Elliptic Algebra $U_{q,p}(\mathfrak{g})$

Recent Advances for Superalgebra $\mathfrak{g} = \widehat{sl}$ (MIN)

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Summary

[I] We construct free field realizations of quantum and elliptic superalgebra $U_q(\widehat{sl}(N|1))$, $U_{q,p}(\widehat{sl}(N|1))$.

[II] We overview future research projects for elliptic superalgebra $U_{q,p}(\widehat{sl}(M|N))$

(vertex operator etc.)

Elliptic Algebra $U_{q,p}(\mathfrak{g})$

Free Field Realization

Free Field Realization

- Realization of algebra by using differential operators

"$x, \frac{\partial}{\partial x}$" Boson

Free Field Realization of $U_q(\mathfrak{g})$

Level k

Center $C \in U_q(\mathfrak{g})$

$C = k \in \mathbb{C}$

- Free field realization for level $k \in \mathbb{C}$ (Wakimoto) is completely different from those for level $k = 1$ (Frenkel-Kac).

Level $k=1$ Frenkel-Kac realization

[Frenkel, Kac, Invent. Math. 62, 23-66 (1980)]

q-Version [Frenkel, Jing, Proc. Natl. Acad. Sci. 85, 9373-9377 (1988)]

Level k : general Wakimoto realization

[Wakimoto, CMP104, 605-609 (1986)]

q-Version [Shiraishi, Phys. Lett. A171, 243-248 (1992)]
[Matsuo, CMP161, 33-48 (1994)]

- To understand that two realizations are completely different, we overview two realizations for $U_q(\widehat{sl}(2))$.

Quantum Group	Elliptic Algebra
$U_q(\mathfrak{g})$	$U_{qp}(\mathfrak{g})$
deformation	
$(z_1 - q^2 z_2) X^+(z_1) X^+(z_2)$	$[u_1 - u_2 + 1] F(z_1) F(z_2)$
$= (q^2 z_1 - z_2) X^+(z_2) X^+(z_1)$	$= [u_1 - u_2 - 1] F(z_2) F(z_1)$
Trigonometric function	Elliptic function
6-vertex model	ABF-face model

Quantum Group $U_q(\widehat{sl}(2))$

Drinfeld Realization

- $a_n,\ x_m^{\pm},\ h,\ C \quad (n \in \mathbb{Z}_{\neq 0},\ m \in \mathbb{Z})$
- $[a_m, a_n] = \dfrac{[2m]_q [cm]_q}{m} \delta_{m+n,0}$
- $(z_1 - q^{\pm 2} z_2) X^{\pm}(z_1) X^{\pm}(z_2)$

$$= (q^{\pm 2} z_1 - z_2) X^{\pm}(z_2) X^{\pm}(z_1)$$

$$\left([a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}, \quad X^{\pm}(z) = \sum_{m \in \mathbb{Z}} X_m^{\pm} z^{-m-1} \right)$$

Quantum Group $U_q(\widehat{sl}(2))$

- $[\, X^+(z_1), X^-(z_2) \,] = \dfrac{1}{(q-q^{-1})z_1 z_2} \times$

$\times \left(\delta\left(q^c \dfrac{z_2}{z_1}\right) \psi_+(q^{\frac{c}{2}} z_2) - \delta\left(q^{-c} \dfrac{z_2}{z_1}\right) \psi_-(q^{-\frac{c}{2}} z_2) \right)$

- $\psi^+(q^{\frac{c}{2}} z) = q^h \exp\left((q-q^{-1}) \sum_{m>0} a_m z^{-m} \right)$
- $\psi^-(q^{-\frac{c}{2}} z) = q^{-h} \exp\left(-(q-q^{-1}) \sum_{m>0} a_{-m} z^{m} \right)$

$$\delta(z) = \sum_{m \in \mathbb{Z}} z^m$$

q-Frenkel-Kac realization

$U_q(\widehat{sl(2)})$

Level $k = 1$

[Frenkel, Jing, Proc. Natl. Acad. Scie. 85 9373-9377 (1988)]

- $$X^{\pm}(z) = \exp\left(\pm \sum_{m>0} \frac{a_m}{[m]_q} q^{\mp \frac{m}{2}} z^m\right) \times \exp\left(\mp \sum_{m>0} \frac{a_m}{[m]_q} q^{\mp \frac{m}{2}} z^{-m}\right) e^{\alpha} z^{\pm \partial}$$

- a_m : Boson in Drinfeld realization

$$[a_m, a_n] = \frac{[2m]_q [m]_q}{m} \delta_{m+n,0}$$

q-Wakimoto realization

[Shiraishi, Phys. Lett. A171 243-248 (1992)]

Level k : general

$U_q(\widehat{sl}(2))$

Boson $\alpha_m, \beta_m, \gamma_m, Q_\beta, Q_\gamma$

$$[\alpha_m, \alpha_n] = \frac{[(k+2)m]_q [2m]_q}{m} \delta_{m+n,0}$$

$$[\beta_m, \beta_n] = -\frac{[m]_q^2}{m} \delta_{m+n,0}$$

$$[\gamma_m, \gamma_n] = \frac{[m]_q^2}{m} \delta_{m+n,0}$$

$$[\beta_0, Q_\beta] = -1 \quad , \quad [\gamma_0, Q_\gamma] = 1 .$$

q-Wakimoto realization

$U_q(\widehat{sl}(2))$

$$\beta(z) = -\sum_{m\neq 0} \frac{\beta_m}{[m]_q} z^{-m} + Q_\beta + \beta_0 \log z$$

$$\gamma(z) = -\sum_{m\neq 0} \frac{\gamma_m}{[m]_q} z^{-m} + Q_\gamma + \gamma_0 \log z$$

$$\beta_\pm(z) = \pm (q-q^{-1}) \sum_{\pm m>0} \beta_m z^{-m} \pm \beta_0 \log q$$

$$\alpha_\pm(z) = \pm (q-q^{-1}) \sum_{\pm m>0} \alpha_m z^{-m}$$

q-Wakimoto realization

[Shiraishi, Phys. Lett. A171, 243-248 (1992)]

- $X^+(z) = \frac{1}{(q-q^{-1})z}\Big(:\exp\big(\beta_+(z) - (\beta+\gamma)(qz)\big): - :\exp\big(\beta_-(z) - (\beta+\gamma)(q^{-1}z)\big): \Big)$

- $X^-(z) = \frac{1}{(q-q^{-1})z}\Big(:\exp\big(\alpha_-(q^{-\frac{k+2}{2}}z) + \beta_-(q^{-k-2}z) + (\beta+\gamma)(q^{-k-1}z)\big): - :\exp\big(\alpha_+(q^{\frac{k+2}{2}}z) + \beta_+(q^{k+2}z) + (\beta+\gamma)(q^{k+1}z)\big): \Big)$

- $a_m = q^{-\frac{k+2}{2}|m|}\alpha_m + \big(q^{-k|m|} + q^{-(k+2)|m|}\big)\beta_m$

$U_q(\widehat{sl}(2))$

q-Frenkel-Kac realization

$U_q(\widehat{sl(2)})$

Level $k = 1$

[Frenkel, Jing, Proc. Natl. Acad. Scie. 85 9373-9377 (1988)]

- $$X^{\pm}(z) = \exp\left(\pm \sum_{m>0} \frac{a_m}{[m]_q} q^{\mp \frac{m}{2}} z^m\right) \times \exp\left(\mp \sum_{m>0} \frac{a_m}{[m]_q} q^{\mp \frac{m}{2}} z^{-m}\right) e^{\alpha} z^{\pm Q}$$

- a_m : Boson in Drinfeld realization

$$[a_m, a_n] = \frac{[2m]_q [m]_q}{m} \delta_{m+n,0}$$

Table of Free Field Realization $U_q(\mathfrak{g})$

Level $k=1$ q-Frenkel-Kac realization

$$\mathfrak{g} = (ADE)^{(1)}, (BC)^{(1)}, G_2^{(1)}, \widehat{sl}(M|N), \widehat{osp}(2|2)^{(2)}$$

Level k : general q-Wakimoto realization

$$\mathfrak{g} = \widehat{sl}(N), \widehat{sl}(N|1)$$

- Now we realized that

  Wakimoto realization (level k)

  is completely different from

  Frenkel-Kac realization (level $k=1$).

- Next we construct free field realization of

  quantum superalgebra $U_q(\widehat{sl}(N|1))$ and

  elliptic superalgebra $U_{q,p}(\widehat{sl}(N|1))$

  for an arbitrary level k.

Quantum Superalgebra $U_q(\hat{sl}(MIN))$

[Kac, Adv. Math 26, 8-96 (1977)]

[J. Van. de. Leur, Commun. Alg. 17, 1815-1841 (1989)]

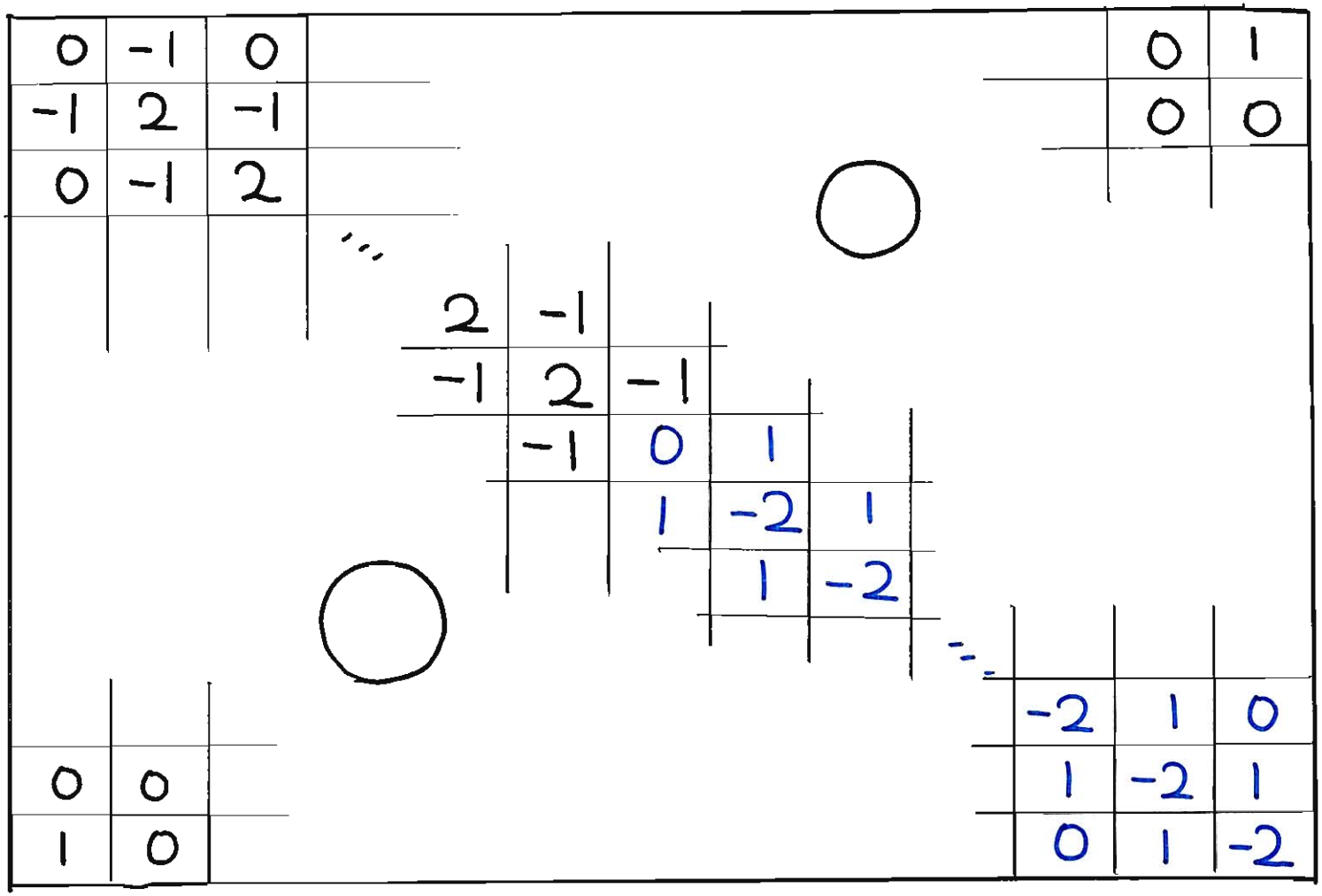
[Yamane, Publ. RIMS 35, 321-390 (1999)]

$$[x, y] = x \cdot y - y \cdot x$$

$$\{x, y\} = x \cdot y + y \cdot x$$

Superalgebra $\widehat{sl}(M|N)$

Cartan matrix $(A_{\bar{i}\bar{j}})_{0\le \bar{i},\bar{j}\le M+N-1}$



Quantum Superalgebra $U_q(\widehat{sl}(M|N))$

Drinfeld's generators

$$X^{\pm}_{i,m}\ ,\ a_{i,n}\ ,\ h_i\ ,\ C \qquad (1 \le i \le M+N-1,\ m \in \mathbb{Z},\ n \in \mathbb{Z}_{\neq 0})$$

Generating functions

$$X^{\pm}_i(z) = \sum_{m \in \mathbb{Z}} X^{\pm}_{i,m}\, z^{-m-1} \qquad (1 \le i \le M+N-1)$$

$$\psi^{+}_i(q^{\frac{c}{2}} z) = q^{h_i} \exp\Big((q-q^{-1}) \sum_{m>0} a_{i,m} z^{-m} \Big)$$

$$\psi^{-}_i(q^{-\frac{c}{2}} z) = q^{-h_i} \exp\Big(-(q-q^{-1}) \sum_{m>0} a_{i,-m} z^{m} \Big)$$

Quantum Superalgebra $U_q(\widehat{sl}(M|N))$

Defining relations

for $|A_{\bar{i}\bar{j}}| \neq 0$

- $(z_1 - q^{\pm A_{\bar{i}\bar{j}}} z_2)\, X^{\pm}_{\bar{i}}(z_1) X^{\pm}_{\bar{j}}(z_2) = (q^{\pm A_{\bar{j}\bar{i}}} z_1 - z_2)\, X^{\pm}_{\bar{j}}(z_2) X^{\pm}_{\bar{i}}(z_1)$

- $[\, X^{\pm}_{\bar{i}}(z_1)\,,\, X^{\pm}_{\bar{j}}(z_2)\,] = 0$ for $|A_{\bar{i}\bar{j}}| = 0\,,\ (\bar{i},\bar{j}) \neq (M,M)$

- $\{\, X^{\pm}_{M}(z_1)\,,\, X^{\pm}_{M}(z_2)\,\} = 0$

- $[\, X^{+}_{\bar{i}}(z_1)\,,\, X^{-}_{\bar{j}}(z_2)\,] = \dfrac{\delta_{\bar{i}\bar{j}}}{(q-q^{-1})z_1 z_2} \times$ for $(\bar{i},\bar{j}) \neq (M,M)$

  $\times \left(\delta\left(q^{c}\dfrac{z_2}{z_1}\right) \psi^{+}_{\bar{i}}\left(q^{\frac{c}{2}} z_2\right) - \delta\left(q^{-c}\dfrac{z_2}{z_1}\right) \psi^{-}_{\bar{i}}\left(q^{-\frac{c}{2}} z_2\right) \right)$

- $\{\, X^{+}_{M}(z_1)\,,\, X^{-}_{M}(z_2)\,\} = \dfrac{1}{(q-q^{-1})z_1 z_2} \times$

  $\times \left(\delta\left(q^{c}\dfrac{z_2}{z_1}\right) \psi^{+}_{M}\left(q^{\frac{c}{2}} z_2\right) - \delta\left(q^{-c}\dfrac{z_2}{z_1}\right) \psi^{-}_{M}\left(q^{-\frac{c}{2}} z_2\right) \right)$ etc.

Quantum Superalgebra $U_q(\widehat{sl}(M|N))$

Serre relations

- $\left(X_i^{\pm}(z_1) X_i^{\pm}(z_2) X_j^{\pm}(z) - (q+q^{-1}) X_i^{\pm}(z_1) X_j^{\pm}(z) X_i^{\pm}(z_2) \right.$
  $\left. + X_j^{\pm}(z) X_i^{\pm}(z_1) X_i^{\pm}(z_2) \right) + (z_1 \leftrightarrow z_2) = 0 \quad$ for $|A_{ij}| = 1, \; i \neq M$

- $\Big(X_M^{\pm}(z_1) X_{M+1}^{\pm}(w_1) X_M^{\pm}(z_2) X_{M-1}^{\pm}(w_2) - q^{-1} X_M^{\pm}(z_1) X_{M+1}^{\pm}(w_1) X_{M-1}^{\pm}(w_2) X_M^{\pm}(z_2)$
  $- q X_M^{\pm}(z_1) X_M^{\pm}(z_2) X_M^{\pm}(w_2) X_{M+1}^{\pm}(w_1) + X_M^{\pm}(z_1) X_{M-1}^{\pm}(w_2) X_M^{\pm}(z_2) X_{M+1}^{\pm}(w_1)$
  $+ X_{M+1}^{\pm}(w_1) X_M^{\pm}(z_2) X_{M-1}^{\pm}(w_2) X_M^{\pm}(z_1) - q^{-1} X_{M+1}^{\pm}(w_1) X_{M-1}^{\pm}(w_2) X_M^{\pm}(z_2) X_M^{\pm}(z_1)$
  $- q X_M^{\pm}(z_2) X_{M-1}^{\pm}(w_2) X_{M+1}^{\pm}(w_1) X_M^{\pm}(z_1) + X_{M-1}^{\pm}(w_2) X_M^{\pm}(z_2) X_{M+1}^{\pm}(w_1) X_M^{\pm}(z_1) \Big)$
  $+ (z_1 \leftrightarrow z_2) = 0$

Free Field Realization

$U_q(\widehat{sl}(N|1))$

Boson $a_m^{\bar{i}}$ $(1 \le \bar{i} \le N)$ $b_m^{\bar{i}\bar{j}}$, $c_m^{\bar{i}\bar{j}}$ $(1 \le \bar{i} < \bar{j} \le N+1)$

$$\cdot [a_m^{\bar{i}}, a_n^{\bar{j}}] = \frac{[(k+N-1)m]_q [A_{\bar{i}\bar{j}} m]_q}{m} \delta_{m+n,0}$$

$$\cdot [b_m^{\bar{i}\bar{j}}, b_n^{\bar{i}'\bar{j}'}] = -\nu_{\bar{i}}\nu_{\bar{j}} \frac{[m]_q^2}{m} \delta_{\bar{i}\bar{i}'} \delta_{\bar{j}\bar{j}'} \delta_{m+n,0}$$

$$\cdot [c_m^{\bar{i}\bar{j}}, c_n^{\bar{i}'\bar{j}'}] = \nu_{\bar{i}}\nu_{\bar{j}} \frac{[m]_q^2}{m} \delta_{\bar{i}\bar{i}'} \delta_{\bar{j}\bar{j}'} \delta_{m+n,0}$$

$\nu_1 = \nu_2 = \cdots = \nu_N = +$, $\nu_{N+1} = -$

[Kojima, arXiv. 1105.5772]

[Awata, Odake, Shiraishi, LMP42, 271-279 (1997)] $\widehat{sl}(2|1)$

Free Field Realization

Zero-Mode $Q_b^{\bar{i}\bar{j}}$, $Q_c^{\bar{i}\bar{j}}$

- $[\, b_0^{\bar{i}\bar{j}}, Q_b^{\bar{i}'\bar{j}'}\,] = -\nu_{\bar{i}}\nu_{\bar{j}}\,\delta_{\bar{i}\bar{i}'}\,\delta_{\bar{j}\bar{j}'}$

- $[\, c_0^{\bar{i}\bar{j}}, Q_c^{\bar{i}'\bar{j}'}\,] = \nu_{\bar{i}}\nu_{\bar{j}}\,\delta_{\bar{i}\bar{i}'}\,\delta_{\bar{j}\bar{j}'}$

Cocycle Condition

- $[\, Q_b^{\bar{i}\bar{j}}, Q_b^{\bar{i}'\bar{j}'}\,] = \delta_{\bar{j}\,N+1}\,\delta_{\bar{j}'\,N+1}\,\log(\pi\sqrt{-1}\,)$

Free Field Realization

- $$b^{i\bar{a}}(z) = -\sum_{m\neq 0} \frac{b_m^{i\bar{a}}}{[m]_q} z^{-m} + Q_b^{i\bar{a}} + b_0^{i\bar{a}} \log z$$
- $$c^{i\bar{a}}(z) = -\sum_{m\neq 0} \frac{c_m^{i\bar{a}}}{[m]_q} z^{-m} + Q_c^{i\bar{a}} + c_0^{i\bar{a}} \log z$$
- $$b_\pm^{i\bar{a}}(z) = \pm(q-q^{-1}) \sum_{\pm m>0} b_m^{i\bar{a}} z^{-m} \pm b_0^{i\bar{a}} \log q$$
- $$a_\pm^{\bar{i}}(z) = \pm(q-q^{-1}) \sum_{\pm m>0} a_m^{\bar{i}} z^{-m}$$

Free Field Realization

$(1 \le \bar{i} \le N-1)$

$$\cdot X_{\bar{i}}^{+}(z) = \frac{1}{(q-q^{-1})z} \sum_{\bar{j}=1}^{\bar{i}} \Big(:\exp\Big((b+c)^{\bar{j}\bar{i}}(q^{\bar{j}-1}z) + b_{+}^{\bar{j}\bar{i}+1}(q^{\bar{j}-1}z) - (b+c)^{\bar{j}\bar{i}+1}(q^{\bar{j}}z) + \sum_{l=1}^{\bar{j}-1} \big(b_{+}^{l\bar{i}+1}(q^{l-1}z) - b_{+}^{l\bar{i}}(q^{l}z) \big) \Big) : - :\exp\Big((b+c)^{\bar{j}\bar{i}}(q^{\bar{j}-1}z) + b_{-}^{\bar{j}\bar{i}+1}(q^{\bar{j}-1}z) - (b+c)^{\bar{j}\bar{i}+1}(q^{\bar{j}-2}z) + \sum_{l=1}^{\bar{j}-1} \big(b_{+}^{l\bar{i}+1}(q^{l-1}z) - b_{+}^{l\bar{i}}(q^{l}z) \big) \Big) : \Big).$$

$$\cdot X_{N}^{+}(z) = \sum_{\bar{j}=1}^{N} :\exp\Big((b+c)^{\bar{j}N}(q^{\bar{j}-1}z) + b^{\bar{j}N+1}(q^{\bar{j}-1}z) - \sum_{l=1}^{\bar{j}-1} \big(b_{+}^{l,N+1}(q^{l}z) + b_{+}^{lN}(q^{l}z) \big) \Big) : .$$

Free Field Realization

$(1 \le i \le N-1)$

$$\cdot X_i^-(z) = \frac{1}{(q-q^{-1})z}\Bigg\{ \sum_{j=1}^{i-1} : \exp\Big(a_-^i(q^{-\frac{k+N-1}{2}}z) + (b+c)^{ji+1}(q^{-k-j}z)$$

$$+ \sum_{l=j+1}^{i} \Big(b_-^{li+1}(q^{-k-l+1}z) - b_-^{li}(q^{-k-l}z) \Big) + \sum_{l=i+1}^{N} \Big(b_-^{il}(q^{-k-l}z) - b_-^{i+1\,l}(q^{-k-l-1}z) \Big) + b_-^{i\,N+1}(q^{-k-N}z) - b_-^{i+1\,N+1}(q^{-k-N+1}z) \Big)$$

$$\times \Big(\exp\big(-b_-^{ji}(q^{-k-j}z) - (b+c)^{ji}(q^{-k-j+1}z)\big) - \exp\big(-b_+^{ji}(q^{-k-j}z) - (b+c)^{ji}(q^{-k-j-1}z)\big) \Big) :$$

$$- \sum_{j=i+1}^{N-1} : \exp\Big(a_+^i(q^{\frac{k+N-1}{2}}z) + (b+c)^{ij+1}(q^{k+j}z)$$

$$+ \sum_{l=j+1}^{N} \Big(b_+^{il}(q^{k+l}z) - b_+^{i+1\,l}(q^{k+l-1}z) \Big) + b_+^{i\,N+1}(q^{k+N}z) - b_+^{i+1\,N+1}(q^{k+N-1}z) \Big)$$

$$\times \Big(\exp\big(b_+^{i+1\,j+1}(q^{k+j}z) - (b+c)^{i+1\,j+1}(q^{k+j+1}z)\big) - \exp\big(b_-^{i+1\,j+1}(q^{k+j}z) - (b+c)^{i+1\,j+1}(q^{k+j-1}z)\big) \Big) : +$$

$$\cdots$$

Free Field Realization

$$+ : \exp\Big(a_-^{\bar{i}}(q^{-\frac{k+N-1}{2}}z) + (b+c)^{\bar{i}\bar{i}+1}(q^{-k-\bar{i}}z) + \sum_{l=\bar{i}+1}^{N} \big(b_-^{\bar{i}l}(q^{-k-l}z) - b_-^{\bar{i}+1\,l}(q^{-k-l+1}z) \big)$$
$$+ b_-^{\bar{i}\,N+1}(q^{-k-N}z) - b_-^{\bar{i}+1\,N+1}(q^{-k-N+1}z) \Big) :$$

$$- : \exp\Big(a_+^{\bar{i}}(q^{\frac{k+N-1}{2}}z) + (b+c)^{\bar{i}\bar{i}+1}(q^{k+\bar{i}}z) + \sum_{l=\bar{i}+1}^{N} \big(b_+^{\bar{i}l}(q^{k+l}z) - b_+^{\bar{i}+1\,l}(q^{k+l-1}z) \big)$$
$$+ b_+^{\bar{i}\,N+1}(q^{k+N}z) - b_+^{\bar{i}+1\,N+1}(q^{k+N-1}z) \Big) : \Big\}$$

$$+ : \exp\Big(a_+^{\bar{i}}(q^{\frac{k+N-1}{2}}z) - b^{\bar{i}\,N+1}(q^{k+N-1}z) - b_+^{\bar{i}+1\,N+1}(q^{k+N-1}z) + b_+^{\bar{i}+1\,N+1}(q^{k+N}z) \Big) : .$$

Free Field Realization

$$\cdot\ X_N^-(z) = \frac{1}{(q-q^{-1})z}\Bigg\{ \sum_{j=1}^{N-1} \Big(-q^{j-1} :\exp\Big(a_-^N(q^{-\frac{k+N-1}{2}}z)$$

$$-b_-^{jN}(q^{-k-j}z) - (b+c)^{jN}(q^{-k-j+1}z) - b_-^{jN+1}(q^{-k-j}z) - b^{jN+1}(q^{-k-j+1}z) - \sum_{l=j+1}^{N-1}\Big(b_-^{lN}(q^{-k-l}z) + b_-^{lN+1}(q^{-k-l}z)\Big)\Big):$$

$$+q^{j-1} :\exp\Big(a_-^N(q^{-\frac{k+N-1}{2}}z)$$

$$-b_+^{jN}(q^{-k-j}z) - (b+c)^{jN}(q^{-k-j-1}z) - b_+^{jN+1}(q^{-k-j}z) - b^{jN+1}(q^{-k-j-1}z) - \sum_{l=j+1}^{N-1}\Big(b_-^{lN}(q^{-k-l}z) + b_-^{lN+1}(q^{-k-l}z)\Big)\Big):$$

$$-q^{N-1} :\exp\Big(a_-^N(q^{-\frac{k+N-1}{2}}z) - b^{N,N+1}(q^{-k-N+1}z)\Big):$$

$$+q^{N-1} :\exp\Big(a_+^N(q^{\frac{k+N-1}{2}}z) - b^{NN+1}(q^{k+N-1}z)\Big):\quad \Bigg\}.$$

Quantum Superalgebra $U_q(\widehat{sl}(M|N))$

Defining relations

for $|A_{ij}| \neq 0$

- $(z_1 - q^{\pm A_{ij}} z_2)\, X_i^{\pm}(z_1) X_j^{\pm}(z_2) = (q^{\pm A_{ji}} z_1 - z_2)\, X_j^{\pm}(z_2) X_i^{\pm}(z_1)$
- $[\, X_i^{\pm}(z_1)\,,\, X_j^{\pm}(z_2)\,] = 0$ for $|A_{ij}| = 0\,,\ (i,j) \neq (M,M)$
- $\{\, X_M^{\pm}(z_1)\,,\, X_M^{\pm}(z_2)\,\} = 0$
- $[\, X_i^{+}(z_1)\,,\, X_j^{-}(z_2)\,] = \dfrac{\delta_{ij}}{(q-q^{-1}) z_1 z_2} \times$ for $(i,j) \neq (M,M)$

  $\times \left(\delta\left(q^{c} \dfrac{z_2}{z_1}\right) \psi_i^{+}(q^{\frac{c}{2}} z_2) - \delta\left(q^{-c} \dfrac{z_2}{z_1}\right) \psi_i^{-}(q^{-\frac{c}{2}} z_2) \right)$
- $\{\, X_M^{+}(z_1)\,,\, X_M^{-}(z_2)\,\} = \dfrac{1}{(q-q^{-1}) z_1 z_2} \times$

  $\times \left(\delta\left(q^{c} \dfrac{z_2}{z_1}\right) \psi_M^{+}(q^{\frac{c}{2}} z_2) - \delta\left(q^{-c} \dfrac{z_2}{z_1}\right) \psi_M^{-}(q^{-\frac{c}{2}} z_2) \right)$ etc.

- We have constructed a free field realization of quantum superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level k.

- Next we give a free field realization of elliptic superalgebra $U_{q,p}(\widehat{sl}(N|1))$ for an arbitrary level k.

$$U_q(\widehat{sl}(M|N)) \xrightarrow{\text{deformation}} U_{q,p}(\widehat{sl}(M|N))$$

[Kojima, arXiv. 1103.5527]

[Jimbo, Konno, Odake, Shiraishi, CMP 199, 605-647 (1999)]

Elliptic Theta Function

- $[u] = q^{\frac{u^2}{r}-u} \dfrac{\textcircled{H}_{q^{2r}}(q^{2u})}{(q^{2r};q^{2r})_\infty^3}$

- $\textcircled{H}_p(z) = (z;p)_\infty (p/z;p)_\infty (p;p)_\infty$
- $(z;p)_\infty = \prod_{m=0}^{\infty} (1 - p^m z)$

Quasi-Periodicity

- $[u+r] = -[u]$
- $[u+r\tau] = -e^{2\pi\sqrt{-1}(u+\frac{\tau}{2})/r}[u]$ $\quad (\tau = \pi r\sqrt{-1}/\log q)$

Elliptic Superalgebra $U_{q,p}(\widehat{sl}(N\|1))$

$p = q^{2r}$, $r, r^* = r-k$ $(\mathrm{Re}(r), \mathrm{Re}(r^*) > 0)$

- $U_{\bar{i}}^{+}(z) = \exp\left(\sum_{m>0} \frac{q^{rm}}{[rm]_q} B_{\bar{i},-m} z^m \right)$

- $U_{\bar{i}}^{-}(z) = \exp\left(-\sum_{m>0} \frac{q^{r^*m}}{[rm]_q} B_{\bar{i},m} z^{-m} \right)$

- $B_{\bar{j}m} = \begin{cases} \frac{[r^*m]_q}{[rm]_q} a_{\bar{j}m} & (m>0), \\ q^{k|m|} a_{\bar{j}m} & (m<0). \end{cases}$

- $a_{\bar{i},m} = q^{-\frac{k+N-1}{2}|m|} a_m^{\bar{i}} + \sum_{l=1}^{\bar{i}} \left(q^{-(k+l-1)|m|} b_m^{l\,\bar{i}+1} - q^{-(k+l)|m|} b_m^{l\,\bar{i}} \right)$
$+ \sum_{l=\bar{i}+1}^{N} \left(q^{-(k+l)|m|} b_m^{\bar{i}\,l} - q^{-(k+l-1)|m|} b_m^{\bar{i}+1\,l} \right) + q^{-(k+N)|m|} b_m^{\bar{i}\,N+1} - q^{-(k+N-1)|m|} b_m^{l\,N+1}$,

- $a_{N,m} = q^{-\frac{k+N-1}{2}|m|} a_m^N - \sum_{l=1}^{N-1} \left(q^{-(k+l)|m|} b_m^{l\,N} + q^{-(k+l)|m|} b_m^{l\,N+1} \right)$.

Elliptic Superalgebra $U_{qp}(\widehat{sl}(N|1))$

- $E_i(z) = U_i^+(z) X_i^+(z) e^{2Q_i} z^{-\frac{1}{r^*}P_i}$,
- $F_i(z) = X_i^-(z) U_i^-(z) z^{\frac{1}{r}(P_i+h_i)}$, $(1 \le i \le N)$
- $H_i^{\pm}(z) = H_i(q^{\pm(r-\frac{k}{2})}z)$,
- $H_i(z) = :\exp\left(-\sum_{m\neq 0} \frac{B_{im}}{[r^*m]_q} z^{-m}\right): e^{2Q_i} z^{-\frac{1}{rr^*}P_i + \frac{1}{r}h_i}$.

$$[P_i, Q_j] = -\frac{A_{ij}}{2} \qquad (1 \le i, j \le N)$$

Elliptic Superalgebra $U_{qp}(\widehat{sl}(N|1))$

- $\left[u_1-u_2-\frac{A_{i\bar{j}}}{2}\right]^* E_i(z_1)E_{\bar{j}}(z_2) = \left[u_1-u_2+\frac{A_{i\bar{j}}}{2}\right]^* E_{\bar{j}}(z_2)E_i(z_1)$,

- $\left[u_1-u_2+\frac{A_{i\bar{j}}}{2}\right] F_i(z_1)F_{\bar{j}}(z_2) = \left[u_1-u_2-\frac{A_{i\bar{j}}}{2}\right] F_{\bar{j}}(z_2)F_i(z_1)$,

- $\left[E_i(z_1), E_{\bar{j}}(z_2)\right] = \frac{\delta_{i\bar{j}}}{(q-q^{-1})z_1z_2} \times$ $\qquad (i,j)\neq(N,N)$

$$\times\left(\delta\left(q^{-k}\frac{z_1}{z_2}\right)H_i(q^r z_2) - \delta\left(q^{k}\frac{z_1}{z_2}\right)H_i(q^{-r}z_2)\right),$$

- $\left\{E_N(z_1), F_N(z_2)\right\} = \frac{1}{(q-q^{-1})z_1z_2} \times$

$$\times\left(\delta\left(q^{-k}\frac{z_1}{z_2}\right)H_N(q^r z_2) - \delta\left(q^{k}\frac{z_1}{z_2}\right)H_N(q^{-r}z_2)\right),$$

- $\left\{E_N(z_1), E_N(z_2)\right\} = 0$, $\left\{F_N(z_1), F_N(z_2)\right\} = 0$. etc.

- We have constructed free field realizations of $U_q(\widehat{sl}(N|1))$ and $U_{q,p}(\widehat{sl}(N|1))$ for an arbitrary level k.

- We can give Level $k=1$ free field realization of $U_{q,p}(\widehat{sl}(M|N))$ by using

[Kimura, Shiraishi, Uchiyama, CMP 188, 367-378 (1997)]

Level $k=1$ $U_q(\widehat{sl}(M|N))$

and

[Kojima, arXiv. 1103.5527 (2011)] elliptic deformation.

- In what follows we demonstrate future research projects for $U_{qp}(\widehat{sl}(MIN))$.

- The following talk is based on $U_{qp}(\widehat{sl}(N))$.

- Cartan matrix of $\widehat{sl}(N)$

$$(A_{ij})_{0\le i,j\le N} = \begin{bmatrix} 2 & -1 & & & & 0 & -1 \\ -1 & 2 & -1 & & & 0 & 0 \\ 0 & -1 & 2 & & \bigcirc & & \\ & & & \ddots & & & \\ & & \bigcirc & & \ddots & & \\ & & & & 2 & -1 & 0 \\ 0 & 0 & & & -1 & 2 & -1 \\ -1 & 0 & & & 0 & -1 & 2 \end{bmatrix}$$

Elliptic Algebra

$U_{q,p}(\widehat{gl(N)})$ $F_j(z), E_j(z)$ $(1 \le j \le N-1)$

- $\left[u_1 - u_2 - \frac{\delta}{N}\right] F_j(z_1) F_{j+1}(z_2) = \left[u_2 - u_1 + \frac{\delta}{N} - 1\right] F_{j+1}(z_2) F_j(z_1)$

- $\left[u_1 - u_2 + 1\right] F_j(z_1) F_j(z_2) = \left[u_1 - u_2 - 1\right] F_j(z_2) F_j(z_1)$

- $\left[u_1 - u_2 + 1 - \frac{\delta}{N}\right]^* E_j(z_1) E_{j+1}(z_2) = \left[u_2 - u_1 + \frac{\delta}{N}\right]^* E_{j+1}(z_2) E_j(z_1)$

- $\left[u_1 - u_2 - 1\right]^* E_j(z_1) E_j(z_2) = \left[u_1 - u_2 + 1\right]^* E_j(z_2) E_j(z_1)$

- $\left[E_i(z_1), F_j(z_2)\right] = \frac{\delta_{ij}}{q - q^{-1}} \times$

$\times \left(\delta\left(q^k \frac{z_2}{z_1}\right) H_i(q^r z_2) - \delta\left(q^{-k} \frac{z_2}{z_1}\right) H_i(q^{-r} z_2) \right).$

- Construction of

  Infinitely many commuting operators G_m

  Elliptic deformation of KdV Theory

  $U_{qp}(\widehat{sl}(N))$

  [Kojima, Shiraishi, CMP283, 795-851 (2008)]

Free Field Realization

Level $k = 1$

q-Frenkel-Kac realization

Boson

$$[B_m^{\bar{i}}, B_n^{\bar{j}}] = m \frac{[(r-1)m]_q}{[rm]_q} \times \begin{cases} \dfrac{[(s-1)m]_q}{[sm]_q} & (\bar{i} = \bar{j}), \\ (-) \dfrac{[m]_q}{[sm]_q} q^{sm \cdot \mathrm{sgn}(\bar{i}-\bar{j})} & (\bar{i} \neq \bar{j}). \end{cases}$$

Zero-Mode

$$[iP_\lambda, Q_\mu] = (\lambda|\mu) \qquad (\lambda, \mu \in \bigoplus_{\bar{j}=1}^{N} \mathbb{Z}\bar{\epsilon}_{\bar{j}})$$

$$(\epsilon_{\bar{i}}|\epsilon_{\bar{j}}) = \delta_{\bar{i}\bar{j}}, \qquad \bar{\epsilon}_{\bar{j}} = \epsilon_{\bar{j}} - \frac{1}{N} \sum_{\bar{j}=1}^{N} \epsilon_{\bar{j}}.$$

Free Field Realization

$U_{qp}(\widehat{gl}_N)$ $\qquad \alpha_j = \epsilon_j - \epsilon_{j+1}$

- $F_j(z) = e^{-i\sqrt{\frac{r-1}{r}}Q\alpha_j}\left(\chi^{(\frac{2s}{N}-1)j}z\right)^{\sqrt{\frac{r-1}{r}}P_{\alpha_j}+\frac{r-1}{r}}$

  $\times : \exp\left(\sum_{m\neq 0}\frac{1}{m}(\beta_m^j-\beta_m^{j+1})(\chi^{\frac{2s}{N}}z)^{-m}\right) :$

- $E_j(z) = e^{-i\sqrt{\frac{r}{r-1}}Q\alpha_j}\left(\chi^{(\frac{2s}{N}-1)j}z\right)^{-\sqrt{\frac{r}{r-1}}P_{\alpha_j}+\frac{r}{r-1}}$

  $\times : \exp\left(-\sum_{m\neq 0}\frac{1}{m}\frac{[rm]_q}{[(r-1)m]_q}(\beta_m^j-\beta_m^{j+1})(\chi^{\frac{2s}{N}}z)^{-m}\right) :$

$(1\leq j\leq N-1)$

Extended Current $F_N(z)$, $E_N(z)$

- $\left[u_1-u_2-\frac{s}{N}\right] F_N(z_1)F_1(z_2) = \left[u_2-u_1+\frac{s}{N}-1\right] F_1(z_2)F_N(z_1)$
- $\left[u_1-u_2-\frac{s}{N}\right] F_{N-1}(z_1)F_N(z_2) = \left[u_2-u_1+\frac{s}{N}-1\right] F_N(z_2)F_{N-1}(z_1)$
- $\left[u_1-u_2+1\right] F_N(z_1)F_N(z_2) = \left[u_1-u_2-1\right] F_N(z_2)F_N(z_1)$

Free Field Realization

$$F_N(z) = e^{i\sqrt{\frac{r-1}{r}}Q\alpha_N} \left(q^{2s-\frac{N}{2}}\right)^{\sqrt{\frac{r-1}{r}}P_{\epsilon_1}+\frac{r-1}{2r}} z^{-\sqrt{\frac{r-1}{r}}P_{\epsilon_1}+\frac{r-1}{2r}}$$

$$\times : \exp\left(\sum_{m\neq 0}\frac{1}{m}\left(q^{-2sm}\beta_m^N-\beta_m^1\right)z^{-m}\right) :$$

Commuting Operators

Elliptic Deformation of KdV theory

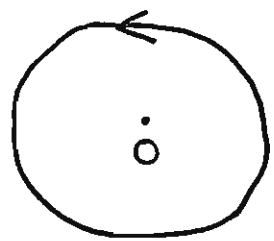
$$\cdot\ G_m = \oint \cdots \oint_C \prod_{t=1}^{N} \prod_{j=1}^{m} \frac{dz_j^{(t)}}{z_j^{(t)}} F_1(z_1^{(1)}) F_1(z_2^{(1)}) \cdots F_1(z_m^{(1)})$$

$$\times F_2(z_1^{(2)}) F_2(z_2^{(2)}) \cdots F_2(z_m^{(2)}) \cdots F_N(z_1^{(N)}) F_N(z_2^{(N)}) \cdots F_N(z_m^{(N)})$$

$$\times \frac{\prod_{t=1}^{N} \prod_{1 \le j < k \le m} \left[u_j^{(t)} - u_k^{(t)} \right] \left[u_k^{(t)} - u_j^{(t)} - 1 \right]}{\prod_{t=1}^{N-1} \prod_{j,k=1}^{m} \left[u_j^{(t)} - u_k^{(t+1)} + 1 - \frac{\hbar}{N} \right] \prod_{j,k=1}^{m} \left[u_j^{(1)} - u_k^{(N)} + \frac{\hbar}{N} \right]}$$

$$\times \prod_{t=1}^{N} \left[\sum_{j=1}^{m} \left(u_j^{(t)} - u_{j+1}^{(t+1)} \right) - \sqrt{\hbar(\hbar-1)}\, P_{\epsilon_{t+1}} \right] .$$

$C : |z_j^{(t)}| = 1$



Theorem

$$[G_m, G_n] = 0 \qquad (m, n \in \mathbb{N})$$

[Kojima, Shiraishi, CMP283, 795-851 (2008)]

- Construction of $\mathcal{L}$ operator for elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{sl}(N))$

$\mathcal{L}$-operator, Vertex operator, ABF-model

[Kojima, Konno, CMP239, 405-447 (2003)]

[Jimbo, Konno, Odake, Shiraishi, CMP199, 605-647 (1999)]

R-matrix　Yang-Baxter equation　$U_{qp}(\widehat{sl}(N))$

$$\cdot R(u,s) = \rho(u) \times \Big\{ \sum_{j=1}^{N} E_{jj} \otimes E_{jj} + \sum_{1 \le j < \ell \le N} \Big(b(u, s_{j\ell}) E_{jj} \otimes E_{\ell\ell} + \overline{b}(u) E_{\ell\ell} \otimes E_{jj} \Big) + \sum_{1 \le j < \ell \le N} \Big(c(u, s_{j\ell}) E_{j\ell} \otimes E_{\ell j} + \overline{c}(u, s_{j\ell}) E_{\ell j} \otimes E_{j\ell} \Big) \Big\}$$

$$s_{j\ell} = \sum_{m=j}^{\ell-1} s_m \quad , \quad s = (s_1, s_2, \cdots, s_N)$$

$$b(u,s) = \frac{[s+1][s-1][u]}{[s]^2[u+1]} \quad , \quad \overline{b}(u) = \frac{[u]}{[u+1]}$$

$$c(u,s) = \frac{[1][s+u]}{[s][u+1]} \quad , \quad \overline{c}(u,s) = \frac{[1][s-u]}{[s][u+1]}$$

$$\rho(u) = q^{\frac{N-1}{N}} z^{\frac{N-1}{2N}} \frac{\{pq^2 z\}\{pq^{2N-2} z\}\{1/z\}\{q^{2N}/z\}}{\{pz\}\{pq^{2N} z\}\{q^2/z\}\{q^{2N-2}/z\}} \quad , \quad \{z\} = (z; p, q^{2N})_\infty$$

YBE
The first inversion
The second inversion

L-operator $\hat{L}(u)$

RLL relation $RLL = LLR^*$

$(R^*(u) = R(u)|_{h \to h^* = h - k})$

$$R^{(1,2)}(u_1 - u_2, Pth)\hat{L}^{(1)}(u_1)\hat{L}^{(2)}(u_2) = \hat{L}^{(2)}(u_2)\hat{L}^{(1)}(u_1)R^{*(1,2)}(u_1 - u_2, P)$$

RLL relation characterize

the elliptic quantum group $\mathcal{B}_{q\lambda}(\hat{sl}(N))$

[Jimbo, Konno, Odake, Shiraishi, Transf. Group 4, 303–327 (1999)]

$\boxed{\text{L-operator}}$ $\quad \hat{L}(u) \qquad B_{q\lambda}(\widehat{sl}(N)).$

$$\hat{L}(u) = \begin{bmatrix} 1 & F_{12}(u) & F_{13}(u) & \cdots & F_{1N}(u) \\ & 1 & F_{23}(u) & \cdots & F_{2N}(u) \\ & & 1 & \cdots & \\ & \bigcirc & & \ddots & \\ & & & 1 & F_{N-1\,N}(u) \\ & & & & 1 \end{bmatrix} \times \begin{bmatrix} K_1(u) & & & \\ & K_2(u) & & \bigcirc \\ & & \ddots & \\ \bigcirc & & & K_N(u) \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & & \\ E_{21}(u) & 1 & & \bigcirc & \\ E_{31}(u) & E_{32}(u) & \ddots & & \\ \vdots & & \ddots & \ddots & \\ & & & & 1 \\ E_{N1}(u) & E_{N2}(u) & & E_{NN-1}(u) & 1 \end{bmatrix}$$

$\boxed{\text{Gauß decomposition}}$

[Kojima, Konno, CMP239, 405-447 (2003)]

L-operator

Level-k

$$\cdot F_{\bar{j}\ell}(u) = \oint \cdots \oint \prod_{m=\bar{j}}^{\ell-1} \frac{dz_m}{2\pi i z_m} F_{\ell-1}(u_{\ell-1}) F_{\ell-2}(u_{\ell-2}) \cdots F_{\bar{j}}(u_{\bar{j}})$$

$$\times \frac{[u-u_{\ell-1}+P_{\bar{j}\ell}+h_{\bar{j}\ell}+\frac{\ell-N}{2}-1][1]}{[u-u_{\ell-1}+\frac{\ell-N}{2}][P_{\bar{j}\ell}+h_{\bar{j}\ell}-1]}$$

$$\times \prod_{m=\bar{j}}^{\ell-2} \frac{[u_{m+1}-u_m+P_{\bar{j}\,m+1}+h_{\bar{j}\,m+1}-\frac{1}{2}][1]}{[u_{m+1}-u_m+\frac{1}{2}][P_{\bar{j}\,m+1}+h_{\bar{j}\,m+1}]}$$

$$\cdot E_{\bar{j}\ell}(u) = \oint \cdots \oint \prod_{m=\bar{j}}^{\ell-1} \frac{dz_m}{2\pi i z_m} E_{\bar{j}}(u_{\bar{j}}) E_{\bar{j}+1}(u_{\bar{j}+1}) \cdots E_{\ell-1}(u_{\ell-1})$$

$$\times \frac{[u-u_{\ell-1}-P_{\bar{j}\ell}+\frac{\ell-N}{2}+\frac{k}{2}+1]^*[1]^*}{[u-u_{\ell-1}+\frac{\ell-N}{2}+\frac{k}{2}]^*[P_{\bar{j}\ell}-1]^*}$$

$$\times \prod_{m=\bar{j}}^{\ell-2} \frac{[u_{m+1}-u_m-P_{\bar{j}\,m+1}+\frac{1}{2}]^*[1]^*}{[u_{m+1}-u_m+\frac{1}{2}]^*[P_{\bar{j}\,m+1}-1]^*}$$

[Kojima, Konno, CMP239, 405-447, (2003)]

RLL relation $RLL = LLR^*$

$$R^{(1,2)}(u_1-u_2, P+h)\,\hat{L}^{(1)}(u_1)\,\hat{L}^{(2)}(u_2)$$
$$= \hat{L}^{(2)}(u_2)\,\hat{L}^{(1)}(u_1)\,R^{*(1,2)}(u_1-u_2, P)$$

- Currents $E_j(z), F_j(z)$ play the same role as Drinfeld currents $X_j^{\pm}(z)$ of $U_q(\widehat{sl}(N))$ in construction of L-operator.
- Current $E_j(z), F_j(z)$ are "Drinfeld current" of elliptic quantum group $B_{q,\lambda}(\widehat{sl}(N))$.

Vertex Operator $\widehat{\Phi}(u)$

Vertex Operator $\widehat{\Phi}(u)$ is obtained by intertwing relation of $\mathcal{L}$-operator.

$$\widehat{\Phi}^{(3)}(u_2)\,\widehat{L}^{(1)}(u_1) = R^{(13)}(u_1-u_2, P+h)\,\widehat{L}^{(1)}(u_1)\,\widehat{\Phi}^{(3)}(u_2)$$

Upon $k=1$ we have free field realizations of vertex operators $\widehat{\Phi}(u) = \sum_{j=1}^{\widetilde{N}} \Phi_j(u) \otimes V_j$ as follows.

Vertex Operator

$\Phi_{\bar{j}}(z)$ $\qquad z = q^{2u}$

$$\Phi_{\bar{j}}(z) = \oint \cdots \oint_C \prod_{m=\bar{j}}^{N-1} \frac{dz_m}{z_m}\, \Phi_N(z) F_{N-1}(z_{N-1}) \cdots F_{\bar{j}}(z_{\bar{j}})$$

$$\times \prod_{m=\bar{j}}^{N-1} \frac{[u_{m+1} - u_m + P_{\bar{j},m+1} + h_{\bar{j},m+1} - \frac{1}{2}][1]}{[u_{m+1} - u_m + \frac{1}{2}][P_{\bar{j}\,m+1} + h_{\bar{j}\,m+1}]} \qquad (1 \le \bar{j} \le N-1)$$

$$\Phi_N(z) = :\exp\Big(-\sum_{m\neq 0} \frac{1}{m} B_m^N z^{-m}\Big): = e^{\bar{\Lambda}_{N-1}} z^{(1-\frac{1}{r})h_{\bar{N}} - \frac{1}{r}P_{\bar{N}}} z^{(1-\frac{1}{r})\frac{N-1}{2N}}$$

This vertex operator $\Phi_{\bar{j}}(z)$ coincides with those for sl_N ABF-model. [Kojima, Konno, CMP239 (2003)]

[Asai, Jimbo, Miwa, Pugai, JPA29, 6595–6616 (1996)].

VO has many physical applications.

- Diagonalization of boundary transfermatrix for $U_{qp}(\widehat{sl}(N))$.

Boundary state, sl_N-ABF model.

[Kojima, JMP52, 013501, (2011)]

[Miwa, Weston, Nucl. Phys. B486, 517- (1997)]

K-matrix

Boundary Yang-Baxter equation

$U_{qp}(\widehat{sl}(N))$

$(0<c<1)$

$$\cdot K\left(a+\bar{\epsilon}_\mu \begin{matrix} a \\ b \end{matrix} \middle| u\right)$$

$$= z^{\frac{b-1}{r}\frac{N-1}{N}-\frac{2}{r}a_1}\ \frac{h(z)}{h(z^{-1})}\ \frac{[c-u][a_\mu+c+u]}{[c+u][a_\mu+c-u]}\ \delta_{a,b}.$$

$$\cdot\ h(z)=\frac{(q^{2r+2N-2}/z^2;q^{2r},q^{4N})_\infty(q^{2N+2}/z^2;q^{2r},q^{4N})_\infty}{(q^{2r}/z^2;q^{2r},q^{4N})_\infty(q^{4N}/z^2;q^{2r},q^{4N})_\infty}$$

$$\times\frac{(q^{2N+2c}/z;q^{2r},q^{2N})_\infty(q^{2r-2c}/z;q^{2r},q^{2N})_\infty}{(q^{2N+2r-2c-2}/z;q^{2r},q^{2N})_\infty(q^{2c+2}/z;q^{2r},q^{2N})_\infty}$$

$$\times\prod_{j=2}^{N}\frac{(q^{2r+2N-2c-2a_ij}/z;q^{2r},q^{2N})_\infty(q^{2c+2a_ij}/z;q^{2r},q^{2N})_\infty}{(q^{2r+2N-2c-2a_ij-2}/z;q^{2r},q^{2N})_\infty(q^{2c+2+2a_ij}/z;q^{2r},q^{2N})_\infty}$$

BYBE
Unitary
Initial

Transfer matrix $T_B(z)$ sl_N-ABF boundary

Let us set $(z = q^{2u})$

$$T_B(z) = \sum_{\mu=1}^{N} \Phi^*_\mu(z^{-1}) K\left(a + \bar{\epsilon}_\mu \begin{matrix} a \\ a \end{matrix} \middle| u\right) \Phi_\mu(z).$$

We have

$$[\, T_B(z_1), T_B(z_2) \,] = 0$$

Linear Algebra

undergraduate-course

① $A, B \in \mathrm{End}(V)$: Diagonalizable

② $[A, B] = 0$: Commutative

$\Rightarrow$

$\exists P \in GL(V)$ s.t.

$$P^{-1}AP = \begin{pmatrix} \alpha_1 & & 0 \\ & \alpha_2 & \\ 0 & & \ddots \\ & & & \alpha_n \end{pmatrix}, \quad P^{-1}BP = \begin{pmatrix} \beta_1 & & 0 \\ & \beta_2 & \\ 0 & & \ddots \\ & & & \beta_n \end{pmatrix}$$

Main Result

Level $k=1$

[Kojima, JMP52, 013501, (2011)]

$$T_B(z)|\ell\rangle_B = |\ell\rangle_B$$

An Eigenvector

$$|\ell\rangle_B = e^F\, |\bar{\ell}, \ell\rangle$$

$$F = -\frac{1}{2}\sum_{m>0}\sum_{\bar{j}=1}^{N-1}\sum_{k=1}^{N-1}\frac{1}{m}\frac{[km]_q}{[(k-1)m]_q}\, I_{\bar{j}k}(m)\,\alpha_{-m}^{\bar{j}}\alpha_{-m}^{k}$$

$$+\sum_{m>0}\sum_{\bar{j}=1}^{N-1}\frac{1}{m}\, D_{\bar{j}}(m)\,\beta_{-m}^{\bar{j}}$$

$$I_{\bar{j}k}(m) = I_{k\bar{j}}(m) = \frac{[\bar{j}m]_q[(N-k)m]_q}{[m]_q[Nm]_q} \qquad (1\le\bar{j}\le k\le N-1)$$

$$\alpha_m^{\bar{j}} = q^{-\bar{j}m}(\beta_m^{\bar{j}} - \beta_m^{\bar{j}+1})$$

$$D_{\tilde{j}}(m) = -\theta_m\left(\frac{[(N-\tilde{j})m/2]_q [hm/2]_q^+}{[(h-1)m/2]_q}\, \frac{(3\tilde{j}-N-1)}{2}m\right)$$

$$+ q^{(\tilde{j}-1)m}\, \frac{[(-h+2\pi i\tilde{j}+2c-\tilde{j}+2)m]_q}{[(h-1)m]_q}$$

$$+ q^{(h-2c+2\tilde{j}-2)m}\, \frac{[m]_q}{[(h-1)m]_q}\left(\sum_{k=j+1}^{N-1} q^{-2m\pi i k}\right)$$

$$+ q^{(2\tilde{j}-N)m}\, \frac{[(h-2\pi i N-2c+N-1)m]_q}{[(h-1)m]_q} \,.$$

$$[a]_q^+ = q^a + q^{-a}\,, \qquad \theta_m(x) = \begin{cases} x & m:\text{even} \\ 0 & m:\text{odd} \end{cases}$$

Type-II Vertex Operators $\Psi^{*(a,b)}(z)$

$$\Phi^{(dc)}(z_1)\Psi^{*(ba)}(z_2) = \chi(z_2/z_1)\Psi^{*(ba)}(z_2)\Phi^{(dc)}(z_1)$$

$$\Phi^{*(cd)}(z_1)\Psi^{*(ba)}(z_2) = \chi(z_1/z_2)\Psi^{*(ba)}(z_2)\Phi^{*(cd)}(z_1)$$

$$\chi(z) = z^{-\frac{N-1}{N}} \frac{(-q^{2N-1}/z\,;x^{2N})_\infty (-qz\,;x^{2N})_\infty}{(-q/z\,;x^{2N})_\infty (-q^{2N-1}z\,;x^{2N})_\infty} \quad \text{etc.}$$

Free Field Realization

$$\Psi^{*(l+\bar{\epsilon}_1,l)}(-z^{-1}) = z^{\frac{N}{2(k-1)}\frac{N-1}{N}} e^{i\sqrt{\frac{k}{k-1}}Q\bar{\epsilon}_1} z^{\sqrt{\frac{k}{k-1}}P_{\bar{\epsilon}_1}}$$

$$\times \exp\left(-\sum_{m>0}\frac{[km]_q}{m[(k-1)m]_q}\beta^1_{-m}z^m\right)\exp\left(\sum_{m>0}\frac{[km]_q}{m[(k-1)m]_q}\beta^1_m z^{-m}\right)$$

[Furutsu, Kojima, Quano, Int. J. Mod. Phys. A15, 1533 (2000)] etc.

Excitations

Physical State

$$|\xi_1 \xi_2 \cdots \xi_M\rangle_{\mu_1 \mu_2 \cdots \mu_M}$$

$$= \Psi^{*(l+\bar{\epsilon}_{\mu_1}+\cdots+\bar{\epsilon}_{\mu_M},\, l+\bar{\epsilon}_{\mu_2}+\cdots+\bar{\epsilon}_{\mu_M})}(\xi_1) \cdots$$

$$\cdots \Psi^{*(l+\bar{\epsilon}_{\mu_{M-1}}+\bar{\epsilon}_{\mu_M},\, l+\bar{\epsilon}_{\mu_M})}(\xi_{M-1}) \Psi^{*(l+\bar{\epsilon}_{\mu_M},\, l)}(\xi_M) |k\rangle_B$$

$$T_B(z) |\xi_1 \xi_2 \cdots \xi_M\rangle_{\mu_1 \mu_2 \cdots \mu_M}$$

$$= \prod_{j=1}^{M} \chi(\xi_j / z)\, \chi(1/\xi_j z) \; |\xi_1 \xi_2 \cdots \xi_M\rangle_{\mu_1 \mu_2 \cdots \mu_M}.$$

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Diagonalization of infinite transfer matrix of boundary $U_{q,p}(A^{(1)}_{N-1})$ face model

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We study infinitely many commuting operators $T_B(z)$, which we call infinite transfer matrix of boundary $U_{q,p}(A^{(1)}_{N-1})$ face model. We diagonalize the infinite transfer matrix $T_B(z)$ by using free field realizations of the vertex operators of the elliptic quantum group $U_{q,p}(A^{(1)}_{N-1})$. © *2011 American Institute of Physics*. [doi:10.1063/1.3521604]

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APPENDIX A: SOME FORMULAE

In this appendix we summarize some formulae,

Summary

- We constructed free field realizations of superalgebra $U_q(\widehat{sl}(N|1))$, $U_{qp}(\widehat{sl}(N|1))$ for an arbitrary level k.

- We demonstrate future research projects for superalgebra $U_{qp}(\widehat{sl}(M|N))$.

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