-x^{-2r+2})\delta\left(x^r\frac{z_2}{z_1}\right) : \Lambda_1(z_1)F_N(z_2) : . \quad (2.27)$$

# 3 Deformed $W$-Algebra $W_{q,t}(\widehat{sl_N})$

In this section we review the deformed $W$-algebra $W_{q,t}(\widehat{sl_N})$ [3, 4, 5].

## 3.1 Deformation of $W$-Algebra

In this section we give the deformation of the $W$-algebra.

**Definition 3.1** *Let us set the operator* $T_j(z)$, $(1 \leqq j \leqq N)$ *by*

$$T_j(z) = \sum_{1\leqq s_1<s_2<\cdots<s_j\leqq N} : \Lambda_{s_1}(x^{-j+1})\Lambda_{s_2}(x^{-j+3})\cdots\Lambda_{s_j}(x^{j-1}z) : . \quad (3.1)$$

**Proposition 3.1** *The bosonic operators* $T_j(z)$, $(1 \leqq j \leqq N)$ *satisfy the following relations.*

$$\begin{aligned} & f_{i,j}(z_2/z_1)T_i(z_1)T_j(z_2) - f_{j,i}(z_1/z_2)T_j(z_2)T_i(z_1) \\ = \; & c\sum_{k=1}^{i}\prod_{l=1}^{k-1}\Delta(x^{2l+1}) \times \left(\delta\left(\frac{x^{j-i+2k}z_2}{z_1}\right) f_{i-k,j+k}(x^{-j+i})T_{i-k}(x^{-k}z_1)T_{j+k}(x^k z_2)\right. \\ - \; & \left.\delta\left(\frac{x^{-j+i-2k}z_2}{z_1}\right) f_{i-k,j+k}(x^{j-i})T_{i-k}(x^k z_1)T_{j+k}(x^{-k}z_2)\right), \quad (1 \leqq i \leqq j \leqq N), (3.2) \end{aligned}$$

*where* $\delta(z)=\sum_{n\in\mathbb{Z}} z^n$.
*Here we set the constnt* $c$ *and the auxiliary function* $\Delta(z)$ *by*

$$c=-\frac{(1-x^{2r})(1-x^{-2r+2})}{(1-x^2)},\quad \Delta(z)=\frac{(1-x^{2r-1}z)(1-x^{1-2r}z)}{(1-xz)(1-x^{-1}z)}. \tag{3.3}$$

*Here we set the structure functions,*

$$f_{i,j}(z)=\exp\left(\sum_{m=1}^{\infty}\frac{1}{m}(1-x^{2rm})(1-x^{-2(r-1)m})\frac{(1-x^{2mMin(i,j)})(1-x^{2m(s-Max(i,j))})}{(1-x^{2m})(1-x^{2sm})}x^{|i-j|m}z^m\right). \tag{3.4}$$

**Example** For $N=2$ the operators $T_1(z),T_2(z)$ satisfy

$$\begin{aligned} & f_{1,1}(z_2/z_1)T_1(z_1)T_1(z_2)-f_{1,1}(z_1/z_2)T_1(z_2)T_1(z_1) \\ = \; & c(\delta(x^2z_2/z_1)T_2(xz_2)-\delta(x^2z_1/z_2)T_2(x^{-1}z_2)), \end{aligned} \tag{3.5}$$

$$f_{1,2}(z_2/z_1)T_1(z_1)T_2(z_2)=f_{2,1}(z_1/z_2)T_2(z_2)T_1(z_1), \tag{3.6}$$

$$f_{2,2}(z_2/z_1)T_2(z_1)T_2(z_2)=f_{2,2}(z_1/z_2)T_2(z_2)T_2(z_1). \tag{3.7}$$

**Example** For $N=3$ the operators $T_1(z),T_2(z),T_3(z)$ satisfy

$$\begin{aligned} & f_{1,1}(z_2/z_1)T_1(z_1)T_1(z_2)-f_{1,1}(z_1/z_2)T_1(z_2)T_1(z_1) \\ = \; & c(\delta(x^2z_2/z_1)T_2(xz_2)-\delta(x^2z_1/z_2)T_2(x^{-1}z_2)), \end{aligned} \tag{3.8}$$

$$\begin{aligned} & f_{1,2}(z_2/z_1)T_1(z_1)T_2(z_2)-f_{2,1}(z_1/z_2)T_2(z_2)T_1(z_1) \\ = \; & c(\delta(x^3z_2/z_1)T_3(xz_2)-\delta(x^3z_1/z_2)T_3(x^{-1}z_2)), \end{aligned} \tag{3.9}$$

$$\begin{aligned} & f_{2,2}(z_2/z_1)T_2(z_1)T_2(z_2)-f_{2,2}(z_1/z_2)T_2(z_2)T_2(z_1) \\ = \; & cf_{1,3}(1)(\delta(x^2z_2/z_1)T_1(xz_2)T_3(xz_2)-\delta(x^2z_1/z_2)T_1(x^{-1}z_2)T_3(x^{-1}z_2)), \end{aligned} \tag{3.10}$$

$$f_{1,3}(z_2/z_1)T_1(z_1)T_3(z_2)=f_{3,1}(z_1/z_2)T_3(z_2)T_1(z_1), \tag{3.11}$$

$$f_{2,3}(z_2/z_1)T_2(z_1)T_3(z_2)=f_{3,2}(z_1/z_2)T_3(z_2)T_2(z_1), \tag{3.12}$$

$$f_{3,3}(z_2/z_1)T_3(z_1)T_3(z_2)=f_{3,3}(z_1/z_2)T_3(z_2)T_3(z_1). \tag{3.13}$$

**Definition 3.2** *The three parameter deformed* $W$*-algebra is an associative algebra generated by* $T_n^{(j)}$, $(n\in\mathbb{Z},1\leqq j\leqq N)$. *Defining relations are given by (3.2). The elements* $T_n^{(j)}$ *are Fourier coefficients of* $T_j(z)=\sum_{n\in\mathbb{Z}}T_n^{(j)}z^{-n}$.

For general $N$, upon specialization $s=N$, the operator $T_N(z)$ degenerate to scalar 1. Let us set parameters $q=x^{2r}$ and $t=x^{2r-2}$.

**Theorem 3.2** *Upon specialization $s = N$, the operator $T_j(z)$ $(1 \leqq j \leqq N)$ give a free field realization of the deformed $W$-algebra $W_{q,t}(\widehat{sl_N})$.*

**Example** For $N = 2$ and $s = 2$, the operator $T_2(z)$ degenerates to scalar 1, and Fourier coefficients of $T_1(z) = \sum_{n \in \mathbb{Z}} T_n z^{-n}$ satisfy the defining relation of the deformed Virasoro algebra $Vir_{q,t} = W_{q,t}(\widehat{sl_2})$ [3].

$$[T_n, T_m] = -\sum_{l=0}^{\infty} f_l (T_{n-l} T_{m+l} - T_{m-l} T_{n+l}) + c((q/t)^n - (t/q)^n) \delta_{n+m,0}, \tag{3.14}$$

where we set the structure constant $f_l$ by $f_{11}(z) = 1 + \sum_{l=1}^{\infty} f_l z^l$. Let us set $q = e^h$ and $t = q^\beta$, $(\beta = \frac{r-1}{r})$, and take the limit $h \to 0$ under the following $h$-expansion,

$$T_n = 2\delta_{n,0} + \beta \left( L_n + \frac{(1-\beta)^2}{4\beta} \delta_{n,0} \right) h^2 + O(h^4), \tag{3.15}$$

we have the defining relation of the Virasoro algebra,

$$[L_m, L_n] = (n-m) L_{m+n} + \frac{C_{CFT}}{12} n(n^2 - 1) \delta_{n+m,0}, \quad C_{CFT} = 1 - \frac{6(1-\beta)^2}{\beta}. \tag{3.16}$$

## 3.2 Central Extension

In this section we study the realization of the deformed $W$-algebra. Let us set the element $\mathcal{C}_m$ by

$$\mathcal{C}_m = \sum_{j=1}^{N} x^{(N-2j+1)m} \beta_m^j. \tag{3.17}$$

This element $\mathcal{C}_m$ is $\eta$-invariant, $\eta(\mathcal{C}_m) = \mathcal{C}_m$. Let us divide $\Lambda_j(z)$ into $\Lambda_j^{DWA}(z)$ and $\mathcal{Z}(z)$.

$$\Lambda_j(z) \ = \ \Lambda_j^{DWA}(z) \mathcal{Z}(z), \quad (1 \leqq j \leqq N), \tag{3.18}$$

where we set

$$\Lambda_j^{DWA}(z) \ = \ x^{-2\sqrt{r(r-1)} P_{\bar{\epsilon}_j}} : \exp \left( \sum_{m \neq 0} \frac{x^{rm} - x^{-rm}}{m} \left( \beta_m^j - \frac{[m]_x}{[Nm]_x} \mathcal{C}_m \right) z^{-m} \right) : \tag{3.19}$$

$$\mathcal{Z}(z) \ = \ : \exp \left( \sum_{m \neq 0} \frac{x^{rm} - x^{-rm}}{m} \frac{[m]_x}{[Nm]_x} \mathcal{C}_m z^{-m} \right) : . \tag{3.20}$$

Let us set

$$T_j^{DWA}(z) = \sum_{1 \leqq s_1 < s_2 < \cdots < s_j \leqq N} : \Lambda_{s_1}^{DWA}(x^{-j+1} z) \Lambda_{s_2}^{DWA}(x^{-j+3} z) \cdots \Lambda_{s_j}^{DWA}(x^{j-1} z) : . \tag{3.21}$$

**Theorem 3.3** *The operators* $T_j^{DWA}(z)$, $(1 \leqq j \leqq N-1)$ *give a free field realization of the deformed* $W$*-algebra* $W_{q,t}(\widehat{sl_N})$.

**Proposition 3.4** *The operators* $T_j^{DWA}(z)$ *and* $\mathcal{Z}(z)$ *commutes with each other.*

$$T_j^{DWA}(z_1)\mathcal{Z}(z_2) = \mathcal{Z}(z_2)T_j^{DWA}(z_1), \quad (1 \leqq j \leqq N-1). \tag{3.22}$$

Three paramedter deformed $W$-algebra defined in the previous subsection (3.2), can be regarded as central extension of $W_{q,t}(\widehat{sl_N})$, therefore, we sometime call three parameter deformed $W$-algebra, "the deformed $W$-algebra $W_{q,t}(\widehat{sl_N})$". In what follows, mainly, we consider three parameter deformed $W$-algebra. It is simpler to show the commutation relations of the Integrals of Motion $[\mathcal{I}_m, \mathcal{I}_n] = [\mathcal{G}_m, \mathcal{G}_n] = [\mathcal{I}_m, \mathcal{G}_n] = 0$ for three parameter $(x, r, s)$ deformed case than those for two parameter $(x, r, s = N)$ deformed case. One additionnal parameter $s$ resolves singularity in the Integrals of Motion, and make problem simpler.

# 4 Local Integrals of Motion

In this section we give explicit formulae of the Local Integrals of Motion $\mathcal{I}_n$.

In what follows we use the notation of the ordered product.

$$\prod_{\overrightarrow{l \in L}} T_j(z_l) = T_j(z_{l_1})T_j(z_{l_2}) \cdots T_j(z_{l_m}), \quad (L = \{l_1, l_2, \cdots, l_m | l_1 < l_2 < \cdots < l_m\}). \tag{4.1}$$

For formal power series $\mathcal{A}(z_1, z_2, \cdots, z_n) = \sum_{k_1, k_2, \cdots, k_n \in \mathbb{Z}} a_{k_1, k_2, \cdots, k_n} z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$, we set the symbol $[\cdots]_{1, z_1, z_2, \cdots, z_n}$.

$$[\mathcal{A}(z_1, z_2, \cdots, z_n)]_{1, z_1, z_2, \cdots, z_n} = a_{0,0,\cdots,0}. \tag{4.2}$$

Let us set the auxiliary function $g_{i,j}(z)$ by fusion of $g_{1,1}(z) = f_{1,1}(z)$.

$$\begin{aligned} g_{i,1}(z) &= g_{1,1}(x^{-i+1}z)g_{1,1}(x^{-i+3}z) \cdots g_{1,1}(x^{i-1}z), \\ g_{i,j}(z) &= g_{i,1}(x^{-j+1}z)g_{i,1}(x^{-j+3}z) \cdots g_{i,1}(x^{j-1}z). \end{aligned} \tag{4.3}$$

**Definition 4.1** *We set the operator* $\mathcal{O}_n(z_1, z_2, \cdots, z_n)$ *by*

$$\mathcal{O}_n(z_1, z_2, \cdots, z_n) = \sum_{\substack{\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_N \geqq 0 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + N\alpha_N = n}} \sum_{\substack{A_1^{(1)}, \cdots, A_{\alpha_1}^{(1)}, \cdots, A_1^{(N)}, \cdots, A_{\alpha_N}^{(N)} \subset \{1, 2, \cdots, n\} \\ |A_j^{(t)}| = t, \quad \oplus_{j,t} A_j^{(t)} = \{1, 2, \cdots, n\}}}$$

$$\times \prod_{\overrightarrow{j\in A_{Min}^{(1)}}} T_1(z_j) \prod_{\overrightarrow{j\in A_{Min}^{(2)}}} T_2(x^{-1}z_j)\cdots \prod_{\overrightarrow{j\in A_{Min}^{(t)}}} T_t(x^{-1+t-2[\frac{t}{2}]}z_j)\cdots \prod_{\overrightarrow{j\in A_{Min}^{(N)}}} T_N(x^{-1+N-2[\frac{N}{2}]}z_j)$$
$$\times \prod_{t=1}^{N}\left((-c)^{t-1}\prod_{u=1}^{t-1}\Delta(x^{2u+1})^{t-u-1}\right)^{\alpha_t} \prod_{t=1}^{N} \prod_{\substack{j=1\\ j_1=A_{j,1}^{(t)}\\ \cdots \\ j_t=A_{j,t}^{(t)}}}^{\alpha_t} \sum_{\substack{\sigma\in S_t\\ \sigma(1)=1}} \prod_{\substack{u=1\\ u\neq[\frac{t}{2}]+1}}^{t} \delta\left(\frac{x^2 z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}}\right)$$
$$\times \prod_{t=1}^{N} \prod_{\substack{j<k\\ j,k\in A_{Min}^{(t)}}} g_{t,t}\left(\frac{z_k}{z_j}\right) \prod_{1\leqq t<u\leqq N} \prod_{\substack{j\in A_{Min}^{(t)}\\ k\in A_{Min}^{(u)}}} g_{t,u}\left(x^{u-t-2[\frac{u}{2}]+2[\frac{t}{2}]}\frac{z_k}{z_j}\right). \qquad (4.4)$$

*Here we have set the constant $c$ and the function $\Delta(z)$ in (3.3). When the index set $A_j^{(t)} = \{j_1, j_2, \cdots, j_t | j_1 < j_2 < \cdots < j_t\}$, $(1 \leqq t \leqq N, 1 \leqq j \leqq \alpha_t)$, we set $A_{j,k}^{(t)} = j_k$, and $A_{Min}^{(t)} = \{A_{1,1}^{(t)}, A_{2,1}^{(t)}, \cdots, A_{t,1}^{(t)}\}$.*

**Example**

$$\mathcal{O}_1(z) = T_1(z), \qquad (4.5)$$
$$\mathcal{O}_2(z_1,z_2) = g_{1,1}(z_2/z_1)T_1(z_1)T_1(z_2) - c\delta(x^2z_2/z_1)T_2(x^{-1}z_1), \qquad (4.6)$$
$$\begin{aligned}\mathcal{O}_3(z_1,z_2,z_3) &= g_{11}(z_2/z_1)g_{1,1}(z_3/z_1)g_{1,1}(z_3/z_2)T_1(z_1)T_1(z_2)T_1(z_3)\\ &- cg_{1,2}(x^{-1}z_2/z_1)T_1(z_1)\delta(x^2z_3/z_2)T_2(x^{-1}z_2)\\ &- cg_{1,2}(x^{-1}z_1/z_2)T_1(z_2)\delta(x^2z_3/z_1)T_2(x^{-1}z_1)\\ &- cg_{1,2}(x^{-1}z_1/z_3)T_1(z_3)\delta(x^2z_2/z_1)T_2(x^{-1}z_1)\\ &+ c^2\Delta(x^3)(\delta(x^2z_2/z_1)\delta(x^2z_1/z_3)+\delta(x^2z_1/z_2)\delta(x^2z_3/z_1))T_3(z_1). \qquad (4.7)\end{aligned}$$

Let us set the auxiliary function $s(z) = s(1/z)$ by

$$s(z) = \frac{(z;x^{2s})_\infty (x^{2s-2r}z;x^{2s})_\infty}{(x^{2s-2}z;x^{2s})_\infty (x^{-2r+2}z;x^{2s})_\infty} \times \frac{(1/z;x^{2s})_\infty (x^{2s-2r}/z;x^{2s})_\infty}{(x^{2s-2}/z;x^{2s})_\infty (x^{-2r+2}/z;x^{2s})_\infty}. \qquad (4.8)$$

**Definition 4.2** *For $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(r) < 0$, we define a family of the operators $\mathcal{I}_n$, $(n = 1, 2, \cdots)$ by*

$$\mathcal{I}_n = \left[\prod_{1\leqq j<k\leqq n} s(z_k/z_j)\mathcal{O}_n(z_1,\cdots,z_n)\right]_{1,z_1,\cdots,z_n}. \qquad (4.9)$$

*For generic* $\mathrm{Re}(s) > 0$ *and* $r \in \mathbb{C}$*, the definition of* $\mathcal{I}_n$ *should be understood as analytic continuation. We call the operator* $\mathcal{I}_n$ *the Local Integral Motion for the deformed* $W$*-algebra.*

**Proposition 4.1** *The operator* $\mathcal{O}_n(z_1, z_2, \cdots, z_n)$ *is* $S_n$*-invariant in "weakly sense".*

$$\prod_{1 \leqq j<k \leqq n} s(z_k/z_j)\mathcal{O}_n(z_1, z_2, \cdots, z_n) = \prod_{1 \leqq j<k \leqq n} s(z_{\sigma(k)}/z_{\sigma(j)})\mathcal{O}_n(z_{\sigma(1)}, z_{\sigma(2)}, \cdots, z_{\sigma(n)}), \ (\sigma \in S_n). \tag{4.10}$$

The following is one of **Main Results**.

**Theorem 4.2** *The Local Integrals of Motion* $\mathcal{I}_n$ $(n = 1, 2, \cdots)$ *commute with each other.*

$$[\mathcal{I}_m, \mathcal{I}_n] = 0, \quad (m, n = 1, 2, \cdots). \tag{4.11}$$

By using $S_n$-invariance of $\mathcal{O}_n(z_1, z_2, \cdots, z_n)$ in "weakly sense", the above theorem is reduced to the following theta function identity, which is shown by induction [6].

$$\begin{aligned} &\sum_{\substack{J \subset \{1,2,\cdots,n+m\} \\ |J|=n}} \prod_{j \in J} \prod_{k \notin J} \frac{[u_k - u_j + 1]_s [u_k - u_j + r - 1]_s}{[u_k - u_j]_s [u_k - u_j + r]_s}, \\ = &\sum_{\substack{J^c \subset \{1,2,\cdots,n+m\} \\ |J^c|=m}} \prod_{j \in J^c} \prod_{k \notin J^c} \frac{[u_k - u_j + 1]_s [u_k - u_j + r - 1]_s}{[u_k - u_j]_s [u_k - u_j + r]_s}. \end{aligned} \tag{4.12}$$

**Conjecture 4.3** *The Local Integrals of Motion are* $\eta$*-invariant.*

$$\eta(\mathcal{I}_n) = \mathcal{I}_n, \quad (n = 1, 2, \cdots). \tag{4.13}$$

We have checked for small $n$. We have already shown $\eta$-invariance conjecture, $\eta(\mathcal{I}_n) = \mathcal{I}_n$, for the deformed Virasoro algebra case, $Vir_{q,t} = W_{q,t}(\widehat{sl_2})$ [8].

# 5 Nonlocal Integrals of Motion

In this section we give explicit formulae of the Nonlocal Integrals of Motion. Let us set the theta function $\vartheta(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$ by following conditions.

$$\vartheta(u^{(1)}|\cdots|u^{(t)} + r|\cdots|u^{(N)}) = \vartheta(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}), \quad (1 \leqq t \leqq N) \tag{5.1}$$

$$\begin{aligned} & \vartheta(u^{(1)}|\cdots|u^{(t)}+r\tau|\cdots|u^{(N)}) \\ = \ & e^{-2\pi i\tau-\frac{2\pi i}{r}(u_{t-1}-2u_t+u_{t+1}+\sqrt{r(r-1)}P_{\alpha_t})}\vartheta(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}), \quad (1 \leqq t \leqq N), \quad (5.2) \\ & \vartheta(u^{(1)}+k|\cdots|u^{(N)}+k)=\vartheta(u^{(1)}|\cdots|u^{(N)}), \quad (k\in\mathbb{C}), \quad (5.3) \\ & \eta(\vartheta(u^{(1)}|\cdots|u^{(N)}))=\vartheta(u^{(N)}|u^{(1)}|\cdots|u^{(N-1)}). \quad (5.4) \end{aligned}$$

**Example** For $N=2$ case, we have

$$\begin{aligned} \vartheta(u_1|u_2) &= [u_1-u_2-\sqrt{r(r-1)}P_{\alpha_1}+\alpha]_r[u_1-u_2-\alpha]_r \\ &+ [u_1-u_2-\sqrt{r(r-1)}P_{\alpha_1}-\alpha]_r[u_1-u_2+\alpha]_r, \quad (\alpha\in\mathbb{C}). \quad (5.5) \end{aligned}$$

**Definition 5.1** *For* $\mathrm{Re}(r)\neq 0$ *and* $0<\mathrm{Re}(s)<2$, *we define a family of operators* $\mathcal{G}_m$, $(m=1,2,\cdots)$ *by*

$$\begin{aligned} \mathcal{G}_m &= \prod_{t=1}^{N}\prod_{j=1}^{m}\oint_C \frac{dz_j^{(t)}}{2\pi\sqrt{-1}z_j^{(t)}}F_1(z_1^{(1)})\cdots F_1(z_m^{(1)})F_2(z_1^{(2)})\cdots F_2(z_m^{(2)})\cdots F_N(z_1^{(N)})\cdots F_N(z_m^{(N)}) \\ &\times \frac{\displaystyle\prod_{t=1}^{N}\prod_{1\leqq i<j\leqq m}\left[u_i^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_i^{(t)}-1\right]_r}{\displaystyle\prod_{t=1}^{N-1}\prod_{i,j=1}^{m}\left[u_i^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_r\prod_{i,j=1}^{m}\left[u_i^{(1)}-u_j^{(N)}+\frac{s}{N}\right]_r} \\ &\times \vartheta\left(\sum_{j=1}^m u_j^{(1)}\middle|\sum_{j=1}^m u_j^{(2)}\middle|\cdots\middle|\sum_{j=1}^m u_j^{(N)}\right). \quad (5.6) \end{aligned}$$

*Here the integral contour* $C$ *is given by*

$$|x^{\frac{2s}{N}}z_j^{(t+1)}|<|z_i^{(t)}|<|x^{-2+\frac{2s}{N}}z_j^{(t+1)}|, \quad (1\leqq t\leqq N-1, 1\leqq i,j\leqq m), \quad (5.7)$$

$$|x^{2-\frac{2s}{N}}z_j^{(1)}|<|z_i^{(N)}|<|x^{-\frac{2s}{N}}z_j^{(1)}|, \quad (1\leqq i,j\leqq m). \quad (5.8)$$

*For generic* $s\in\mathbb{C}$, *the definition of* $\mathcal{G}_n$ *should be understood as analytic continuation. We call the operator* $\mathcal{G}_n$ *the Nonlocal Integrals of Motion for the deformed* $W$*-algebra.*

**Example** For $N=2$ and $m=1$ case, we have

$$\mathcal{G}_1=\int\int_C \frac{dz_1}{2\pi\sqrt{-1}z_1}\frac{dz_2}{2\pi\sqrt{-1}z_2}F_1(z_1)F_0(z_2)\frac{\vartheta(u_1|u_2)}{[u_1-u_2+\frac{s}{2}]_r[u_1-u_2-\frac{s}{2}+1]_r}. \quad (5.9)$$

Here $C$ is given by $|x^s z_2| < |z_1| < |x^{-2+s} z_2|$.

The following is one of **Main Results**.

**Theorem 5.1** *The Nonlocal Integrals of Motion $\mathcal{G}_n$, $(n = 1, 2, \cdots)$ commute with each other.*

$$[\mathcal{G}_m, \mathcal{G}_n] = 0, \quad (m, n = 1, 2, \cdots). \tag{5.10}$$

By using commutation relations of the screening currents $F_j(z)$, the above theorem is reduced to the following theta function identity, which is shown by induction [6].

$$\begin{aligned}
&\sum_{\sigma_1 \in S_{m+n}} \sum_{\sigma_2 \in S_{m+n}} \cdots \sum_{\sigma_N \in S_{m+n}} \vartheta_\alpha \left( \sum_{j=1}^{m} u^{(1)}_{\sigma_1(j)} \middle| \sum_{j=1}^{m} u^{(2)}_{\sigma_2(j)} \middle| \cdots \middle| \sum_{j=1}^{m} u^{(N)}_{\sigma_N(j)} \right) \\
\times\ &\vartheta_\beta \left( \sum_{j=m+1}^{m+n} u^{(1)}_{\sigma_1(j)} \middle| \sum_{j=m+1}^{m+n} u^{(2)}_{\sigma_2(j)} \middle| \cdots \middle| \sum_{j=m+1}^{m+n} u^{(N)}_{\sigma_N(j)} \right) \\
\times\ &\prod_{t=1}^{N} \prod_{i=1}^{m} \prod_{j=m+1}^{m+n} \frac{\left[ u^{(t)}_{\sigma_t(i)} - u^{(t+1)}_{\sigma_{t+1}(j)} - \frac{s}{N} \right]_r \left[ u^{(t)}_{\sigma_t(j)} - u^{(t+1)}_{\sigma_{t+1}(i)} + 1 - \frac{s}{N} \right]_r}{\left[ u^{(t)}_{\sigma_t(i)} - u^{(t)}_{\sigma_t(j)} \right]_r \left[ u^{(t)}_{\sigma_t(j)} - u^{(t)}_{\sigma_t(i)} - 1 \right]_r} \\
=\ &\sum_{\sigma_1 \in S_{m+n}} \sum_{\sigma_2 \in S_{m+n}} \cdots \sum_{\sigma_N \in S_{m+n}} \vartheta_\beta \left( \sum_{j=1}^{n} u^{(1)}_{\sigma_1(j)} \middle| \sum_{j=1}^{m} u^{(2)}_{\sigma_2(j)} \middle| \cdots \middle| \sum_{j=1}^{n} u^{(N)}_{\sigma_N(j)} \right) \\
\times\ &\vartheta_\alpha \left( \sum_{j=n+1}^{m+n} u^{(1)}_{\sigma_1(j)} \middle| \sum_{j=n+1}^{m+n} u^{(2)}_{\sigma_2(j)} \middle| \cdots \middle| \sum_{j=n+1}^{m+n} u^{(N)}_{\sigma_N(j)} \right) \\
\times\ &\prod_{t=1}^{N} \prod_{i=1}^{n} \prod_{j=n+1}^{m+n} \frac{\left[ u^{(t)}_{\sigma_t(i)} - u^{(t+1)}_{\sigma_{t+1}(j)} - \frac{s}{N} \right]_r \left[ u^{(t)}_{\sigma_t(j)} - u^{(t+1)}_{\sigma_{t+1}(i)} + 1 - \frac{s}{N} \right]_r}{\left[ u^{(t)}_{\sigma_t(i)} - u^{(t)}_{\sigma_t(j)} \right]_r \left[ u^{(t)}_{\sigma_t(j)} - u^{(t)}_{\sigma_t(i)} - 1 \right]_r},
\end{aligned} \tag{5.11}$$

where $\vartheta_\alpha(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$ and $\vartheta_\beta(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$ are not necesarry the same theta functions.

**Theorem 5.2** *The Nonlocal Integrals of Motion are $\eta$-invariant.*

$$\eta(\mathcal{G}_m) = \mathcal{G}_m, \quad (m = 1, 2, \cdots). \tag{5.12}$$

**Conjecture 5.3** *The Local Integrals of Motion* $\mathcal{I}_n$, $(n=1,2,\cdots)$ *and Nonlocal Integrals of Motion* $\mathcal{G}_m$, $(m=1,2,\cdots)$ *commute with each other.*

$$[\mathcal{I}_n, \mathcal{G}_m] = 0, \quad (m,n=1,2,\cdots). \tag{5.13}$$

We have already shown the commutation relations $[\mathcal{I}_n, \mathcal{G}_m] = 0, (m,n = 1,2,\cdots)$ for the deformed Virasoro algebra case, $Vir_{q,t} = W_{q,t}(\widehat{sl_2})$ [8]. We have already shown the commutation relations $[\mathcal{I}_1, \mathcal{G}_m] = 0$, $(m = 1,2,\cdots)$ for general $W_{q,t}(\widehat{sl_N})$ case. If we assume $\eta$-invariance of the Local Integrals of Motion $\eta(\mathcal{I}_n) = \mathcal{I}_n$, the commutation relation $[\mathcal{I}_n, \mathcal{G}_m] = 0$, $(m,n = 1,2,\cdots)$ for general $W_{q,t}(\widehat{sl_N})$, can be shown by simple argument.

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